μ-convexity and generalized spherical mean value operators on Euclidean space

By

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Abstract

We briefly report the results of [15], in which we consider the Neumann version of the spherical mean value operator and its variants in the spaces of smooth functions, distributions and compactly supported ones. The notion of μ-convexity for supports and singular supports is essential in our argument. Our treatment of it in the present paper is slightly more general than that in [15]. In §7, we calculate the Fourier transform of the kernel of the spherical mean value operator for the sake of completeness.

§ 1. Introduction

Convolution equations are natural extensions of linear partial differential equations with constant coefficients. Many authors studied convolutions equations in various function spaces in the real or complex domains. See Malgrange [13], Ehrenpreis [3], Hörmander [5], Korobeĭnik [9], Kawai [10], Ishimura-Okada [8], Abanin-Ishimura-Khoi [1], Langenbruch [11] and the references therein.

General theory has been the center of research on this topic and inspiring concrete examples are rarely mentioned. An important exception is the paper by Lim [12], in which the author studied the spherical mean value operator on Euclidean space.

In [12], the spherical mean value operator involves Dirichlet boundary values on spheres. In the present paper, we study the surjectivity of the Neumann mean value operator on Euclidean space and its higher order variants. The main result is that they are surjections on the space of smooth functions and that of distributions. Moreover, we give a characterization of the ranges in the case of compact supports.

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§ 2. Distributions, convolution and the Fourier transform

Recall that the surface area of $S(0,r)=\{x\in\mathbb{R}^n;\ |x|=r\}$ is $\sigma_n r^{n-1}$, where

$$\sigma_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

The spherical mean value operator M_r is defined by

$$M_r u(x)=\begin{cases}\dfrac{1}{2}\{u(x-r)+u(x+r)\} & (n=1),\\ \sigma_n^{-1}r^{-n+1}\displaystyle\int_{S(x,r)}u(y)\,dS_{x,r}(y) & (n\geq 2),\end{cases}$$

where $dS_{x,r}$ is the surface area measure of $S(x,r)=\{y\in\mathbb{R}^n;\ |y-x|=r\}$. Let $\delta_{S(0,r)}$ be the distribution defined by $\delta_{S(0,r)}\colon \mathcal{C}^\infty(\mathbb{R}^n)\to\mathbb{C},\ u(x)\mapsto M_r u(0)$. Then we have $\left\langle \delta_{S(0,r)},\,1\right\rangle=1$ and $M_r u(x)=\delta_{S(0,r)}*u(x)$, where $*$ denotes convolution.

Let $n=n_y$ be the outer unit normal of $S(x,r)$ at $y\in S(x,r)$. The Neumann version of the spherical mean value operator and its generalization are defined by

$$\begin{aligned}M_r^{(\ell)}u(x)&=\sigma_n^{-1}r^{-n+1}\int_{S(x,r)}\left(\frac{\partial}{\partial n_y}\right)^{\ell}u(y)\,dS_{x,r}(y)\\ &=\sigma_n^{-1}\int_{S(0,1)}\frac{\partial^\ell}{\partial r^\ell}u(x+r\omega)\,dS_{0,1}(\omega)\end{aligned}$$

for a smooth function u on $\mathbb{R}^n$ and a non-negative integer ℓ. We have

$$M_r^{(\ell)}u(x)=\frac{\partial^\ell\delta_{S(0,r)}}{\partial r^\ell}*u(x) \tag{2.1}$$

for a smooth function u on $\mathbb{R}^n$ and a non-negative integer ℓ. The convolution operator $M_r^{(\ell)}=(\partial^\ell\delta_{S(0,r)}/\partial r^\ell)*$ can be defined as endomorphisms on $\mathcal{C}^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{C}_0^\infty(\mathbb{R}^n)$.

By convention, the Fourier transform of a compactly supported distribution u is defined by

$$\hat{u}(\xi)=\left\langle u(x),e^{-i\langle x,\xi\rangle}\right\rangle$$

in the present paper. In some cases, it is denoted by $(u)^\wedge$.

§ 3. Invertibility

The notion of invertibility of a kernel plays the central role in the theory of convolution equations.

Definition 3.1 ([7, Definition 16.3.12])**.** We say that an element u of $\mathcal{E}'(\mathbb{R}^n)$ is invertible if there exists a constant $A > 0$ such that we have

$$\sup\{|\hat{u}(\zeta)|\,;\,\zeta \in \mathbb{C}^n, |\zeta - \xi| < A\log(2+|\xi|)\} > (A+|\xi|)^{-A}$$

for any $\xi \in \mathbb{R}^n$.

We employ the normalized Bessel function $j_\nu(z)$ defined by

$$j_\nu(z) = \Gamma(\nu+1)\left(\frac{2}{z}\right)^{\nu} J_\nu(z), \quad \nu > -1,$$

where $J_\nu(z)$ is the usual Bessel function of the first kind of order ν. We have

$$j_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(\nu+1)}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k}, \, z \in \mathbb{C}$$

and it is an even entire function.

For $\nu > -1$ and $\ell \geq 0$, there exist constants $C_{\ell,k} = C^{(\nu)}_{\ell,k}$ such that

$$j_\nu^{(\ell)}(z) = \begin{cases} \displaystyle\sum_{k=\ell/2}^{\ell} C_{\ell,k} z^{2k-\ell} j_{\nu+k}(z) & (\ell : \text{even}), \\ \displaystyle\sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} z^{2k-\ell} j_{\nu+k}(z) & (\ell : \text{odd}). \end{cases} \tag{3.1}$$

On the real axis, the dominant term as $x \to \infty$ is the one corresponding to $k = \ell$ in either case. Fix $r > 0$ and $n \geq 1$. Let $\nu = n/2 - 1$. Then there exist constants $A, B > 0$ such that

$$\sup\left\{\left|j_\nu^{(\ell)}(r|\eta|)\right|\,;\,\eta \in \mathbb{R}^n, |\eta - \xi| < A\log(2+|\xi|)\right\} > (A+|\xi|)^{-A} \tag{3.2}$$

for any $\xi \in \mathbb{R}^n$ with $|\xi| > B$.

Theorem 3.2. *For any $\xi \in \mathbb{R}^n$, we have*

$$\widehat{\delta}_{S(0,r)}(\xi) = j_{n/2-1}(r|\xi|). \tag{3.3}$$

For $\ell \geq 0$, the Fourier transform of $\partial\delta^\ell_{S(0,r)}/\partial r^\ell$ is

$$\begin{aligned} \left(\frac{\partial^\ell \delta_{S(0,r)}}{\partial r^\ell}\right)^{\wedge}(\xi) &= |\xi|^\ell j^{(\ell)}_{n/2-1}(r|\xi|) \\ &= \begin{cases} \displaystyle\sum_{k=\ell/2}^{\ell} C_{\ell,k} r^{2k-\ell}|\xi|^{2k} j_{n/2-1+k}(r|\xi|) & (\ell : \textit{even}), \\ \displaystyle\sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} r^{2k-\ell}|\xi|^{2k} j_{n/2-1+k}(r|\xi|) & (\ell : \textit{odd}). \end{cases} \end{aligned} \tag{3.4}$$

Proof. By [4, Introduction, Lemma 3.6],

$$\widehat{\delta}_{S(0,1)}(\xi) = 2^{n/2-1}\Gamma(n/2)\frac{J_{n/2-1}(|\xi|)}{|\xi|^{n/2-1}} = j_{n/2-1}(|\xi|). \tag{3.5}$$

For the convenience of the interested reader, we give a proof of (3.5) in §7.

For general $r > 0$,

$$\begin{aligned}\widehat{\delta}_{S(0,r)}(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\delta_{S(0,r)}(x)\,dx = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} r^{-n}\delta_{S(0,1)}(x/r)\,dx \\ &= \int_{\mathbb{R}^n} e^{-ir\langle y,\xi\rangle}\delta_{S(0,1)}(y)\,dy = \widehat{\delta}_{S(0,1)}(r\xi) = j_{n/2-1}(r|\xi|).\end{aligned}$$

The formula (3.3) has been proved and (3.4) follows from (3.3) and (3.1). ☐

Proposition 3.3. *The distribution $\partial^\ell \delta_{S(0,r)}/\partial r^\ell$ $(\ell \geq 0)$ is invertible.*

Proof. The proposition follows from (3.2) and Theorem 3.2. ☐

§ 4. μ-convexity for supports and singular supports

We introduce the notion of μ-convexity for supports of a pair of open sets (X_1, X_2). When $\mu \in \mathcal{E}'(\mathbb{R}^n)$, we set

$$\check{\mu}(\phi) = \mu(\check{\phi}),\ \check{\phi}(x) = \phi(-x),$$

where $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is a test function. In some cases, $\check{\mu}$ is denoted by $(\mu)^\vee$.

Definition 4.1 ([7, Definition 16.5.4])**.** Assume $\mu \in \mathcal{E}'(\mathbb{R}^n)$. Let X_1 and X_2 be non-empty open subsets of $\mathbb{R}^n$ satisfying $X_2 - \operatorname{supp}\mu \subset X_1$. We say that (X_1, X_2) is μ-convex for supports if for every compact set $K_1 \subset X_1$ one can find a compact set $K_2 \subset X_2$ such that $\operatorname{supp} v \subset K_2$ if $v \in \mathcal{C}_0^\infty(X_2)$ and $\operatorname{supp}\check{\mu} * v \subset K_1$.

We will need the case of $X_1 = X_2 = \mathbb{R}^n$ only. The condition $X_2 - \operatorname{supp}\mu \subset X_1$ is trivial in that case.

Propositions 4.2 and 4.5 below are generalizations of Lemmas 2.11 and 2.11 in [12].

Proposition 4.2. *Let μ be an arbitrary compactly supported distribution with $\mu \neq 0$. Then the pair $(\mathbb{R}^n, \mathbb{R}^n)$ is μ-convex for supports.*

Proof. Recall that

$$\operatorname{ch}\operatorname{supp} u_1 * u_2 = \operatorname{ch}\operatorname{supp} u_1 + \operatorname{ch}\operatorname{supp} u_2$$

holds for any $u_1, u_2 \in \mathcal{E}'(\mathbb{R}^n)$ ([6, Theorem 4.3.3]), where ch denotes the convex hull. It implies $\operatorname{ch}\operatorname{supp} u_1 \subset \operatorname{ch}\operatorname{supp} u_1 * u_2 - \operatorname{ch}\operatorname{supp} u_2$ if $u_2 \neq 0$.

It is trivial that $\check{\mu}$ is a compactly supported distribution.

Let $K_1 \subset \mathbb{R}^n$ be an arbitrary compact set and assume that $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp} \check{\mu} * v \subset K_1$. We have

$$\begin{aligned} \operatorname{supp} v &\subset \operatorname{ch}\operatorname{supp} \check{\mu} * v - \operatorname{ch}\operatorname{supp} \check{\mu} \\ &\subset \operatorname{ch} K_1 - \operatorname{ch}\operatorname{supp} \check{\mu}. \end{aligned}$$

Recall that the convex hull of a compact subset in $\mathbb{R}^n$ is compact. Therefore $K_2 = \operatorname{ch} K_1 - \operatorname{ch}\operatorname{supp} \check{\mu}$ is a compact set independent of v. □

Definition 4.3 ([7, Definition 16.5.13])**.** Assume $\mu \in \mathcal{E}'(\mathbb{R}^n)$. Let X_1, X_2 be non-empty open subsets of $\mathbb{R}^n$ satisfying $X_2 - \operatorname{sing\,supp} \mu \subset X_1$. We say that (X_1, X_2) is μ-convex for singular supports if for every compact set $K_1 \subset X_1$ one can find a compact set $K_2 \subset X_2$ such that $\operatorname{sing\,supp} v \subset K_2$ if $v \in \mathcal{E}'(X_2)$ and $\operatorname{sing\,supp} \check{\mu} * v \subset K_1$.

Theorem 4.4 ([7, Corollary 16.3.15])**.** *Assume that $u \in \mathcal{E}'(\mathbb{R}^n)$ is invertible. Then we have*

$$\operatorname{ch}\operatorname{sing\,supp} v \subset \operatorname{ch}\operatorname{sing\,supp}(u * v) - \operatorname{ch}\operatorname{sing\,supp} u, \quad v \in \mathcal{E}'(\mathbb{R}^n).$$

The following is an analogue of Proposition 4.2.

Proposition 4.5. *Assume that a compactly supported distribution μ is invertible. Then the pair $(\mathbb{R}^n, \mathbb{R}^n)$ is μ-convex for singular supports.*

Proof. The proof is almost the same as that of Proposition 4.2. We can use Proposition 3.3 and Theorem 4.4. □

§ 5. Surjectivity

The following two theorems give sufficient conditions for surjectivity in the spaces of smooth functions and distributions.

Theorem 5.1 ([7, Theorem 16.5.7])**.** *Assume $\mu \in \mathcal{E}'(\mathbb{R}^n)$. Let X_1 and X_2 be non-empty open subsets of $\mathbb{R}^n$ satisfying $X_2 - \operatorname{supp} \mu \subset X_1$. Then the following two statements are equivalent.*

(i) The convolution operator $\mu * \colon \mathcal{C}^\infty(X_1) \to \mathcal{C}^\infty(X_2)$ is surjective.

(ii) The distribution μ is invertible and the pair (X_1, X_2) is μ-convex for supports.

Theorem 5.2 ([7, Corollary 16.5.19]). *Assume $\mu \in \mathcal{E}'(\mathbb{R}^n)$. Let X_1 and X_2 be non-empty open subsets of $\mathbb{R}^n$. Assume $X_2 - \operatorname{supp}\mu \subset X_1$. Then the following two statements are equivalent.*

(i) The convolution operator $\mu *\colon \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)$ is surjective.

(ii) The distribution μ is invertible and (X_1, X_2) is μ-convex for supports and singular supports.

The following two theorems are our main results about surjectivity.

Theorem 5.3. *Let $r > 0$. The convolution operator $\partial^\ell \delta_{S(0,r)}/\partial r^\ell *\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is surjective.*

Proof. Apply Theorem 5.1 and Propositions 3.3 and 4.2. □

Theorem 5.4. *The convolution operator $\partial^\ell \delta_{S(0,r)}/\partial r^\ell *\colon \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is surjective.*

Proof. Apply Theorem 5.2 and Propositions 3.3, 4.2 and 4.5. □

§ 6. Range characterization

In this section, we restrict our consideration to the cases of $\ell = 0, 1$ and $n \geq 2$. The endomorphisms $\partial^\ell \delta_{S(0,r)}/\partial r^\ell *\colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$ and $\partial^\ell \delta_{S(0,r)}/\partial r^\ell *\colon \mathcal{C}_0^\infty(\mathbb{R}^n) \to \mathcal{C}_0^\infty(\mathbb{R}^n)$ are not surjective. We characterize their ranges in terms of the zeros of Bessel functions.

Theorem 6.1. *Assume $\ell = 0, 1$ and $n \geq 2$. Then we have the following characterization of the range of the endomorphism $\partial^\ell \delta_{S(0,r)}/\partial r^\ell *$ on $\mathcal{E}'(\mathbb{R}^n)$.*

$$\begin{aligned} &\operatorname{range}\left(\partial^\ell \delta_{S(0,r)}/\partial r^\ell *\colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)\right) \\ &= \begin{cases} \{w \in \mathcal{E}'(\mathbb{R}^n);\ \hat{w}(\zeta) = 0 \text{ if } \zeta^2 = x_1^2/r^2, x_2^2/r^2, x_3^2/r^2, \dots\} & (\ell = 0), \\ \{w \in \mathcal{E}'(\mathbb{R}^n);\ \hat{w}(\zeta) = 0 \text{ if } \zeta^2 = 0, \tilde{x}_1^2/r^2, \tilde{x}_2^2/r^2, \tilde{x}_3^2/r^2, \dots\} & (\ell = 1), \end{cases} \end{aligned} \tag{6.1}$$

where $\pm x_j$ and $\pm \tilde{x}_j$ $(j = 1, 2, \dots)$ are the zeros of $J_{n/2-1}(z)$ and $J_{n/2}(z)$ respectively. For the case $\ell = 0, n \geq 1$, see Theorem 2.17 of [12].

Theorem 6.2. *Assume $\ell = 0, 1$ and $n \geq 2$. Then we have the following characterization of the range of the endomorphism $\partial^\ell \delta_{S(0,r)}/\partial r^\ell *$ on $\mathcal{C}_0^\infty(\mathbb{R}^n)$.*

$$\begin{aligned} &\operatorname{range}\left(\partial^\ell \delta_{S(0,r)}/\partial r^\ell *\colon \mathcal{C}_0^\infty(\mathbb{R}^n) \to \mathcal{C}_0^\infty(\mathbb{R}^n)\right) \\ &= \begin{cases} \{w \in \mathcal{C}_0^\infty(\mathbb{R}^n);\ \hat{w}(\zeta) = 0 \text{ if } \zeta^2 = x_1^2/r^2, x_2^2/r^2, x_3^2/r^2, \dots\} & (\ell = 0), \\ \{w \in \mathcal{C}_0^\infty(\mathbb{R}^n);\ \hat{w}(\zeta) = 0 \text{ if } \zeta^2 = 0, \tilde{x}_1^2/r^2, \tilde{x}_2^2/r^2, \tilde{x}_3^2/r^2, \dots\} & (\ell = 1), \end{cases} \end{aligned} \tag{6.2}$$

For the case $\ell = 0, n \geq 1$, see Theorem 2.23 of [12].

§ 7. Appendix

In this appendix, we give a proof of (3.5).

First, we calculate the Fourier transform of the indicator function of a ball. Let $B(0,r)$ be the ball $\{x \in \mathbb{R}^n;\ |x| \le r\}$ $(r>0)$ and V_n be the volume of $B(0,1)$. We have $V_n = \pi^{n/2}/\Gamma(n/2+1)$. We denote by $\chi_{B(0,r)}(x)$ the indicator function of $B(0,r)$.

Proposition 7.1. $\widehat{\chi_{B(0,1)}}(\xi) = (2\pi)^{n/2}|\xi|^{-n/2}J_{n/2}(|\xi|)$.

Proof. The Fourier transform of a radial function is also radial. We have only to evaluate $\widehat{\chi_{B(0,1)}}(\rho \boldsymbol{e}_n)$, where $\rho>0$, $\boldsymbol{e}_n = (0,0,\dots,0,1)$. We get

$$\begin{aligned}\widehat{\chi_{B(0,1)}}(\rho \boldsymbol{e}_n) &= \int_{|x|\le 1} \exp(-ix_n\rho) = \int_{-1}^{1} V_{n-1}(1-x_n^2)^{(n-1)/2}\exp(-ix_n\rho)\,dx_n \\ &= \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}\int_{-1}^{1}(1-x_n^2)^{(n-1)/2}\exp(-i\rho x_n)\,dx_n\end{aligned} \tag{7.1}$$

By [14, 10.9.4], we have

$$\begin{aligned}J_{n/2}(z) &= \frac{2(z/2)^{n/2}}{\pi^{1/2}\Gamma((n+1)/2)}\int_0^1 (1-t^2)^{(n-1)/2}\cos(zt)\,dt \\ &= \frac{(z/2)^{n/2}}{\pi^{1/2}\Gamma((n+1)/2)}\int_{-1}^1 (1-t^2)^{(n-1)/2}\exp(-izt)\,dt.\end{aligned} \tag{7.2}$$

By (7.1) and (7.2), we have

$$\widehat{\chi_{B(0,1)}}(\rho \boldsymbol{e}_n) = (2\pi)^{n/2}\rho^{-n/2}J_{n/2}(\rho).$$

Therefore, we get

$$\widehat{\chi_{B(0,1)}}(\xi) = (2\pi)^{n/2}|\xi|^{-n/2}J_{n/2}(|\xi|).$$ □

Proposition 7.2.

$$\widehat{\chi_{B(0,r)}}(\xi) = (2\pi)^{n/2}r^{n/2}|\xi|^{-n/2}J_{n/2}(r|\xi|).$$

Proof. Set $x = ry, \eta = r\xi$. Then we have

$$\begin{aligned}\widehat{\chi_{B(0,r)}}(\xi) &= \int_{B(0,r)} e^{-i\xi\cdot x}\,dx = \int_{B(0,1)} e^{-i\eta\cdot y}\,r^n dy \\ &= r^n\widehat{\chi_{B(0,1)}}(\eta) = r^n\widehat{\chi_{B(0,1)}}(r\xi).\end{aligned} \tag{7.3}$$

It immediately follows that $\widehat{\chi_{B(0,r)}}(\xi) = (2\pi)^{n/2}r^{n/2}|\xi|^{-n/2}J_{n/2}(r|\xi|)$. □

Next, we calculate the Fourier transform of $\delta_{S(0,r)}$. Recall that $\delta_{S(0,r)}$ is so normalized that $\langle \delta_{S(0,r)}, 1 \rangle = 1$. For a continuous function $\varphi(x)$ on $\mathbb{R}^n$, we have

$$\int_{S(0,r)} \varphi(x)\, dS_{0,r} = \sigma_n r^{n-1} \left\langle \delta_{S(0,r)},\, \varphi(x) \right\rangle,$$

where $dS_{0,r}$ is the surface area measure of $S(0,r)$.

Proposition 7.3. $\widehat{\delta_{S(0,r)}}(\xi) = j_{n/2-1}(r|\xi|)$.

Proof. Let $\varphi(x)$ be a test function in $C_0^\infty(\mathbb{R}^n)$. Since

$$\left\langle \chi_{B(0,r)}, \varphi(x) \right\rangle = \int_{|x| \le r} \varphi(x)\, dx = \int_0^r ds \int_{S(0,s)} \varphi(x) dS_{0,s},$$

we get

$$\frac{\partial}{\partial r} \chi_{B(0,r)} = \sigma_n r^{n-1} \delta_{S(0,r)}. \tag{7.4}$$

Notice that (7.4) can be proved in a more sophisticated way: it is essentially the pull-back of $Y'(s) = \delta(s)$ by the mapping $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}_{>0} \ni (x, r) \mapsto r - |x| \in \mathbb{R}$.

The formula (7.4) gives, by Proposition 7.2,

$$\sigma_n r^{n-1} \widehat{\delta_{S(0,r)}}(\xi) = (2\pi)^{n/2} \frac{\partial}{\partial r} \left\{ r^{n/2} |\xi|^{-n/2} J_{n/2}(r|\xi|) \right\}. \tag{7.5}$$

Recall the formula $\frac{d}{ds} \{ s^\nu J_\nu(s) \} = s^\nu J_{\nu-1}(s)$ ([14, 11.6.6]). By setting $s = r|\xi|$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial r} \left\{ r^{n/2} |\xi|^{-n/2} J_{n/2}(r|\xi|) \right\} = |\xi|^{-n+1} \frac{\partial}{\partial s} \left\{ s^{n/2} J_{n/2}(s) \right\} \\ &= |\xi|^{-n+1} s^{n/2} J_{n/2-1}(s) = r^{n/2} |\xi|^{-n/2+1} J_{n/2-1}(r|\xi|). \end{aligned} \tag{7.6}$$

By (7.5) and (7.6), we have

$$\begin{aligned} &\widehat{\delta_{S(0,r)}}(\xi) = \frac{(2\pi)^{n/2}}{\sigma_n r^{n-1}} r^{n/2} |\xi|^{-n/2+1} J_{n/2-1}(r|\xi|) \\ &= \frac{(2\pi)^{n/2}}{\sigma_n (r|\xi|)^{n/2-1}} J_{n/2-1}(r|\xi|) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{(2\pi)^{n/2}}{(r|\xi|)^{n/2-1}} J_{n/2-1}(r|\xi|) \\ &= \frac{2^{n/2-1}\Gamma(n/2)}{(r|\xi|)^{n/2-1}} J_{n/2-1}(r|\xi|) = j_{n/2-1}(r|\xi|). \end{aligned}$$

□

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