Comparison between WKB solutions and convergent solutions at a regular singular point of simple pole type via the confluence

By

Yoshitsugu TAKEI*

Abstract

Motivated by a physical problem, we discuss a relation between WKB solutions and convergent solutions of a particular second-order linear ordinary differential equation near a simple-pole type turning point in this paper. By applying the confluence process, we obtain an explicit relation between these two solutions.

§ 1. Motivation

To study the mass of meson, which is an interacting particle of mass intermediate between proton and neutron, Shigaki discusses an eigenvalue problem for the following seond-order ordinary differential equation from the viewpoint of the exact WKB analysis in [S]:

$$\phi'' - \eta^2 \left(\frac{9(s^2-1)}{4s(1-s^3)} + \eta^{-1} \frac{(-9\tilde{\lambda})}{4s(1-s^3)} + \eta^{-2} \frac{5-64s^3+32s^6}{16s^2(1-s^3)^2} \right) \phi = 0, \tag{1.1}$$

where s is an independent variable, η is a large parameter and $\tilde{\lambda}$ denotes a spectral parameter. (Eigenvalues are given by $1+\eta^{-1}\tilde{\lambda}$ in terms of $\tilde{\lambda}$.) One peculiar point of this problem is that (1.1) has a special kind of turning points, which is introduced by Koike and called a ghost in [Ko2], at $s=1$. Making full use of this fact that (1.1) has a ghost at $s=1$, Shigaki shows the following result in [S]: Let

$$\phi_{\pm}(s,\eta) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int^s S_{\text{odd}}\, ds \right) \tag{1.2}$$

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*Department of Mathematical Sciences, Doshisha University, Kyotanabe, Kyoto 610-0394, Japan.

be WKB solutions of (1.1), where $S_{\rm odd}$ denotes the odd part of the formal power series solutions of the Riccati equation associated with (1.1), and assume that $\arg \eta$ is contained in a sufficiently small neighborhood of $-\pi/4$. Then, modulo exponentially small terms, the secular equation for (1.1) is given by

$$\kappa + \frac{1}{2} \pm \frac{1}{4} \in \{\, 0, -1, -2, \dots \}, \tag{1.3}$$

where

$$\kappa = -\frac{1}{2\pi i} \oint_{\text{around } s=1} S_{\rm odd}\, ds. \tag{1.4}$$

On the other hand, Shigaki also computes the asymptotics of large eigenvalues numerically in [S]. This amounts to considering Eq. (1.1) in the following form, that is, with keeping η to be a finite parameter and regarding $\nu^2 = \eta\tilde{\lambda}$ as a new large parameter instead of η :

$$\varphi'' - \left[\nu^2 \frac{(-9)}{4s(1-s^3)} + \left(\eta^2 \frac{9(s^2-1)}{4s(1-s^3)} + \frac{5 - 64s^3 + 32s^6}{16s^2(1-s^3)^2} \right) \right] \varphi = 0. \tag{1.5}$$

Note that $s = 1$ is not a turning point of ghost type, but of simple pole type in (1.5). To compare these numerical computations with the above result for the secular equation (1.3), we thus need to consider the following questions:

(1) What occurs in the exact WKB analysis of second-order linear ordinary differential equations when a large parameter η is replaced by a new large parameter, for example, ν in the case of (1.1) and (1.5) ?

(2) In particular, we want to know the asymptotic expansion of $\oint S_{\rm odd}\, ds$ of (1.1) with respect to a new large parameter ν.

(3) More generally, we want to clarify explicit relations between WKB solutions $\phi_\pm(s, \eta)$ of (1.1) and $\varphi_\pm(s, \nu)$ of (1.5).

At the present stage we do not have a complete answer to these questions. In this paper, as the first step toward answering these questions, we discuss a relation between WKB solutions $\varphi_\pm(s, \nu)$ and convergent solutions $u_\pm(s)$ of (1.5) around $s = 1$ via the confluence.

§ 2. Preliminaries

In what follows we use $x = s - 1$ as an independent variable and consider

$$\frac{d^2\varphi}{dx^2} - \left[\nu^2 \frac{9}{4x(x+1)(x^2+3x+3)} + \left(\eta^2 \frac{9(x+2)}{4(x+1)(x^2+3x+3)} + \frac{32x^6+192x^5+480x^4+576x^3+288x^2-27}{16x^2(x+1)^2(x^2+3x+3)^2}\right)\right]\varphi = 0, \tag{2.1}$$

or more generally

$$\frac{d^2\varphi}{dx^2} - \nu^2 Q(x,\nu)\varphi = 0 \tag{2.2}$$

with

$$Q(x,\nu) = \frac{Q_0(x)}{x} + \nu^{-2}\frac{Q_2(x)}{x^2} \tag{2.3}$$

around a turning point $x = 0$ of simple pole type. Here $Q_0(x)$ and $Q_2(x)$ are holomorphic functions at $x = 0$. As is shown by Koike ([Ko1], [Ko3]), one Stokes curve defined by

Figure 1 : Stokes curve (2.4). (A wiggly line designates a cut for $\sqrt{Q_0(x)/x}$.)

$$\Im \int_0^x \sqrt{\frac{Q_0(x)}{x}}\, dx = 0 \tag{2.4}$$

emanates from such a turning point $x = 0$ of simple pole type and the following connection formula holds on (2.4):

$$\begin{cases} \varphi_+^{(0)} \longmapsto \varphi_+^{(0)} + c\varphi_-^{(0)}, \\ \varphi_-^{(0)} \longmapsto \varphi_-^{(0)}, \end{cases} \tag{2.5}$$

where $\varphi_{\pm}^{(0)}$ denote the WKB solutions of (2.2) normalized at $x=0$, that is,

$$\varphi_{\pm}^{(0)}(x,\nu)=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_0^x S_{\mathrm{odd}}\,dx\right), \tag{2.6}$$

and the Stokes constant (or connection constant) c is explicitly given by

$$c=2i\cos\left(2\pi\sqrt{r_2+\frac{1}{4}}\right)\quad\text{with}\quad r_2=\left(\nu^2x^2Q(x,\nu)\right)\Big|_{x=0}=Q_2(0). \tag{2.7}$$

(Here, and in what follows, we assume that the branch of $\sqrt{Q_0(x)/x}$ is chosen so that $\varphi_{+}^{(0)}$ is dominant over $\varphi_{-}^{(0)}$ on the Stokes curve (2.4).) In particular, $r_2=-3/16$ and $c=0$ in the case of (2.1). Otherwise stated, no Stokes phenomenon occurs on the Stokes curve (2.4) for Borel resummed WKB solutions of Eq. (2.1).

On the other hand, $x=0$ is a regular singular point of (2.2) and hence there exist convergent power series solutions at $x=0$. In the case of (2.1) characteristic exponents at $x=0$ are readily computed to be $1/4$ and $3/4$. Therefore (2.1) has the following pair of linearly independent convergent solutions $u_{\pm}(x)$ near $x=0$:

$$u_{+}(x)=x^{3/4}\big(1+O(x)\big),\quad u_{-}(x)=x^{1/4}\big(1+O(x)\big). \tag{2.8}$$

Here, by considering local monodromy of $\varphi_{\pm}^{(0)}(x,\nu)$ around $x=0$, we find that the relation

$$\begin{cases}\varphi_{+}^{(0)}(x,\nu)-\varphi_{-}^{(0)}(x,\nu)=C_{+}u_{+}(x),\\ \varphi_{+}^{(0)}(x,\nu)+\varphi_{-}^{(0)}(x,\nu)=C_{-}u_{-}(x)\end{cases} \tag{2.9}$$

holds with some constants $C_{\pm}$ between $\varphi_{\pm}^{(0)}(x,\nu)$ and $u_{\pm}(x)$. What we want to discuss in this paper is the explicit determination of these constants $C_{\pm}$.

§ 3. Confluence

To determine the constants $C_{\pm}$ appearing in the relation (2.9), we make use of the so-called confluence process in this paper. To be more specific, we replace the potential (2.3) of Eq. (2.1) by

$$Q^{(\rho)}(x,\nu)=\frac{x-\rho}{x^2}Q_0(x)+\nu^{-2}\frac{Q_2(x)}{x^2} \tag{3.1}$$

and take the limit $\rho\to 0$. In this section we first discuss what occurs with WKB solutions in this confluence process.

For $\rho\neq 0$ we consider

$$\frac{d^2\psi}{dx^2}-\nu^2Q^{(\rho)}(x,\nu)\psi=0 \tag{3.2}$$

with the potential $Q^{(\rho)}(x,\nu)$ given by (3.1). Eq. (3.2) has a simple turning point at $x=\rho$, while $x=0$ is a double pole of (3.2). Let $\hat{\theta}=\hat{\theta}^{(\rho)}$ denote the residue of S_{odd} at $x=0$, that is,

$$\hat{\theta}=\hat{\theta}^{(\rho)}=\operatorname*{Res}_{x=0} S_{\mathrm{odd}}=\sqrt{r_2+\frac{1}{4}-\frac{3}{4}\rho\nu^2}. \tag{3.3}$$

Then the characteristic exponents of (3.2) at $x=0$ are expressed as $1/2\pm\hat{\theta}$. Furthermore, letting $\psi_{\pm}^{(\rho)}(x,\nu)$ be WKB solutions of (3.2) normalized at the simple turning point $x=\rho$, we find that the formal analytic continuation (i.e., formal local monodromy) of $\psi_{\pm}^{(\rho)}(x,\nu)$ around $x=0$ is given by

$$\lambda_{\pm}\psi_{\pm}^{(\rho)}(x,\nu), \tag{3.4}$$

where

$$\lambda_{\pm}=\exp\left(2\pi i\left(\frac{1}{2}\pm\hat{\theta}\right)\right)=-\exp\left(\pm 2\pi i\hat{\theta}\right). \tag{3.5}$$

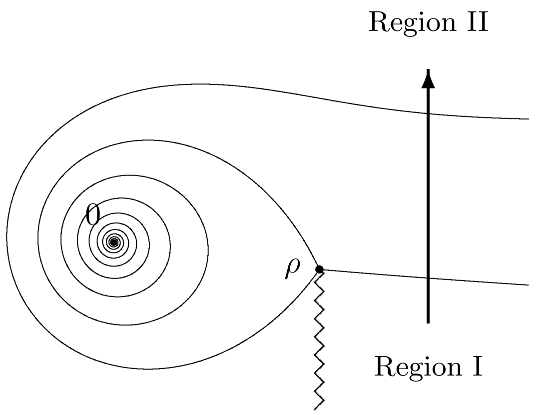

Figure 2 : Stokes curves of Eq. (3.2) near $x=0$ and $x=\rho$.

Since the connection formula at a simple turning point for $\psi_{\pm}^{(\rho)}$ is well known (cf. [KT], [V]), the analytic continuation of (the Borel sum of) $\psi_{\pm}^{(\rho)}$ along any path can be readily computed. For example, the analytic continuation of $\psi_{+}^{(\rho)}$ from Region I to Region II indicated in Figure 2 is explicitly described as follows:

$$\psi_{+}^{(\rho)}\longmapsto\psi_{+}^{(\rho)}+i\left(1+\frac{\lambda_+}{\lambda_-}\right)\psi_{-}^{(\rho)}. \tag{3.6}$$

We now rewrite (3.6) in terms of differently normalized WKB solutions $\psi_{\pm}^{(0)}$ defined by

$$\psi_{\pm}^{(0)}(x,\nu) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm\frac{1}{2}\int_{\gamma^{(0)}} S_{\text{odd}}\,dx\right), \tag{3.7}$$

where the integration path $\gamma^{(0)}$ is indicated in Figure 3. The WKB solutions $\psi_{\pm}^{(0)}$ can be

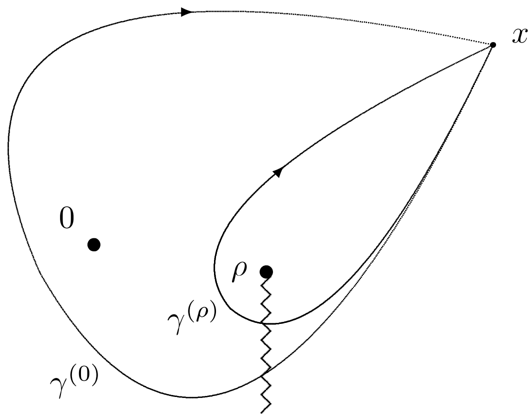

Figure 3 : Integration paths $\gamma^{(0)}$ and $\gamma^{(\rho)}$.

called "WKB solutions normalized at $x=0$" as $\psi_{\pm}^{(0)} \to \varphi_{\pm}^{(0)}$ holds for the limit $\rho \to 0$. Furthermore, since $\psi_{\pm}^{(\rho)}$ is expressed as

$$\psi_{\pm}^{(\rho)}(x,\nu) = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm\frac{1}{2}\int_{\gamma^{(\rho)}} S_{\text{odd}}\,dx\right) \tag{3.8}$$

with $\gamma^{(\rho)}$ being also indicated in Figure 3, we have

$$\psi_{\pm}^{(0)}(x,\nu) = \exp(\mp\pi i\hat{\theta})\psi_{\pm}^{(\rho)}(x,\nu). \tag{3.9}$$

Hence, combining (3.6) with (3.9) and using (3.5), we obtain

$$\psi_{+}^{(0)} \longmapsto \psi_{+}^{(0)} + 2i\cos(2\pi\hat{\theta})\psi_{-}^{(0)} \tag{3.10}$$

for the analytic continuation from Region I to Region II. Taking the limit $\rho \to 0$, we find (3.10) is consistent with Koike's connection formula (2.5) & (2.7) as $\hat{\theta}$ tends to $\sqrt{r_2 + 1/4}$ when $\rho \to 0$. Thus Koike's connection formula at a simple-pole type turning point can be derived from the well-known connection formula at a simple turning point via the confluence.

§ 4. Comparison with convergent solutions

Finally, making use of the confluence process introduced in the preceding section, we explicitly determine the constants $C_\pm$ in the relation (2.9) between WKB solutions and convergent solutions of (2.1) in this section.

To this end, we need a regularization of S_{odd} at a double pole $x = 0$. However, a simple-minded regularization like

$$S_{\mathrm{odd}} = \frac{\hat{\theta}}{x} + (\text{regular part at } x = 0) \tag{4.1}$$

is not good enough as (4.1) destroys the odd character of S_{odd} at $x = \rho$. Another candidate for the regularization

$$S_{\mathrm{odd}} = \frac{\hat{\theta}}{x}\sqrt{1 - \frac{x}{\rho}} + (\text{regular part at } x = 0) \tag{4.2}$$

also has a problem in the sense that it creates a new divergent term with

$$\int^x \frac{\hat{\theta}}{x}\sqrt{1 - \frac{x}{\rho}}\,dx = \log\frac{x}{2\rho} + \cdots . \tag{4.3}$$

($\log x/(2\rho)$ is divergent when $\rho \to 0$.) Having these difficulties in mind, we define the regularization of S_{odd} at $x = 0$ as

$$S_{\mathrm{odd}} = \frac{\sqrt{-3\nu^2}}{2x}\sqrt{\tilde{\rho} - x} + \widetilde{S}_{\mathrm{odd}} \quad \text{with} \quad \tilde{\rho} = \rho - \frac{1}{12\nu^2} \tag{4.4}$$

by neglecting the divergent term $1/(4\rho)$ in the expression

$$\frac{\hat{\theta}}{x}\sqrt{1 - \frac{x}{\rho}} = \frac{1}{2x}\sqrt{\left(\frac{1}{4} - 3\rho\nu^2\right) - \left(\frac{1}{4\rho} - 3\nu^2\right)x}. \tag{4.5}$$

The corresponding WKB solutions denoted by $psi_\pm^{(\mathrm{reg})}$ are then defined by

$$\begin{aligned}(4.6)\quad \psi_\pm^{(\mathrm{reg})} = &\frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp \pm\Bigg\{ \frac{\sqrt{-3\nu^2}}{2}\Big(2\sqrt{\tilde{\rho} - x} \\ &+ \sqrt{\tilde{\rho}}\log\big(\sqrt{\tilde{\rho}} - \sqrt{\tilde{\rho} - x}\big) - \sqrt{\tilde{\rho}}\log\big(\sqrt{\tilde{\rho}} + \sqrt{\tilde{\rho} - x}\big)\Big) + \int_0^x \widetilde{S}_{\mathrm{odd}}\,dx \Bigg\}.\end{aligned}$$

Note that, since the residue at $x = 0$ of the first term of the right-hand side of (4.4) is $\hat{\theta}$ and coincident with that of S_{odd}, $\widetilde{S}_{\mathrm{odd}}$ is holomorphic at $x = 0$.

Once a regularization of S_{odd} is introduced in an appropriate manner, it becomes possible to compare WKB solutions and convergent solutions at the double pole $x = 0$. As a matter of fact, computing the local expansion of S_{odd} at $x = 0$ explicitly by using (4.4), we obtain the following

Proposition 4.1. *Let*

$$u^{(\rho)}_{\pm}(x) = x^{1/2\pm\hat{\theta}}\big(1+O(x)\big) \tag{4.7}$$

be convergent solutions of (3.2) *at* $x=0$. *Then the following relations hold in a sufficiently small neighborhood of* $x=0$*:*

$$\psi^{(\mathrm{reg})}_{+}(x,\nu) = A^{(\rho)}_{+}u^{(\rho)}_{+}(x), \quad \psi^{(\mathrm{reg})}_{-}(x,\nu) + \textit{(exp. small terms)} = A^{(\rho)}_{-}u^{(\rho)}_{-}(x), \tag{4.8}$$

where

$$A^{(\rho)}_{\pm} = \frac{1}{\sqrt{\hat{\theta}}}\exp\Big(\pm\,\hat{\theta}(2-\log(4\tilde{\rho}))\Big). \tag{4.9}$$

Remark. The relation (4.8) is verified by the comparison of the local expansion of $\psi^{(\mathrm{reg})}_{\pm}$ and that of $u^{(\rho)}_{\pm}$ at $x=0$. However, since a dominant WKB solution $\psi^{(\mathrm{reg})}_{-}$ are divergent at $x=0$, to obtain an exact relation we need to specify the Stokes region so that the Borel sum of $\psi^{(\mathrm{reg})}_{-}$ may be defined without any ambiguity. This is the reason why the second relation of (4.8) holds only modulo exponentially small terms. Note that these exponentially small terms are constant multiple of a subdominant WKB solution $\psi^{(\mathrm{reg})}_{+}$ and the multiplicative constsnt can be fixed once the Stokes region is specified. On the other hand, since the other WKB solution $\psi^{(\mathrm{reg})}_{+}$ is subdominant and Borel summable in a neighborhood of $x=0$, the first relation of (4.8) holds modulo no exponentially small terms.

Furthermore, computing $\displaystyle\int_{\gamma^{(\rho)}} S_{\mathrm{odd}}\,dx$ explicitly by using (4.4), we also obtain the following relations between $\psi^{(\mathrm{reg})}_{\pm}$ and $\psi^{(*)}_{\pm}$ ($*=\rho$ or 0).

Proposition 4.2. *The following relations hold:*

$$\psi^{(\mathrm{reg})}_{\pm}(x,\nu) = \psi^{(\rho)}_{\pm}(x,\nu)\exp(\pm\,I^{(\rho)}) = \psi^{(0)}_{\pm}(x,\nu)\exp\pm(\pi i\hat{\theta}+I^{(\rho)}), \tag{4.10}$$

where

$$I^{(\rho)} = \int_0^{\rho}\widetilde{S}_{\mathrm{odd}}\,dx. \tag{4.11}$$

Finally, making use of these propositions, we determine a relation between WKB solutions and convergent solutions in Region I indicated in Figure 4, which is stable during the confluence process. We first use the relation (4.8) in Region III which is also indicated in Figure 4. Taking Remark after Proposition 4.1 into account and combining with Proposition 4.2, we have

$$\begin{aligned} A^{(\rho)}_{+}u^{(\rho)}_{+} &= \psi^{(\mathrm{reg})}_{+} = e^{I^{(\rho)}}\psi^{(\rho)}_{+}, \\ A^{(\rho)}_{-}u^{(\rho)}_{-} &= \psi^{(\mathrm{reg})}_{-} + \alpha\psi^{(\mathrm{reg})}_{+} = e^{-I^{(\rho)}}\psi^{(\rho)}_{-} + \alpha e^{I^{(\rho)}}\psi^{(\rho)}_{+} \end{aligned} \tag{4.12}$$

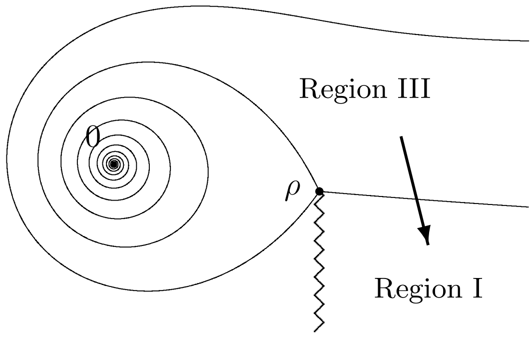

Figure 4

with some constant α. Then the analytic continuation to Region I is described by

$$A_+^{(\rho)} u_+^{(\rho)} \longmapsto e^{I^{(\rho)}} (\psi_+^{(\rho)} - i\psi_-^{(\rho)}) = e^{I^{(\rho)}} \left(e^{\pi i \hat{\theta}} \psi_+^{(0)} - i e^{-\pi i \hat{\theta}} \psi_-^{(0)} \right), \tag{4.13}$$

$$A_-^{(\rho)} u_-^{(\rho)} \longmapsto e^{-I^{(\rho)}} \psi_-^{(\rho)} + \alpha e^{I^{(\rho)}} (\psi_+^{(\rho)} - i\psi_-^{(\rho)})$$
$$= \alpha e^{I^{(\rho)} + \pi i \hat{\theta}} \psi_+^{(0)} + \left(e^{-I^{(\rho)}} - i\alpha e^{I^{(\rho)}} \right) e^{-\pi i \hat{\theta}} \psi_-^{(0)}.$$

Thus, as $u_\pm^{(\rho)}(x)$ are convergent solutions, we have the following relation in Region I:

$$A_+^{(\rho)} u_+^{(\rho)} = e^{I^{(\rho)}} \left(e^{\pi i \hat{\theta}} \psi_+^{(0)} - i e^{-\pi i \hat{\theta}} \psi_-^{(0)} \right), \tag{4.14}$$

$$A_-^{(\rho)} u_-^{(\rho)} = \alpha e^{I^{(\rho)} + \pi i \hat{\theta}} \psi_+^{(0)} + \left(e^{-I^{(\rho)}} - i\alpha e^{I^{(\rho)}} \right) e^{-\pi i \hat{\theta}} \psi_-^{(0)}$$

We now assume that $I^{(\rho)} \to 0$ holds when $\rho \to 0$. (Although we have not verified it rigourously yet, it is reasonable to assume this since $\widetilde{S}_{\mathrm{odd}}$ is holomorphic at $x = 0$.) Then, by considering the limit $\rho \to 0$ and taking account of the fact that $\hat{\theta} \to 1/4$ and $\tilde{\rho} \to -1/(12\nu^2)$ for $\rho \to 0$, we obtain

$$A_+ u_+ = e^{\pi i/4} (\varphi_+^{(0)} - \varphi_-^{(0)}), \quad A_- u_- = \alpha e^{\pi i/4} \varphi_+^{(0)} + (1 - i\alpha) e^{-\pi i/4} \varphi_-^{(0)}, \tag{4.15}$$

where

$$A_\pm = \lim_{\rho \to 0} A_\pm^{(\rho)} = 2 \exp \pm \left(\frac{1}{2} + \frac{1}{4} \log(-3\nu^2) \right). \tag{4.16}$$

Comparing (4.15) with (2.9), we find $\alpha = -i/2$ and consequently we have

$$\begin{cases} \varphi_+^{(0)} - \varphi_-^{(0)} = 2 e^{C_0} e^{-\pi i/4} u_+(x), \\ \varphi_+^{(0)} + \varphi_-^{(0)} = 4 e^{-C_0} e^{\pi i/4} u_-(x) \end{cases} \tag{4.17}$$

with

$$C_0 = \frac{1}{2} + \frac{1}{4}\log(-3\nu^2). \tag{4.18}$$

Thus we have obtained an explicit relation between WKB solutions $\varphi_{\pm}^{(0)}$ and convergent solutions $u_{\pm}(x)$ of (2.1).

§ 5. Concluding remarks

By applying the confluence process, we have succeeded in obtaining an explicit relation between WKB solutions and convergent solutions around a simple-pole type turning point of Eq. (1.5), or equivalently (2.1), in this paper. Although a very particular equation (1.5) or (2.1) is discussed in the paper, the argument employed here is applicable to general equations of the form (2.2).

If we apply a similar confluence argument to Eq. (1.1) near its ghost point, it is expected that a relation between WKB solutions and convergent solutions of (1.1) may be also obtained. Then, by eliminating convergent solutions from these two relations, we can expect to obtain a relation between WKB solutions of (1.1) and those of (1.5). In the case of a ghost point, however, the computation becomes much more complicated and we have not completed it yet. When we obtain some definite results for the confluence process near a ghost point, we will report them somewhere else. Furthermore, it is a very important future problem to discuss the effect of changing a large parameter in the exact WKB analysis in more general setting.

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