Differential-difference equations satisfied by the Voros coefficients at the unit for the generalized hypergeometric differential equation with a large parameter

By

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Abstract

Differential-difference equations satisfied by the Voros coefficients at the unit are given for the generalized hypergeometric differential equation for ${}_3F_2$ with a large parameter.

§ 1. Introduction

The Voros coefficients are primarily introduced for second-order ordinary differential equations with a large parameter [2, 5, 8, 10, 11, 13]. Explicit forms of them for Gauss hypergeometric differential equation and its confluent differential equations have been obtained in those articles. Recently they are introduced for some higher-order ordinary differential equations [3, 4, 12]. The authors has defined them of the origin and the infinity for the generalized hypergeometric differential equations and obtained the explicit forms [3, 4]. In this article, we consider the Voros coefficients of the unit for the generalized hypergeometric differential equation for ${}_3F_2$ with a large parameter positive η:

$$ {}_3P_2\psi = 0, \tag{1.1}$$

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where we set

$$ {}_3P_2 = \eta^{-3}\Big((\vartheta_x + b_1)(\vartheta_x + b_2)\partial_x - (\vartheta_x + a_1)(\vartheta_x + a_2)(\vartheta_x + a_3)\Big), \tag{1.2} $$

$\partial_x = \dfrac{d}{dx}$, $\vartheta_x = x\dfrac{d}{dx}$, $a_i = a_{i0} + a_{i1}\eta$ $(i = 1, 2, 3)$ and $b_j = b_{j0} + b_{j1}\eta$ $(j = 1, 2)$. The generalized hypergeometric series (or function)

$$ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k k!} x^k $$

is a solution to (1.1). The generalized Riemann scheme ([9]) of (1.1) can be written as

$$ \left\{\begin{matrix} x = 0 & x = 1 & x = \infty \\ 0 & [0]_{(2)} & a_1 \\ 1 - b_1 & & a_2 \\ 1 - b_2 & b_1 + b_2 - a_1 - a_2 - a_3 & a_3 \end{matrix}\right\}. \tag{1.3} $$

We may define the Voros coefficients of the unit for (1.1) in a similar way to the case of the origin and the infinity. Computation of their explicit forms contains several difficulties which come from the multiplicity of the eigenvalues of the local monodromy at the unit. However, we have obtained the local behaviors at the unit of the WKB solutions of (1.1) and derived the systems differential-difference equations which characterize the Voros coefficients at the unit. These are main results of this article (Lemma 2.2, Theorem 2.3). To solve them and obtain the explicit forms of the Voros coefficients are our next problems.

§ 2. WKB solutions and Voros coefficients

The total symbol $\sigma({}_3P_2)(x, \xi)$ of ${}_3P_2$ is, by definition, a polynomial obtained by replacing ∂_x by a variable ξ in (1.2) and it is written in the form

$$ \sigma({}_3P_2)(x, \xi) = \sum_{k=0}^{3} \eta^{-k} \sigma_k({}_3P_2)(x, \zeta), $$

where we set $\zeta = \xi/\eta$ ([1, 6]). We call $\sigma_0({}_3P_2)(x, \zeta)$ the principal symbol of ${}_3P_2$. The explicit formula for $\sigma_0({}_3P_2)(x, \zeta)$ is given by the following:

$$ \begin{aligned} \sigma_0({}_3P_2)(x, \zeta) &= \zeta(x\zeta + b_{11})(x\zeta + b_{21}) - (x\zeta + a_{11})(x\zeta + a_{21})(x\zeta + a_{31}) \\ &= x^2(1 - x)\zeta^3 + x\left(b_{11} + b_{21} - x\left(a_{11} + a_{21} + a_{31}\right)\right)\zeta^2 \\ &\quad + \left(b_{11}b_{21} - x\left(a_{11}a_{21} + a_{31}a_{21} + a_{11}a_{31}\right)\right)\zeta - a_{11}a_{21}a_{31}. \end{aligned} $$

A point $x_* \in \mathbb{C}$ is called a turning point of ${}_3P_2$ with the characteristic value ζ_* if

$$\sigma_0({}_3P_2)\,(x_*,\zeta_*) = \partial_\zeta \sigma_0({}_3P_2)\,(x_*,\zeta_*) = 0$$

([1, 6, 7]). The turning point x_* is said to be simple if $\partial_\zeta^2 \sigma_0({}_3P_2)\,(x_*,\zeta_*) \neq 0$ and $\partial_x \sigma_0({}_3P_2)\,(x_*,\zeta_*) \neq 0$. In this article, we assume that all turning points of the equation (1.1) are simple. Equation (1.1) has four simple turning points under suitable assumptions ([3]). Outside the turning points, there are three roots of the cubic equation $\sigma_0({}_3P_2)(x,\zeta) = 0$ in ζ, which are denoted by ζ_ℓ ($\ell = 1,2,3$) and called the characteristic roots of ${}_3P_2$. For a simple turning point x_*, there are two numbers $j,k \in \{1,2,3\}$, $j \neq k$ such that $\zeta_j(x_*) = \zeta_k(x_*)$. Then we say that x_* is a simple turning point of type (j,k) ([1, 6]).

Definition 2.1. The characteristic roots of ${}_3P_2$ are numbered so that $\zeta_1^{(1)}$, $\zeta_2^{(1)}$ and $\zeta_3^{(1)}$ which have local behavior as

$$\begin{aligned}
\zeta_1^{(1)} &= \frac{b_{11}b_{21} - a_{11}a_{21} - a_{21}a_{31} - a_{31}a_{11} + \sqrt{D_1}}{2\,(a_{11} + a_{21} + a_{31} - b_{11} - b_{21})} + O(x-1),\\
\zeta_2^{(1)} &= \frac{b_{11}b_{21} - a_{11}a_{21} - a_{21}a_{31} - a_{31}a_{11} - \sqrt{D_1}}{2\,(a_{11} + a_{21} + a_{31} - b_{11} - b_{21})} + O(x-1),\\
\zeta_3^{(1)} &= \frac{b_{11} + b_{21} - a_{11} - a_{21} - a_{31}}{x-1} + O(1),
\end{aligned}$$

where

$$\begin{aligned}
D_1 = &\,(a_{11}a_{21} + a_{21}a_{31} + a_{31}a_{11} - b_{11}b_{21})^2\\
&- 4a_{11}a_{21}a_{31}\,(a_{11} + a_{21} + a_{31} - b_{11} - b_{21})\,.
\end{aligned}$$

Note that the leading terms of $\zeta_1^{(1)}$ and $\zeta_2^{(1)}$ are the roots of the following quadratic equation:

$$(a_{11} + a_{21} + a_{31} - b_{11} - b_{21})\,\zeta^2 + (a_{11}a_{21} + a_{21}a_{31} + a_{31}a_{11} - b_{11}b_{21})\,\zeta + a_{11}a_{21}a_{31} = 0.$$

The characteristic variety $\mathrm{Ch}({}_3P_2) = \{(x,\zeta) \mid \sigma_0({}_3P_2)(x,\zeta) = 0\}$ can be regarded as a compact Riemann surface Σ. There is a natural projection

$$\pi : \Sigma \to \mathbb{P}^1_{\mathbb{C}},$$

which is a 3–covering map. By using the Riemann-Hurwitz theorem, the genus of Σ equals 0.

A WKB solution ψ of ${}_3P_2\psi = 0$ is a formal solution of the form

$$\psi = \exp\left(\int S dx\right), \qquad S = \eta S_{-1} + S_0 + \eta^{-1} S_1 + \cdots .$$

The leading term $\zeta = S_{-1}$ satisfies the cubic equation $\sigma_0({}_3P_2)(x,\zeta) = 0$. We choose S_{-1} from $\sigma_0({}_3P_2)(x,\zeta) = 0$, we obtain S_j ($j \geq 0$) recursively outside the turning points.

Lemma 2.2. *Let* $S^{(\ell)} = \sum_{j=-1}^{\infty} \eta^{-j} S_j^{(\ell)}$ $(\ell = 1, 2, 3)$ *be the formal solutions to* (1.1) *such that the numbering are consistent with that of the leading terms given in Definition 2.1:*

$$S_{-1}^{(1)} = \zeta_1^{(1)}, \quad S_{-1}^{(2)} = \zeta_2^{(1)}, \quad S_{-1}^{(3)} = \zeta_3^{(1)}.$$

Then the formal solutions $S^{(\ell)}$ $(\ell = 1, 2, 3)$ *have the following local behaviors near* $x = 1$*:*

$$S^{(1)} = \frac{b_1 b_2 - a_1 a_2 - a_2 a_3 - a_3 a_1 - a_1 - a_2 - a_3 - 1 + \sqrt{D}}{2\,(a_1 + a_2 + a_3 - b_1 - b_2 + 2)} + O(x-1), \tag{2.1}$$

$$S^{(2)} = \frac{b_1 b_2 - a_1 a_2 - a_2 a_3 - a_3 a_1 - a_1 - a_2 - a_3 - 1 - \sqrt{D}}{2\,(a_1 + a_2 + a_3 - b_1 - b_2 + 2)} + O(x-1), \tag{2.2}$$

$$S^{(3)} = \frac{b_1 + b_2 - a_1 - a_2 - a_3}{x-1} + O(1), \tag{2.3}$$

where

$$\begin{aligned} D = {} & (a_1 a_2 + a_2 a_3 + a_3 a_1 + a_1 + a_2 + a_3 - b_1 b_2 + 1)^2 \\ & - 4 a_1 a_2 a_3 \,(a_1 + a_2 + a_3 - b_1 - b_2 + 2)\,. \end{aligned}$$

Proof. We consider expanding S_j $(j \geq 0)$ in a Taylor series around $x = 1$, which denotes

$$S_j = \sum_{k=0}^{\infty} s_{j,k}(x-1)^k.$$

By using the equation $\sigma_0({}_3P_2)(x, S_{-1}) = 0$ and recurrence relations to determine S_j $(j \geq 0)$, we have

$$-s_{-1,0}(2A_3 s_{-1,0} + A_1) = -A_3 s_{-1,0}^2 + a_{11}a_{21}a_{31}, \tag{2.4}$$

$$\begin{aligned} -s_{0,0}(2A_3 s_{-1,0} + A_1) = {} & A_2 s_{-1,0}^2 + A_4 s_{-1,0} \\ & + a_{11}a_{21}a_{31} + a_{11}a_{21}a_{31} + a_{11}a_{21}a_{31}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} -s_{1,0}(2A_3 s_{-1,0} + A_1) = {} & 2A_2 s_{0,0} s_{-1,0} + A_3 s_{0,0}^2 + A_4 s_{0,0} + A_5 s_{-1,0} \\ & + a_{11}a_{20}a_{30} + a_{10}a_{21}a_{30} + a_{10}a_{20}a_{31}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} -s_{2,0}(2A_3 s_{-1,0} + A_1) = {} & A_2(s_{0,0}^2 + 2s_{-1,0}s_{1,0}) + 2A_3 s_{0,0} s_{-1,0} + A_4 s_{1,0} \\ & + A_5 s_{0,0} + a_{10}a_{20}a_{30}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} -s_{j,0}(2A_3 s_{-1,0} + A_1) = {} & A_2 \sum_{\substack{-1 \leq k,m \\ k+m=j-2}} s_{k,0} s_{m,0} + A_3 \sum_{\substack{0 \leq k,m \\ k+m=j-1}} s_{k,0} s_{m,0} \\ & + A_4 s_{j-1,0} + A_5 s_{j-2,0} \qquad (j \geq 3), \end{aligned} \tag{2.8}$$

where

$$
\begin{aligned}
A_1 &= a_{11}a_{21} + a_{21}a_{31} + a_{31}a_{11} - b_{11}b_{21},\\
A_2 &= a_{10} + a_{20} + a_{30} - b_{10} - b_{20} + 2,\\
A_3 &= a_{11} + a_{21} + a_{31} - b_{11} - b_{21},\\
A_4 &= a_{11} + a_{21} + a_{31} - b_{11}b_{20} - b_{10}b_{21}\\
&\qquad + a_{11}(a_{20} + a_{30}) + a_{21}(a_{30} + a_{10}) + a_{31}(a_{20} + a_{10}),\\
A_5 &= a_{10} + a_{20} + a_{30} + a_{10}a_{20} + a_{20}a_{30} + a_{30}a_{10} - b_{10}b_{20} + 1.
\end{aligned}
$$

We multiply both sides of (2.4) by η^{-1}, both sides of (2.6) by η, both sides of (2.7) by η^2, both sides of (2.8) by η^j, and then consider their sum. Then we obtain

$$
\begin{aligned}
-s(2A_3 s_{-1,0} + A_1) = \eta^{-2}A_2 s^2 &+ \eta^{-1}A_3(s - \eta s_{-1,0})^2\\
&+ \eta^{-1}A_4 s + \eta^{-2}A_5 s - \eta s_{-1,0}^2 A_3 + \eta^{-2}a_1a_2a_3,
\end{aligned}
\tag{2.9}
$$

where $s = \sum_{j=-1}^{\infty} s_{j,0}\eta^{-j}$. Equation (2.9) can be simplified into the following quadratic equation:

$$
\begin{aligned}
&(a_1 + a_2 + a_3 - b_1 - b_2 + 2)s^2\\
&\qquad + (a_1a_2 + a_2a_3 + a_3a_1 + a_1 + a_2 + a_3 - b_1b_2 + 1)\, s + a_1a_2a_3 = 0.
\end{aligned}
$$

Hence we have (2.1) and (2.2). Similarly, we obtain (2.3). □

The Voros coefficient of type (j, k) at $x = 1$ is defined by

$$
V_1^{(j,k)} = \frac{1}{2}\lim_{x\to 1}\int_{\gamma_{jk}} (S - \eta S_{-1} - S_0)\, dx
$$

([3]). Here γ_{jk} denotes a path on Σ starting from $x = 1$ on the j-th sheet, going to and detouring turning point τ counterclockwise and coming back to $x = 1$ on the k-th sheet.

Theorem 2.3. *Let*

$$
\begin{aligned}
s^{(1)} &= \frac{b_1b_2 - a_1a_2 - a_2a_3 - a_3a_1 - a_1 - a_2 - a_3 - 1 + \sqrt{D}}{2(a_1 + a_2 + a_3 - b_1 - b_2 + 2)},\\
s^{(2)} &= \frac{b_1b_2 - a_1a_2 - a_2a_3 - a_3a_1 - a_1 - a_2 - a_3 - 1 - \sqrt{D}}{2(a_1 + a_2 + a_3 - b_1 - b_2 + 2)},\\
s^{(3)} &= b_1 + b_2 - a_1 - a_2 - a_3,
\end{aligned}
$$

where

$$
\begin{aligned}
D = (&a_1a_2 + a_2a_3 + a_3a_1 + a_1 + a_2 + a_3 - b_1b_2 + 1)^2\\
&- 4a_1a_2a_3(a_1 + a_2 + a_3 - b_1 - b_2 + 2).
\end{aligned}
$$

Then $V_1^{(j,3)}$ $(j = 1, 2)$ *satisfies the following differential-difference equations*

$$2\partial_{a_{i1}}\Delta_{a_{i1}}V_1^{(j,3)} = \partial_{a_{i1}} \log \frac{s^{(3)}}{s^{(j)} + a_i} + \Delta_{b_{m1}} g_{a_{i1}}^{(j,3)} \qquad (i = 1, 2, 3),$$

$$2\partial_{b_{m1}}\Delta_{b_{m1}}V_1^{(j,3)} = -\partial_{b_{m1}} \log \frac{\tilde{s}^{(3)}}{\tilde{s}^{(j)} + b_m} + \Delta_{b_{m1}} g_{b_{m1}}^{(j,3)} \qquad (m = 1, 2).$$

Here $\Delta_\rho = e^{\eta^{-1}\partial_\rho} - 1$ $(\rho = a_{11}, a_{21}, \ldots, b_{21})$, $\tilde{s}^{(\ell)} = s^{(\ell)}\Big|_{b_{m1} \to b_{m1} + \eta^{-1}}$ $(\ell = 1, 2, 3, m = 1, 2)$ *and* $g_\rho^{(j,3)}$ $(\rho = a_{11}, a_{21}, \ldots, b_{21})$ *is a linear functions of* η.

The idea of the proof of Theorem 2.3 is similar to [3, Lemma 3.6], i.e., we derive a system of differential-difference equations which characterizes $V_1^{(j,3)}$ with respect to $a_{11}, a_{21}, \ldots, b_{21}$ by using ladder operators and formal differential operators of infinite order.

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