Relation between the hypergeometric function and WKB solutions in the neighborhood of a simple-pole-type turning point

By

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Abstract

The Gauss hypergeometric differential equation with a large parameter deformable to a Schrödinger equation with a simple pole at the origin is investigated. In this announcement paper, the relations between the standard solutions of the hypergeometric differential equation in the neighborhood of the origin and Borel sums of WKB solutions are given.

§ 1. Introduction

In this article we study a Schrödinger equation with a large parameter η:

$$\Big(-\frac{d^2}{dx^2}+\eta^2\big(\frac{Q_0}{x}+\eta^{-2}\frac{Q_2}{x^2}\big)\Big)\psi=0 \tag{1.1}$$

from the viewpoint of exact WKB analysis. Here

$$Q_0=\frac{(\alpha-\beta)^2x+4\alpha\beta}{4(x-1)^2},\ Q_2=-\frac{x^2-x+1}{4(x-1)^2} \tag{1.2}$$

and α,β are complex parameters. Equation (1.1) comes from the Gauss hypergeometric differential equation with complex parameters a,b and c:

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0. \tag{1.3}$$

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We introduce the large parameter η in (1.3) as

$$a = \frac{1}{2} + \alpha\eta, b = \frac{1}{2} + \beta\eta, c = 1 \tag{1.4}$$

and have the following equation:

$$x(1-x)\frac{d^2w}{dx^2} + (1 - ((\alpha+\beta)\eta + 2)x)\frac{dw}{dx} - (\alpha\eta + \frac{1}{2})(\beta\eta + \frac{1}{2})w = 0. \tag{1.5}$$

Moreover, eliminating the first-order term by taking

$$w = x^{-1/2}(1-x)^{-((\alpha+\beta)\eta+1)/2}\psi \tag{1.6}$$

as a new unknown function, we get (1.1). Equation (1.1) has regular singular points at $0, 1, \infty$ and one simple pole at $x = 0$ in its potential function. The equation (1.5) has fundamental solutions at the origin:

$$w_1 = {}_2F_1(a, b, 1; x), \tag{1.7}$$

$$w_2 = {}_2F_1(a, b, 1; x)\log x + \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(k!)^2} \sum_{l=0}^{k-1} \left(\frac{1}{a+l} + \frac{1}{b+l} - \frac{2}{1+l} \right) x^k, \tag{1.8}$$

where ${}_2F_1(a, b, c; x)$ denotes the hypergeometric function defined by the hypergeometric series:

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n \tag{1.9}$$

which is convergent in the open unit disk with the center at the origin in the complex plane (cf. [2]). Here the Pochhammer symbol $(a)_n$ stands for $a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$.

On the other hand, we can construct formal solutions which are called WKB solutions of equation (1.1). It is known that the WKB solutions normalized appropriately are Borel summable and Borel sums of the WKB solutions are analytic solutions to the equation (1.1) (cf. [4], [6]). In [5], the relations between the standard solutions in the neighborhood of $x = 1$ of (1.5) and the Borel sums of the WKB solutions (cf. [3]) are established for each topological type of the Stokes curves. Moreover, Aoki and the authors obtained the relation between the hypergeometric function (1.9) with

$$a = \alpha_0 + \alpha\eta, \ b = \beta_0 + \beta\eta, \ c = \gamma_0 + \gamma\eta \tag{1.10}$$

and the Borel sums of the WKB solutions normalized at the origin of the hypergeometric equation with η (cf. [1]).

In this report, we give a concrete form of the relations between (w_1, w_2) and Borel sums of WKB solutions. Detailed discussions and proofs will be given in our article in preparation.

§ 2. The hypergeometric differential equation with a large parameter

We set

$$\tilde{E}_0 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid \alpha\beta(\alpha - \beta)(\alpha + \beta) = 0\}, \tag{2.1}$$

$$\tilde{E}_1 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid \mathrm{Re}\alpha\ \mathrm{Re}\beta = 0\}, \tag{2.2}$$

$$\tilde{E}_2 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid \mathrm{Re}(\alpha - \beta)\ \mathrm{Re}(\alpha + \beta) = 0\}. \tag{2.3}$$

If (α, β) is not contained in $\tilde{E}_0$ there is simple zero of Q_0/x which does not coincide with any one of $0, 1, \infty$. The (simple) zero which is denoted by τ and simple pole at the origin are called (simple) turning point of (1.1). Hereafter we assume that $(\alpha, \beta) \notin \tilde{E}_j$ $(j = 0, 1, 2)$. A Stokes curve emanating from $a = \tau, 0$ is, by definition, a curve defined by

$$\mathrm{Im} \int_a^x \sqrt{\frac{Q_0}{x}} dx = 0. \tag{2.4}$$

If (α, β) doesn't belong to $\tilde{E}_1 \cup \tilde{E}_2$, there are no Stokes curves which connect turning point(s). Hence all the Stokes curves flow into some of the singular points $0, 1, \infty$. WKB solutions of (1.1) are, by definition, the formal solutions of (1.1) of the following form:

$$\tilde{\psi}_{a,\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\pm \int_a^x S_{\mathrm{odd}}\, dx\Big), \tag{2.5}$$

where $a = \tau, 0$ and S_{odd} denotes the odd-order part of the formal solution

$$S(x, \eta) = S_{\mathrm{odd}} + S_{\mathrm{even}} = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x) \tag{2.6}$$

in η^{-1} of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q \tag{2.7}$$

to (1.1) having the leading term $S_{-1} = \sqrt{Q_0/x}$. Here the integration in the right-hand side of (2.5) is understood as a half of the integral of $\sqrt{Q_0/x}$ on a contour starting from x in the second sheet of the Riemann surface of $\sqrt{Q_0/x}$, going straight to a and going around it counterclockwise and back to x on the first sheet. Let b be the destination of the Stokes curve emanating from the origin. Since the Stokes geometry is non-degenerate, $b = 1$ or ∞. Let n_1 and n_2 denote the number of Stokes curve(s) emanating from τ that flow(s) into 1 and ∞, respectively. Clearly $n_1 + n_2 = 3$ holds. We set $\tilde{n} = (b; n_1, n_2)$ and call it the type of the Stokes geometry of (1.1). We define the sets $\tilde{\omega}_k$ $(k = 3, 4)$ of the pair of parameters (α, β) as follows:

$$\tilde{\omega}_3 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\beta\}, \tag{2.8}$$

$$\tilde{\omega}_4 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid 0 < \mathrm{Re}\alpha + \mathrm{Re}\beta < \mathrm{Re}\beta\} \tag{2.9}$$

and $\tilde{\Pi}_k$ $(k = 3, 4)$ in $\mathbb{C}^2$ by

$$\tilde{\Pi}_k = \bigcup_{r \in G} r(\tilde{\omega}_k), \tag{2.10}$$

where G is the group generated by the involutions $\tilde{\iota}_l$ $(l = 0, 1)$:

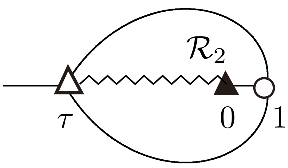
$$\tilde{\iota}_0 : (\alpha, \beta) \mapsto (\beta, \alpha), \tag{2.11}$$

$$\tilde{\iota}_1 : (\alpha, \beta) \mapsto (-\alpha, -\beta). \tag{2.12}$$

We can classify the type of the Stokes geometry of (1.1) by $\tilde{n}$:

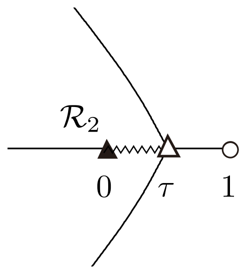
Theorem 2.1. ([5, Proposition 1]) (1) *If* $(\alpha, \beta) \in \tilde{\Pi}_3$, *then* $\tilde{n} = (1; 2, 1)$. (2) *If* $(\alpha, \beta) \in \tilde{\Pi}_4$, *then* $\tilde{n} = (\infty; 1, 2)$.

Some examples are given in Fig. 1 and Fig. 2.



$(\alpha, \beta) = (0.8, 2) \in \tilde{\omega}_3$

Fig. 1



$(\alpha, \beta) = (-0.3, 1.8) \in \tilde{\omega}_4$

Fig. 2

Here the triangle, the black triangle, the circle and the wavy line designate the turning point τ, the simple pole 0, the singular point 1 and the branch cut for $\sqrt{Q_0/x}$, respectively. Moreover, a region whose boundary consists of Stokes curves is called the Stokes region. A region whose boundary consists of Stokes curves and the branch cut are labeled $\mathcal{R}_2$ as shown in Fig. 1 and Fig. 2.

The WKB solutions are Borel summable on $\mathcal{R}_2$. Let $\tilde{\Psi}^{2,k}_{0,\pm}$ $(k = 3, 4)$ denote the Borel sums of $\tilde{\psi}_{0,\pm}$ in $\mathcal{R}_2$. We obtain the relation between (w_1, w_2) and $(\tilde{\Psi}^{2,k}_{0,+}, \tilde{\Psi}^{2,k}_{0,-})$ and have

Theorem 2.2. *Suppose that* $(\alpha, \beta) \in \tilde{\omega}_k$ $(k = 3, 4)$. *Then* (w_1, w_2) *and the Borel sum* $(\tilde{\Psi}^{2,k}_{0,+}, \tilde{\Psi}^{2,k}_{0,-})$ *on* $\mathcal{R}_2$ *of the WKB solution* $(\tilde{\psi}_{0,+}, \tilde{\psi}_{0,-})$ *are related by*

$$(w_1, w_2) = \frac{x^{-1/2}(1-x)^{-(a+b)/2}}{2\sqrt{\pi}} (\tilde{\Psi}^{2,3}_{0,+}, \tilde{\Psi}^{2,3}_{0,-}) \begin{pmatrix} 1 & -(\psi(a) + \psi(b) + 2\gamma) \\ i & -i(\psi(a) + \psi(b) + 2\gamma + 2\pi) \end{pmatrix} \tag{2.13}$$

and

$$(w_1, w_2) = \frac{x^{-1/2}(1-x)^{-(a+b)/2}}{2\sqrt{\pi}} (\tilde{\Psi}^{2,4}_{0,+}, \tilde{\Psi}^{2,4}_{0,-}) \begin{pmatrix} 1 & -(\psi(1-a) + \psi(b) + 2\gamma + \pi i) \\ -i & i(\psi(1-a) + \psi(b) + 2\gamma) - \pi \end{pmatrix}. \tag{2.14}$$

Here $\psi(z)$ *and* γ *are the digamma function defined by*

$$\psi(z) = \frac{d}{dz} \ln(\Gamma(z)) \tag{2.15}$$

and Euler's constant, respectively.

A complete proof of this theorem will be given in our forthcoming paper.

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