On the existence of Laplace hyperfunction solutions for a system of PDEs with constant coefficients

By

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Abstract

We introduce a Laplace transformation of a Laplace hyperfunction with a Čech-Dolbeault representative and its Laplace inverse transformation. As an application, we show the existence of Laplace hyperfunctions solutions for a system of PDEs with constant coefficients.

§ 1. Introduction

The theory of Laplace hyperfunctions of one variable was established by H. Komatsu in order to consider the Laplace transform of a hyperfunction ([5] – [10]). The theory of Laplace hyperfunctions of several variables has been established by the authors ([2], [3], [4]). In the paper [11], intuitive representation of a Laplace hyperfunction was constructed. Thanks to that, we become able to manipulate a Laplace hyperfunction like a function as it is represented by holomorphic functions of exponential type at infinity on wedges of type. Recently, N. Honda, T. Izawa and T. Suwa succeeded in constructing a representation of Sato's hyperfunction by Čech-Dolbeault cohomology groups in the paper [1]. In this note, we introduce a representation of Laplace hyperfunction by Čech-Dolbeault cohomology groups. Furthermore, we construct the Laplace transformation and its Laplace inverse transformation. We also give its application to PDE with constant coefficients. For the detail, we refer the reader to the forthcoming paper [2].

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§ 2. Laplace transformation for Laplace hyperfunctions

In this section, we give a Čech-Dolbeault representation of a Laplace hyperfunction. Moreover, we introduce a Laplace transformation and its Laplace inverse transformation.

§ 2.1. Laplace hyperfunctions

Let M be an n-dimensional real vector space with the norm $\|\bullet\|$ and $E := M \otimes_{\mathbb{R}} \mathbb{C}$. We denote by $\mathbb{D}_E := E \sqcup S^{2n-1}_\infty$ (resp. $\mathbb{D}_M := M \sqcup S^{n-1}$) the radial compactification of E (resp. M)(see Definition 2.1 [3]). Set $M_\infty := \mathbb{D}_M \setminus M$ and $E_\infty := \mathbb{D}_E \setminus E$. We define an $\mathbb{R}_+$-action on $\mathbb{D}_E$ by, for $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{D}_E$,

$$\lambda x = \begin{cases} \lambda x & \text{if } x \in E, \\ x & \text{if } x \in E_\infty. \end{cases}$$

The $\mathbb{R}_+$-action on $\mathbb{D}_M$ is defined to be the restriction of the one in $\mathbb{D}_E$ to $\mathbb{D}_M$. And we also define an addition for $a \in M$ (resp. $a \in E$) and $x \in \mathbb{D}_M$ (resp. $x \in \mathbb{D}_E$) by

$$a + x = \begin{cases} a + x & \text{if } x \in M \text{ (resp. } x \in E), \\ x & \text{if } x \in M_\infty \text{ (resp. } x \in E_\infty). \end{cases}$$

Definition 2.1. Let K be a subset in $\mathbb{D}_M$. We say that K is a cone with vertex $a \in M$ in $\mathbb{D}_M$ if there exists an $\mathbb{R}_+$-conic set $L \subset \mathbb{D}_M$ such that $K = a + L$.

Definition 2.2. Let V be an open subset in E. We define the open subset $\widehat{V}$ in $\mathbb{D}_E$ as follows.

$$\widehat{V} := \mathbb{D}_E \setminus \overline{(E \setminus V)}.$$

Note that we sometimes write ${}^\frown V$ instead of $\widehat{V}$. For an open subset U in M, we can define an open subset $\widehat{U}$ in $\mathbb{D}_M$ in the same way as that in $\mathbb{D}_E$.

Definition 2.3. A closed subset F in $\mathbb{D}_E$ is said to be regular if $\overline{F \cap E} = F$ holds.

Let M^* and E^* be dual vector spaces of M and E, respectively. We denote by $\mathbb{D}_{M^*}$ and $\mathbb{D}_{M^*}$ the radial compactification of E^* and M^*, respectively. Set $M^*_\infty = \mathbb{D}_{M^*} \setminus M^*$ and $E^*_\infty = \mathbb{D}_{E^*} \setminus E^*$. We also define the open subset $\widehat{V}$ in $\mathbb{D}_{E^*}$ for an open subset V in E^* in the same way as that in $\mathbb{D}_E$. The canonical projection $\varpi_\infty : E^*_\infty \setminus \sqrt{-1}M^*_\infty \to M^*_\infty$ is defined by

$$E^*_\infty \setminus \sqrt{-1}M^*_\infty \ni \xi + \sqrt{-1}\eta \ ((\xi, \eta) \in S^{2n-1}, \, \xi \neq 0) \ \mapsto \ \xi/|\xi| \in M^*_\infty. \tag{2.1}$$

Let Z be a subset in $\mathbb{D}_E$.

Definition 2.4. The subset $\mathrm{N}^*_{pc}(Z)$ in E^*_∞ is defined by

$$\{\zeta \in E^*_\infty;\ \mathrm{Re}\,\langle z,\,\zeta\rangle > 0\ (\forall z \in \overline{Z} \cap E_\infty)\}.$$

Definition 2.5. We say that Z is properly contained in a half space of $\mathbb{D}_E$ with direction $\zeta \in E^*_\infty$ if there exists $r \in \mathbb{R}$ such that

$$\overline{Z} \subset {}^\frown\{z \in E;\ \mathrm{Re}\langle z,\,\zeta\rangle > r\}, \tag{2.2}$$

where ζ is regarded as a unit vector in E^*. If a subset Z is properly contained in a half space of $\mathbb{D}_E$ with some direction, then Z is often said to be proper in $\mathbb{D}_E$. Note that $\zeta \in N^*_{pc}(Z)$ if and only if Z is properly contained in a half space of D_E with the direction ζ.

Let $\mathscr{O}^{\exp}_{\mathbb{D}_E}$ (resp. $\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}$) denote the sheaf of holomorphic functions (resp. p-forms) of exponential type (at ∞) on $\mathbb{D}_E$.

Definition 2.6. The sheaf on $\mathbb{D}_M$ of p-forms of Laplace hyperfunctions is defined by

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M} := \mathscr{H}^n_{\mathbb{D}_M}(\mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \otimes_{\mathbb{Z}_{\mathbb{D}_M}} or_{\mathbb{D}_M/\mathbb{D}_E},$$

where $or_{\mathbb{D}_M/\mathbb{D}_E}$ is the relative orientation sheaf over $\mathbb{D}_M$, that is, it is given by $\mathscr{H}^n_{\mathbb{D}_M}(\mathbb{Z}_{\mathbb{D}_E})$.

Let U be an open subset in $\mathbb{D}_M$, and V an open subset in $\mathbb{D}_E$ with $V \cap \mathbb{D}_M = U$. Then we have

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(U) = \mathrm{H}^n_U(V;\ \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E})\ \otimes_{\mathbb{Z}_{\mathbb{D}_M}(U)}\ or_{\mathbb{D}_M/\mathbb{D}_E}(U).$$

§ 2.2. Čech-Dolbeault representation

In this subsection, we give a representation of a Laplace hyperfunction by Čech-Dolbeault cohomology groups.

Let V be an open subset in $\mathbb{D}_E$ and let f be a measurable function on $V \cap E$. We say that f is of exponential type at ∞ on V if, for any compact subset K in V, there exists $H_K > 0$ such that $|\exp(-H_K|z|)\,f(z)|$ is essentially bounded on $K \cap E$, i.e.,

$$||\exp(-H_K|z|)\,f(z)||_{L^\infty(K\cap E)} < +\infty.$$

We define the set $\mathscr{Q}_{\mathbb{D}_E}(V)$ as follows. A C^∞-function f on $V \cap E$ belongs to $\mathscr{Q}_{\mathbb{D}_E}(V)$ if any higher derivative of f with respect to variables z and $\overline{z}$ is of exponential type on V. Then it is easy to see that $\{\mathscr{Q}_{\mathbb{D}_E}(V)\}_V$ forms the sheaf $\mathscr{Q}_{\mathbb{D}_E}$. Let $\mathscr{Q}^{p,q}_{\mathbb{D}_E}$ denote the sheaf on $\mathbb{D}_E$ of $(p,\,q)$-forms with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$. Define

$$\mathscr{Q}^k_{\mathbb{D}_E} := \bigoplus_{p+q=k} \mathscr{Q}^{p,q}_{\mathbb{D}_E}.$$

We define the de-Rham complex $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}$ on $\mathbb{D}_E$ with coefficients in $\mathscr{Q}_{\mathbb{D}_E}$ by

$$0 \longrightarrow \mathscr{Q}^0_{\mathbb{D}_E} \xrightarrow{d} \mathscr{Q}^1_{\mathbb{D}_E} \xrightarrow{d} \dots \xrightarrow{d} \mathscr{Q}^{2n}_{\mathbb{D}_E} \longrightarrow 0,$$

and the Dolbeault complex $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}$ on $\mathbb{D}_E$ by

$$0 \longrightarrow \mathscr{Q}^{p,0}_{\mathbb{D}_E} \xrightarrow{\bar\partial} \mathscr{Q}^{p,1}_{\mathbb{D}_E} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathscr{Q}^{p,n}_{\mathbb{D}_E} \longrightarrow 0.$$

Set $V_0 = V \setminus \mathbb{D}_M$, $V_1 = V$ and $V_{01} = V_0 \cap V_1$. Define the coverings

$$\mathcal{V}_{\mathbb{D}_M} = \{V_0, V_1\}, \qquad \mathcal{V}'_{\mathbb{D}_M} = \{V_1\}.$$

We denote by $\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}')$ the Čech-Dolbeault complex

$$0 \longrightarrow \mathscr{Q}^{p,0}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') \xrightarrow{\overline{\vartheta}} \mathscr{Q}^{p,1}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') \xrightarrow{\overline{\vartheta}} \dots \xrightarrow{\overline{\vartheta}} \mathscr{Q}^{p,n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') \longrightarrow 0$$

which is defined by

$$\begin{aligned}
&\mathscr{Q}^{p,k}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') = \mathscr{Q}^{p,\,k}_{\mathbb{D}_E}(V_1) \oplus \mathscr{Q}^{p,\,k-1}_{\mathbb{D}_E}(V_{01}),\\
&\overline{\vartheta}(\xi_1, \xi_{01}) = (\overline{\partial}\xi_1, \xi_1|_{V_{01}} - \overline{\partial}\xi_{01}) \quad ((\xi_1, \xi_{01}) \in \mathscr{Q}^{p,\,k}_{\mathbb{D}_E}(V_1) \oplus \mathscr{Q}^{p,\,k-1}_{\mathbb{D}_E}(V_{01})).
\end{aligned}$$

We also denote by $\mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}')$ the Čech-de-Rham complex

$$0 \longrightarrow \mathscr{Q}^0_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') \xrightarrow{D} \mathscr{Q}^1_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') \xrightarrow{D} \dots \xrightarrow{D} \mathscr{Q}^{2n}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') \longrightarrow 0,$$

which is defined by

$$\begin{aligned}
&\mathscr{Q}^k_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}') = \mathscr{Q}^k_{\mathbb{D}_E}(V_1) \oplus \mathscr{Q}^{k-1}_{\mathbb{D}_E}(V_{01}),\\
&D(v_1, v_{01}) = (dv_1, v_1|_{V_{01}} - dv_{01}) \quad ((v_1, v_{01}) \in \mathscr{Q}^k_{\mathbb{D}_E}(V_1) \oplus \mathscr{Q}^{k-1}_{\mathbb{D}_E}(V_{01})).
\end{aligned}$$

Let $U = V \cap \mathbb{D}_M$. Then we have

Theorem 2.7. *There exist the canonical quasi-isomorphisms:*

$$\boldsymbol{R}\Gamma_U(V; \mathbb{C}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}'), \qquad \boldsymbol{R}\Gamma_U(V; \mathscr{O}^{\exp,(p)}_{\mathbb{D}_E}) \simeq \mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}').$$

It follows from the above theorem that we have

$$\mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(U) \simeq \mathrm{H}^n(\mathscr{Q}^{p,\bullet}_{\mathbb{D}_E}(\mathcal{V}_{\mathbb{D}_M}, {\mathcal{V}_{\mathbb{D}_M}}')) \otimes_{\mathbb{Z}_{\mathbb{D}_M(U)}} or_{\mathbb{D}_M/\mathbb{D}_E}(U). \tag{2.3}$$

This implies that any Laplace hyperfunction $u \in \mathscr{B}^{\exp,(p)}_{\mathbb{D}_M}(U)$ is represented by a pair (ω_1, ω_{01}) of C^∞-forms which satisfies the following conditions 1. and 2.

1. $\omega_1 \in \mathscr{Q}^{p,n}(V)$ and $\omega_{01} \in \mathscr{Q}^{p,n-1}(V \setminus U)$.
2. $\overline{\partial}\omega_{01} = \omega_1$ on $V \setminus U$.

§ 2.3. Laplace transformation

In this subsection, we introduce a Laplace transformation of a Laplace hyperfunction with a Čech-Dolbeault representative.

Let $h : E^*_\infty \to \{-\infty\} \cup \mathbb{R}$ be an upper semi-continuous function.

Definition 2.8. Let W be an open subset in $\mathbb{D}_{E^*}$ and let f be a holommorphic function on $W \cap E^*$. We say that f is of infra-h-exponential type (at ∞) on W if, for any compact set $K \subset W$ and any $\epsilon > 0$, there exists $C > 0$ such that

$$e^{|\zeta|\, h(\pi(\zeta))}|f(\zeta)| \leq Ce^{\epsilon|\zeta|} \qquad (\zeta \in K \cap (E^* \setminus \{0\})),$$

where $\pi : E^* \setminus \{0\} \to (E^* \setminus \{0\})/\mathbb{R}_+ = E^*_\infty$ is the canonical projection. In particular, we say that f is simply called of infra-exponential type if $h \equiv 0$.

We define the sheaf $\mathscr{O}^{\mathrm{inf}-h}_{E^*_\infty}$ on E^*_∞ as follows. For an open subset Ω in E^*_∞, we set

$$\mathscr{O}^{\mathrm{inf}-h}_{E^*_\infty}(\Omega) := \varinjlim_{W} \{f \in \mathscr{O}(W);\ f \text{ is of infra-}h\text{-exponential type on } W\},$$

where W runs through open neighborhoods of Ω in $\mathbb{D}_{E^*}$. Then the family $\{\mathscr{O}^{\mathrm{inf}-h}_{E^*_\infty}(\Omega)\}_\Omega$ forms the sheaf $\mathscr{O}^{\mathrm{inf}-h}_{E^*_\infty}$ on E^*_∞. Similarly, we define $\mathscr{O}^{\mathrm{inf}}_{E^*_\infty}$ the sheaf of holomrphic functions of infra-exponential type on E^*_∞. We also define the sheaf $\mathscr{A}^{\mathrm{exp}}_{\mathbb{D}_M}$ of real analytic functions of exponential type by

$$\mathscr{A}^{\mathrm{exp}}_{\mathbb{D}_M} := \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}_E}\big|_{\mathbb{D}_M}$$

and the sheaf $\mathscr{V}^{\mathrm{exp}}_{\mathbb{D}_M}$ of real analytic volumes of exponential type by

$$\mathscr{V}^{\mathrm{exp}}_{\mathbb{D}_M} := \mathscr{O}^{\mathrm{exp},(n)}_{\mathbb{D}_E}\Big|_{\mathbb{D}_M} \otimes_{\mathbb{Z}_{\mathbb{D}_M}} or_{\mathbb{D}_M},$$

where $or_{\mathbb{D}_M} := (j_M)_* \, or_M$ is the orientation sheaf on $\mathbb{D}_M$ with the canonical inclusion $j_M : M \hookrightarrow \mathbb{D}_M$. Note that we have

$$or_{\mathbb{D}_M/\mathbb{D}_E} \otimes or_{\mathbb{D}_M} \simeq or_{\mathbb{D}_E}|_{\mathbb{D}_M}, \tag{2.4}$$

where $or_{\mathbb{D}_E}$ is the orientation sheaf on $\mathbb{D}_E$ befined by $(j_E)_* \, or_E$ with the canonical inclusion $j_E : E \hookrightarrow \mathbb{D}_E$. For a subset K in $\mathbb{D}_E$, we define the support function $h_K(\zeta) : E^*_\infty \to \{\pm\infty\} \cup \mathbb{R}$ by

$$h_K(\zeta) = \begin{cases} +\infty & \text{if } K \cap E \text{ is empty}, \\ \inf\limits_{z \in K \cap E} \operatorname{Re}\langle z,\, \zeta\rangle & \text{otherwise}. \end{cases} \tag{2.5}$$

Let K be a closed subset in $\mathbb{D}_M$ such that $\mathrm{N}^*_{pc}(K) \neq \emptyset$. Take a point $\xi_0 \in \mathrm{N}^*_{pc}(K) \cap M^*_\infty$ and an open neighborhood V of K in $\mathbb{D}_E$. Set $U := \mathbb{D}_M \cap V$ and coverings

$$\mathcal{V}_K := \{V_0 := V \setminus K,\, V_1 := V\}, \qquad \mathcal{V}'_K := \{V_0\}. \tag{2.6}$$

For simplicity, we assume that U and V are connected. Note that we have

$$\Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}) \simeq \mathrm{H}^n(\mathscr{Q}^{n,\bullet}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M/\mathbb{D}_E}(U) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M}(U).$$

Let

$$u = \tilde{u} \otimes a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M} \in \Gamma_K(U;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}),$$

where $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M} \in or_{\mathbb{D}_M/\mathbb{D}_E}(U) \underset{\mathbb{Z}_{\mathbb{D}_M}(U)}{\otimes} or_{\mathbb{D}_M}(U)$ and let $\nu = (\nu_1,\, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')$ be a representative of $\tilde{u}$, i.e., $\tilde{u} = [\nu]$. We define the Laplace transform of u as follows.

Definition 2.9. The Laplace transform of u with a Čech-Dolbeault representative $\nu = (\nu_1,\, \nu_{01}) \in \mathscr{Q}^{n,n}_{\mathbb{D}_E}(\mathcal{V}_K,\, \mathcal{V}_K{}')$ is defined by

$$\mathcal{L}(u)(\zeta) := \int_{D \cap E} e^{-z\zeta}\nu_1 - \int_{\partial D \cap E} e^{-z\zeta}\nu_{01}. \tag{2.7}$$

Here D is a contractible open subset in $\mathbb{D}_E$ with (partially) smooth boundary satisfying the following conditions:

1. $K \subset D \subset \overline{D} \subset V$.
2. D is properly contained in a half space of $\mathbb{D}_E$ with direction ξ_0.

Then we have the following theorem.

Theorem 2.10. *Assume $K \cap M \neq \emptyset$. Then we have $\mathcal{L}(u) \in \mathscr{O}^{\inf - h_K}_{E^*_\infty}(N^*_{pc}(K))$.*

Let G be an $\mathbb{R}_+$-conic proper closed subset in M and $a \in M$. We denote by $G^\circ \subset E^*$ the dual open cone of G in E^*, that is,

$$G^\circ := \{\zeta \in E^*;\, \mathrm{Re}\langle \zeta, x\rangle > 0 \ \text{ for any } x \in G\}.$$

Assume $K = \overline{\{a\} + G} \subset \mathbb{D}_M$. Then the above theorem immediately implies the following theorem.

Theorem 2.11. *Under the above situation, we have $e^{a\zeta}\mathcal{L}(u)(\zeta) \in \mathscr{O}^{\inf}_{E^*_\infty}(\widehat{\ }(G^\circ) \cap E^*_\infty)$.*

§ 2.4. Laplace inverse transformation

To construct a Laplace inverse transformation, we prepare some definitions.

Let T be a real analytic manifold and set $Y = T \times \mathbb{D}_E$ and $Y_\infty = T \times (\mathbb{D}_E \setminus E)$. Let W be an open subset in Y and $f(t, z)$ a measurable function on $W \setminus Y_\infty$. We say that $f(t, z)$ is of exponential type on W if, for any compact subset K in W, there exists $H_K > 0$ such that $|\exp(-H_K|z|)\, f(t, z)|$ is essentially bounded on $K \setminus Y_\infty$. We define the set $\mathscr{L}\mathscr{Q}_Y(W)$ as follows. A locally integrable function $f(t, z)$ on $W \setminus Y_\infty$ belongs to $\mathscr{L}\mathscr{Q}_Y(W)$ if any higher derivative of $f(t, z)$ with respect to the variables z and $\overline{z}$ in the sense of distributions is a locally integrable function of exponential type on W. Let $\mathscr{L}\mathscr{Q}_Y^k$ denotes the sheaf on Y of k-forms with respect to the variables in E, and let us define the de-Rham complex $\mathscr{L}\mathscr{Q}_Y^\bullet$ by

$$0 \longrightarrow \mathscr{L}\mathscr{Q}_Y^0 \xrightarrow{d_{\mathbb{D}_E}} \mathscr{L}\mathscr{Q}_Y^1 \xrightarrow{d_{\mathbb{D}_E}} \dots \xrightarrow{d_{\mathbb{D}_E}} \mathscr{L}\mathscr{Q}_Y^{2n} \longrightarrow 0.$$

We also denote by $\mathscr{L}_{loc,T}^\infty$ the sheaf of locally integrable functions on T.

Let I be a connected open subset in M_∞^* and $a \in M$ and let $h : M_\infty^* \to \{-\infty\} \cup \mathbb{R}$ be an upper semi-continuous function such that $h(\xi)$ is continuous on I and $h(\xi) > -\infty$ there. Recall the definition of the map ϖ_∞ given in § 2.1. We define $\hat{h}(\zeta)$ on E_∞^* by, for $\zeta = \xi + \sqrt{-1}\eta \in E_\infty^*$ $((\xi, \eta) \in S^{2n-1})$,

$$\hat{h}(\zeta) = \begin{cases} 0 & (\zeta \in \sqrt{-1}M_\infty^*), \\ |\xi| h(\varpi_\infty(\zeta)) & (\zeta \in E_\infty^* \setminus \sqrt{-1}M_\infty^*). \end{cases}$$

Let $f \in \mathscr{O}_{E_\infty^*}^{\inf -\hat{h}}(\varpi_\infty^{-1}(I))$. By the definition of $\mathscr{O}_{E_\infty^*}^{\inf -\hat{h}}$, we can find continuous functions $\psi : I \times [0, \infty) \to \mathbb{R}_{\geq 0}$ and $\varphi : [0, \infty) \to \mathbb{R}_{\geq 0}$ satisfying the following conditions:

1. For any compact subset $L \subset I$, the function $\sup\limits_{\xi \in L} \psi(\xi,\, t)$ is an infra-linear function of the variable t and f is holomorphic on an open subset $W_\psi \cap E^*$, where

$$W_\psi := {}^\frown\left\{\zeta = t\xi + \sqrt{-1}\eta \in E^*;\, \eta \in M^*,\ \xi \in I,\ t > \psi(\xi,\, |\eta|)\right\}. \tag{2.8}$$

2. $\varphi(t)$ is a continuous infra-linear function on $[0, \infty)$ such that

$$|f(\zeta = \xi + \sqrt{-1}\eta)| \leq e^{-|\xi| h(\pi(\zeta)) + \varphi(|\xi|)} \qquad (\zeta \in W_\psi \cap E^*), \tag{2.9}$$

   where $\pi : M^* \setminus \{0\} \to M_\infty^*$ is the canonical projection.

Define an n-dimensional real chain γ^* in E^* by

$$\gamma^* := \left\{\zeta = \xi + \sqrt{-1}\eta \in E^*;\, \eta \in M^* \setminus \{0\},\, \xi = \psi_{\xi_0}(|\eta|)\, \xi_0\right\}, \tag{2.10}$$

where $\xi_0 \in I$ and $\psi_{\xi_0}(t)$ is a continuous infra-linear function on $[0, \infty)$ with $\psi_{\xi_0}(t) > \psi(\xi_0, t)$ $(t \in [0, \infty))$ and $\psi_{\xi_0}(t)/(\psi(\xi_0, t) + 1) \to \infty$ $(t \to \infty)$. Let $T = S^{n-1}$ and

$Y = S^{n-1} \times \mathbb{D}_E$. Define coverings

$$\mathcal{W} = \{W_0 = Y \setminus (S^{n-1} \times D_M),\, W_1 = Y\}, \qquad \mathcal{W}' = \{W_0\}$$

with $W_{01} = W_0 \cap W_1$. We denote by $p_T : Y \to T$ (resp. $p_{\mathbb{D}_E} : Y \to \mathbb{D}_E$) the canonical projection to T (resp. $\mathbb{D}_E$). Then we have the isomorphisms

(2.11)
$$\Gamma(T;\, \mathscr{L}^{\infty}_{loc,T}) = \Gamma(Y;\, \tilde{p}_T^{-1}\mathscr{L}^{\infty}_{loc,T}) \overset{\sim}{\to} \mathrm{H}^n_{p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M)}(Y;\, p_T^{-1}\mathscr{L}^{\infty}_{loc,T}) = \mathrm{H}^n(\mathscr{L}\mathscr{Q}^{\bullet}_Y(\mathcal{W}, \mathcal{W}')),$$

where $\tilde{p}_T : p^{-1}_{\mathbb{D}_E}(\mathbb{D}_M) = T \times \mathbb{D}_M \to T$ is the canonical projection. Set

$$\Omega := {}^{\frown}\big\{(\eta, z) \in S^{n-1} \times E;\, \langle \eta, \operatorname{Im} z\rangle > 0\big\} \subset Y.$$

Let $j : \Omega \hookrightarrow Y$ be the canonical open inclusion. We can take a specific $\omega = (\omega_1, \omega_{01}) \in \mathscr{L}\mathscr{Q}^n_Y(\mathcal{W}, \mathcal{W}')$ which satisfies the following two conditions:

1. $D_{\mathbb{D}_E}\omega = 0$ and $[\omega]$ is the image of a constant function $1 \in \Gamma(T;\, \mathscr{L}^{\infty}_{loc,T})$ through the above isomorphisms (2.11).
2. $\operatorname{supp}_{W_1}(\omega_1) \subset \Omega$ and $\operatorname{supp}_{W_{01}}(\omega_{01}) \subset \Omega$.

Note that such a representative always exists (Lemma 7.0.2 [2]). Then we define the Laplace inverse transform as follows.

Definition 2.12. The Laplace inverse transform $I\mathcal{L}$ is given by

$$I\mathcal{L}(f) = \big([I\mathcal{L}_\omega(f d\zeta)] \otimes a_{\mathbb{D}_M/\mathbb{D}_E}\big) \otimes \nu_{\mathbb{D}_M}$$

with

$$\begin{aligned} I\mathcal{L}_\omega(f d\zeta) &:= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\gamma^*} \rho(\omega)\big|_{t=\eta/|\eta|}\, e^{\zeta z} f(\zeta) d\zeta \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(\int_{\gamma^*} \rho_n(\omega_1)(\frac{\eta}{|\eta|}, z)\, e^{\zeta z} f(\zeta)d\zeta,\ \int_{\gamma^*} \rho_{n-1}(\omega_{01})(\frac{\eta}{|\eta|}, z)\, e^{\zeta z} f(\zeta) d\zeta\right). \end{aligned}$$

Here $\zeta = \xi + \sqrt{-1}\eta$ are the dual variables of $z = x + \sqrt{-1}y$, $a_{\mathbb{D}_M} \in or_{\mathbb{D}_M}(\mathbb{D}_M)$, $a_{\mathbb{D}_M/\mathbb{D}_E} \in or_{\mathbb{D}_M/\mathbb{D}_E}(\mathbb{D}_M)$ so that $a_{\mathbb{D}_M/\mathbb{D}_E} \otimes a_{\mathbb{D}_M}$ has the same orientation as that of E through the isomorphism $or_{\mathbb{D}_M/\mathbb{D}_E} \otimes or_{\mathbb{D}_M} \simeq or_{\mathbb{D}_E}|_{\mathbb{D}_M}$, and the volume $\nu_{\mathbb{D}_M}$ is defined by $dz \otimes a_{\mathbb{D}_M}$ with $dz = dz_1 \wedge \cdots \wedge dz_n$ and $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$.

We have the following theorem and corollary.

Theorem 2.13. *Let $G \subset M$ be an $\mathbb{R}^+$-conic proper closed convex subset and $a \in M$. Set $K = \overline{a + G} \subset \mathbb{D}_M$. Then*

$$\mathcal{L} : \Gamma_K(\mathbb{D}_M;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})) \to e^{-a\zeta}\mathscr{O}^{\inf}_{E^*_\infty}(N^*_{pc}(K))$$

and

$$\mathcal{IL} : e^{-a\zeta}\mathscr{O}^{\inf}_{E^*_\infty}(N^*_{pc}(K)) \to \Gamma_K(\mathbb{D}_M;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}))$$

are inverse to each other.

Corollary 2.14. *Let $K \subset \mathbb{D}_M$ be a regular closed subset satisfying that $K \cap M$ is convex and $N^*_{pc}(K) \cap M^*_\infty$ is connected (in particular, non-empty). Recall the definition of the map h_K given in* (2.5)*. Then*

$$\mathcal{L} : \Gamma_K(\mathbb{D}_M;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M})) \to \mathscr{O}^{\inf - h_K}_{E^*_\infty}(N^*_{pc}(K))$$

and

$$\mathcal{IL} : \mathscr{O}^{\inf - h_K}_{E^*_\infty}(N^*_{pc}(K)) \to \Gamma_K(\mathbb{D}_M;\, \mathscr{B}^{\exp}_{\mathbb{D}_M} \otimes_{\mathscr{A}^{\exp}_{\mathbb{D}_M}} \mathscr{V}^{\exp}_{\mathbb{D}_M}))$$

are inverse to each other.

§ 3. Application to PDE with constant coefficients

We show the existence of Laplace hyperfunctions solutions for a system of PDEs with constant coefficients.

Let $\mathfrak{R}$ be the polynomial ring $\mathbb{C}[\zeta_1, \cdots, \zeta_n]$ on E^* and let $\mathfrak{D}$ be the ring $\mathbb{C}[\partial_{x_1}, \cdots, \partial_{x_n}]$ of linear differential operators on M with constant coefficients. Define the principal symbol map $\sigma : \mathfrak{D} \to \mathfrak{R}$ by

$$\mathfrak{D} \ni P(\partial) = \sum c_\alpha \partial^\alpha \mapsto \sigma(P)(\zeta) = \sum_{|\alpha| = \mathrm{ord}(P)} c_\alpha \zeta^\alpha \in \mathfrak{R}.$$

For an $\mathfrak{D}$ module $\mathfrak{M} = \mathfrak{D}/\mathfrak{I}$ with the ideal $\mathfrak{I} \subset \mathfrak{D}$, we define the closed subset $\mathrm{Char}_{E^*_\infty}(\mathfrak{M})$ in E^*_∞ by

$$\mathrm{Char}_{E^*_\infty}(\mathfrak{M}) = \{\zeta \in E^*_\infty;\, \sigma(P)(\zeta) = 0 \quad (\forall\, P \in \mathfrak{I})\}.$$

Let $P_1(\partial)$, $\cdots$, $P_\ell(\partial)$ be in $\mathfrak{D}$. Set

$$\mathfrak{M} = \mathfrak{D}/(P_1(\partial), \cdots, P_\ell(\partial)).$$

Then we have the following theorem.

Theorem 3.1. *Let K be a regular closed subset in $\mathbb{D}_M$. Assume that $K \cap M$ is convex and $N^*_{pc}(K) \cap M^*_\infty$ is connected, and that $P_1(\zeta)$, $\cdots$, $P_\ell(\zeta)$ form a regular sequence over $\mathfrak{R}$. Then the condition*

$$N^*_{pc}(K) \cap \mathit{Char}_{E^*_\infty}(\mathfrak{M}) = \emptyset$$

implies

$$\mathit{Ext}^k_{\mathfrak{D}}(\mathfrak{M},\, \Gamma_K(\mathbb{D}_M, \mathscr{B}^{\exp}_{\mathbb{D}_M})) = 0 \qquad (k = 0, 1).$$

It follows from Theorem 3.1 that we have the following corollary.

Corollary 3.2. *Let $P(\partial) \in \mathfrak{D}$, and let K be a regular closed subset in $\mathbb{D}_M$ satisfying that $K \cap M$ is convex and $N^*_{pc}(K) \cap M^*_\infty$ is connected. Then the morphism*

$$\Gamma_K(\mathbb{D}_M, \mathscr{B}^{\exp}_{\mathbb{D}_M}) \xrightarrow{P(\partial)\,\bullet} \Gamma_K(\mathbb{D}_M, \mathscr{B}^{\exp}_{\mathbb{D}_M})$$

becomes isomorphic if $\sigma(P)(\zeta) \neq 0$ holds for any $\zeta \in N^*_{pc}(K)$.

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