$2, 3, n$-independency of tangential weights of G/K

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1. Introduction

This article is the research announcement of the progress work [**KS**]. In this article, we introduce the main result of [**KS**].

1.1. k-independence. Let $S := \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{Z}^n$ be a set of vectors. The set S is said to be *k-independent* for some $2 \leq k \leq n$ if every k vectors $\alpha_{i_1}, \ldots, \alpha_{i_k} \in S$ are linearly independent but there exists $k+1$ vectors $\alpha_{j_1}, \ldots, \alpha_{j_{k+1}} \in S$ which are linearly dependent. In this case, we also say that the set of vectors S has the property U^k_m, i.e., the *uniform matroid* with rank k and with m elements, see [**O06**].

Example 1.1. Let $S := \{e_1, e_2, -e_1 - e_2, e_1 + e_2 + e_3\} \subset \mathbb{R}^3$ be the set of 4 vectors in the 3-dimensional real vector space $\mathbb{R}^3$, where e_i, $i = 1, 2, 3$, is the standard basis of $\mathbb{R}^3$. Then, one can easily check that S has the property U^2_4, i.e., S is 2-independent. Note that the subset $\{e_1, e_2, e_1 + e_2 + e_3\}$ is linearly independent but $\{e_1, e_2, -e_1 - e_2\}$ is linearly dependent.

1.2. Problem and motivation. Let M be a $2m$-dimensional, smooth T-manifold with isolated fixed points M^T, where $\dim T = n$. Assume that the T-action is *almost effective*, i.e., the kernel of the T-action is finite. Under this assumption, the differentiable slice theorem tells us that the inequality $n \leq m$ holds.

If we choose an invariant complex structure on $T_pM \simeq \mathbb{R}^{2m}$ for a fixed point $p \in M^T$, there is the following irreducible decomposition

$$T_pM \simeq \bigoplus_{i=1}^{m} V_{\alpha_{p,i}}, \tag{1.1}$$

where $V_{\alpha_{p,i}}$ is the complex one-dimensional representation of T with the weight vector (called a *tangential weight*) $\alpha_{p,i} \in \mathfrak{t}^*_{\mathbb{Z}} \simeq \mathbb{Z}^n$ for $i = 1, \ldots, m$. Here, the symbol $\mathfrak{t}^*_{\mathbb{Z}}$ represents the character lattice of the dual of the Lie algebra $\mathfrak{t}^*$ of T.

Set $S_p := \{\alpha_{p,i} \mid i = 1, \ldots, m\} \subset \mathbb{Z}^n$. Assume that S_p is $k(p)$-inependent. By definition, $2 \leq k(p) \leq m$ for all $p \in M^T$; therefore, this is equivalent to that we assume M is a *GKM manifold* in [**GKM98, GZ01**]. Notice that if $k(p) = m$ for all $p \in M^T$, M is called a *torus manifold*, defined in [**HM03**]. If a GKM manifold satisfies $H^{odd}(M) = 0$, we call it an *equivariantly formal* GKM manifold.

Mikiya Masuda has asked the following question in the private communication:

Problem 1.2. *Assume that M^{2m} is an equivariantly formal GKM manifold with 4-independent T^n-action. Then, does the T-action on M extend to the almost effective T^m-action? Equivalently, is M a torus manifold?*

This article answers this problem for the homogeneous GKM manifolds. Roughly, the main theorem states that $k(p)$ can be taken only 2, 3 or n. We will prepare to describe the main theorem from the next section.

2. Remarks from GKM theory

We first introduce the following proposition. This answers Toshio Sumi's question during the author's talk in RIMS conference 2024.

PROPOSITION 2.1 (Proposition 4.3 [**Ku19**]). Let M^{2m} be a GKM manifold with T^n-action. Assume that M^{2m} has the extended T^l-action, i.e., the restricted $T^n \subset T^l$ action is the original torus action. Then, the T^l-action on M^{2m} also satisfies the condition of a GKM manifold.

So, the GKM condition is preserved after extending the torus action.

We can construct a GKM manifold M such that there are fixed points $p,\ q \in M^T$ with different properties $U_m^{k(p)}$ and $U_m^{k(q)}$.

EXAMPLE 2.2. Let $\mathbb{C}P^1$ be the complex projective space with the natural T^1-action. We assume that $T^1 \subset T^3$ is the first coordinate. Define the following symbols:

- $p_i : T^3 \to S^1$ for $i = 0, 1, 2, 3$ is the projection onto the ith coordinate, where p_0 is the trivial homomorphism;
- $\epsilon_i = \mathbb{C}P^1 \times \mathbb{C}$ for $i = 0, 1, 2, 3$ is the T^3-equivariant trivial line bundle over $\mathbb{C}P^1$ with the T^3-action on the fiber $\mathbb{C}$ by $p_i : T^3 \to S^1$;
- γ is the tautological line bundle over $\mathbb{C}P^1$.

Let $M^8 := \mathbb{P}((\gamma \otimes \epsilon_2 \otimes \epsilon_3) \oplus \epsilon_2 \oplus \epsilon_3 \oplus \epsilon_0)$ be the projectivization of the rank 4 equivariant complex vector bundle over $\mathbb{C}P^1$, i.e., M^8 is equivariantly diffeomorphic to a $\mathbb{C}P^3$-bundle over $\mathbb{C}P^1$ with T^3-action (see e.g. [**KS**]). Then, by computing the tangential weights around 8 fixed points, one can check that M^8 is a GKM manifold, i.e., for every p, the tangential weights have the property $U_4^{k(p)}$ with $k(p) \geq 2$. Moreover, the tangential weights on the fixed points $([1:0],[0:0:0:1])$ and $([0:1],[0:0:0:1])$ have the property U_4^2 and the property U_4^3 respectively.

In Proposition 3.1, we prove that the number $k(p)$ does not depend on the fixed points for the case of homogeneous GKM manifolds; therefore, we may write it by $k(G/K)$.

For the general GKM manifold which is defined by the 1-skeleton has the structure of a graph, there are several ways to define the axial function on edges. Here, we state the position of this article about the axial functions in the following remark.

REMARK 2.3. Let M be a $2m$-dimensional GKM manifold with the n-dimensional torus T-action. If there is a T-invariant almost complex structure J on M (e.g. $M = G/Z$, where G is a compact, connected Lie group and Z is the centralizer of a maximal torus of G, see [**KKLS20**]), then the identification (1.1) can be induced from J, see [**GZ01, GHZ06**]. For the case when $m = n$, called a *torus manifold* (e.g. $M = S^{2n} \simeq SO(2n+1)/SO(2n)$ with the standard T^n-action for $n \geq 2$), the identification (1.1) is determined by choosing the omni-orientation of the characteristic submanifolds in the torus manifold, see [**MMP07, Ku16**]. For the other cases (e.g. $M = \mathbb{H}P^n \simeq Sp(n+1)/Sp(n) \times Sp(1)$ for $n \geq 2$, see [**GL**]), there is no canonical way to determine the sign of the representations in (1.1). To avoid the sign ambiguities of the representations, we may choose some identification (1.1) for each fixed point. These define the label on edges of GKM graphs without sign ambiguities (called an *axial function*). In the purpose of works in some literature, the axial functions are defined up to signs (e.g. this is called a *pre-axial* in [**GHZ06**]). In particular, this definition is enough to define the graph equivariant cohomology of a GKM graph. In this article, we define the axial function without sign ambiguities by choosing some identification (1.1) for each fixed point.

3. The maximal torus action on the homogeneous space G/K

Let G be a compact, connected, semi-simple Lie group, T be its maximal torus, and K be a closed, connected subgroup of G such that $T \subset K \subset G$, i.e., a *maximal rank subgroup* of G. In this setting, we have that the fundamental group $\pi_1(G)$ is finite. It is known that the homogeneous space G/K is a $2m$-dimensional, simply connected and equivariantly formal manifold, i.e., $H^{odd}(G/K) = 0$, see e.g. [**MT79, B13**]. Due to [**GHZ06**], G/K satisfies the GKM condition. Below we recall the description of the tangential weights of the T-action on G/K by following [**GHZ06**, Theorem 2.4], also see [**MT79, B13**].

We first recall some basic facts from the theory of Lie groups. We denote the Lie algebras of $T \subset K \subset G$ by $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$, respectively. Let $\exp : \mathfrak{g} \to G$ be the exponential map, and $\Lambda_G := \exp^{-1}(e) \subset \mathfrak{t}$ be the *integer lattice*, where $e \in T \subset G$ is the identity element. Then, we have an identification $T \simeq \mathfrak{t}/\Lambda_G$ and $\mathrm{Hom}(T, S^1) \simeq \Lambda_G^* := \mathrm{Hom}(\Lambda_G, \mathbb{Z}) \simeq \mathbb{Z}^n$. The $\mathbb{Z}$-module $\Lambda_G^* \subset \mathfrak{t}^*$ is called a *weight (or character) lattice* of G, where $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ is the dual of $\mathfrak{t}$. Note that if G is simply connected, then Λ_G^* is spanned by the fundamental weight (see [**MT79**, Chapter 5 Theorem 6.36]). If G is not simply connected, then by taking the universal covering $p : \widetilde{G} \to G$ and the dual dp^* of its differential dp on the identities of maximal tori, we may regard $\Lambda_G^* \subset \Lambda_{\widetilde{G}}^*$ for two weight lattices (see [**MT79**, Chapter 5 Theorem 4.9] or [**B13**, Section 19]). Let $\Delta_G \subset \Lambda_G^* \subset \mathfrak{t}^*$ be the root systems of G with respect to the maximal torus T. Because the linear relations among the root systems do not change by taking the finite covering of G, it is enough to consider the case when G is simply connected for the purpose of this article, i.e., computing the $k(p)$-independence for the tangent wights on every $p \in (G/K)^T$.

Assume that G is simply connected, i.e., $\pi_1(G) = 0$. The set of the fixed points $(G/K)^T$ can be identified with the finite set of the quotient W_G/W_K of the Weyl groups for the maximal torus T, where $W_G := N_G(T)/T$ and $W_K := N_K(T)/T$. Let $p_0 = eK \in (G/K)^T \simeq W_G/W_K$ be the coset of the identity $e \in G$. In other words, W_G acts on $(G/K)^T$ transitively, where the isotropy subgroup of p_0 is W_K. Namely, we may write

$$(G/K)^T = \{wp_0 \mid w \in W_G\}.$$

Then, one has the irreducible decomposition

$$T_{p_0}G/K = \mathfrak{g}/\mathfrak{k} = \bigoplus_{[\beta]} V_{[\beta]},$$

of the tangential T-representation at p_0, where $[\beta]$ runs over $\Delta_{G,K}/\{\pm 1\} := (\Delta_G \setminus \Delta_K)/\{\pm 1\}$. If one can choose a section $s_{p_0} : \Delta_{G,K}/\{\pm 1\} \to \Delta_{G,K}$ of the natural projection $\Delta_{G,K} \to \Delta_{G,K}/\{\pm 1\}$ such that W_K acts on $\mathrm{Im}(s_{p_0}) \sqcup -\mathrm{Im}(s_{p_0})$, called a W_K-*equivariant section*, then $[\beta]$ may be regarded as the weight vector $s_{p_0}([\beta])$ in $\Delta_{G,K} \subset \Lambda_G^* \subset \mathfrak{t}^*$. This is equivalent to choosing the T-invariant complex structure on $T_{p_0}G/K$. Then, the set of the tangential weights on p_0 may be regarded as $\Delta_{p_0} := s_{p_0}(\Delta_{G,K}/\{\pm 1\}) \subset \Lambda_G^* \simeq \mathbb{Z}^n$. Let $p = wp_0$ with $[w] \in W_G/W_K$. Then, the tangential representation around wp_0 is

$$T_{wp_0}G/K = \bigoplus_{[\beta] \in \Delta_{G,K}/\{\pm 1\}} V_{[w\beta]}.$$

Similarly, we may define the set of tangential weights around the fixed point wp_0 by $\Delta_{wp_0} := \mathrm{Im}(s_{wp_0})$ for some section $s_{wp_0} : \Delta_{G,K}/\{\pm 1\} \to \Delta_{G,K}$.

Note that Δ_{p_0} and Δ_{wp_0} for every $w \in W_G/W_K$ are isomorphic to $\Delta_{G,K}/\{\pm 1\}$. This implies the following proposition:

Proposition 3.1. Let G be a compact, connected semi-simple Lie group, K be its maximal rank subgroup and $T \subset K \subset G$ be a maximal torus. Then, there exists an integer $k(G/K)$ such that $2 \le k(G/K) \le \dim T$ and the tangential weights of $T_p(G/K)$ are $k(G/K)$-independent for every $p \in (G/K)^T \simeq W_G/W_K$.

This proposition shows that if we compute the $k(G/K)$-independence on $\Delta_G^+ \backslash \Delta_K^+ \simeq \Delta_{G,K}/\{\pm 1\}$ for positive roots $\Delta_K^+ \subset \Delta_G^+$, then we have the $k(G/K)$-independence of all tangent spaces on the fixed points of G/K. (Cf. Example 2.2) In this paper, we determine the k-independence for all maximal rank homogeneous spaces G/K.

3.1. Reducing to the simple cases and main theorem. If we assume G is simply connected, then the standard T-action on G/K is equivariantly diffeomorphic to the following product of the maximal rank homogeneous spaces (see e.g. [**Ku10**, Section 2.2]):

$$G/K \simeq G_1/K_1 \times \cdots \times G_r/K_r, \tag{3.1}$$

where G_i is a compact, simply connected simple Lie group, K_i be its maximal rank subgroup and there is an isomorphism $T \simeq T_1 \times \cdots \times T_r$ such that $T_i \subset K_i \subset G_i$ is a maximal torus for $i = 1, \ldots, r$. Moreover, we can easily show the following proposition:

PROPOSITION 3.2. The following equality holds for the splitting (3.1):

$$k(G/K) = \min\{k(G_i/K_i) \mid 1 \leq i \leq r\}.$$

This proposition says that to compute the independency of G/K, it is enough to compute the independency of the simple factors G_i/K_i. The classification of each pair of $(G_i.K_i)$ is known by the work of Borel-deSiebenthal [**BS49**]. The following list shows their classification of the simple Lie group G and its maximal, maximal rank subgroup K, where the locally isomorphic means the Lie algebras are isomorphic.

Type A_{n+1}: If G is of type A_{n+1} for $n \geq 0$, then K is locally isomorphic to one of the followings:
- **Case 1:** $A_n \times T^1$;
- **Case 2:** $A_{i-1} \times A_{n+1-i} \times T^1$ $(1 < i < n+1)$.

Type D_n: If G is of type D_n for $n \geq 4$, then K is locally isomorphic to one of the followings, where $D_1 = T^1, D_2 = A_1 \times A_1, D_3 = A_3$:
- **Case 1:** $A_{n-1} \times T^1$.
- **Case 2:** $D_{n-1} \times T^1$;
- **Case 3:** $D_i \times D_{n-i}$ $(1 < i < n-1)$;

Type B_n: If G is of type B_n for $n \geq 2$, then K is locally isomorphic to one of the followings, where $B_1 = A_1$:
- **Case 1:** $B_{n-1} \times T^1$;
- **Case 2:** $D_i \times B_{n-i}$ $(1 < i < n)$;
- **Case 3:** D_n.

Type C_n: If G is of type C_n for $n \geq 3$, then K is locally isomorphic to one of the followings, where $C_1 = A_1, C_2 = B_2$:
- **Case 1:** $C_i \times C_{n-i}$ $(1 \leq i < n)$;
- **Case 2:** $A_{n-1} \times T^1$.

Type E_6: If G is of type E_6, then K is locally isomorphic to one of the followings:
- **Case 1:** $D_5 \times T^1$;
- **Case 2:** $A_1 \times A_5$;
- **Case 3:** $A_2 \times A_2 \times A_2$.

Type E_7: If G is of type E_7, then K is locally isomorphic to one of the followings:
- **Case 1:** $D_6 \times A_1$;
- **Case 2:** $A_2 \times A_5$;
- **Case 3:** A_7;
- **Case 3:** $E_6 \times T^1$.

Type E_8: If G is of type E_8, then K is locally isomorphic to one of the followings:
- **Case 1:** D_8;
- **Case 2:** $A_4 \times A_4$;
- **Case 3:** A_8;
- **Case 4:** $E_6 \times A_2$;
- **Case 5:** $E_7 \times A_1$.

Type F_4: If G is of type F_4, then K is locally isomorphic to one of the followings:
- **Case 1:** $C_3 \times A_1$;
- **Case 2:** $A_2 \times A_2$;
- **Case 3:** B_4.

Type G_2: If G is of type G_2, then K is locally isomorphic to one of the followings:
- **Case 1:** A_2;
- **Case 2:** $A_1 \times A_1$.

The following lemma is also useful to reduce the cases.

Lemma 3.3. *Let $H \subset K \subset G$ be a sequence of compact connected maximal rank Lie groups. If $k(G/K) = 2$, then $k(G/H) = 2$.*

Proof. There is the fibration $K/H \to G/H \to G/K$. Let $p_0 = eH$. Then, there is the following decomposition:

$$T_{p_0}G/H \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{g}/\mathfrak{k} \oplus \mathfrak{k}/\mathfrak{h}.$$

Because $k(G/K) = 2$, the subspace $\mathfrak{g}/\mathfrak{k}$ is 2-independent. Since the lowest independency is 2, we have the statement of the lemma. □

Now we may state the main theorem:

Theorem 3.4. *Assume that G is simply connected, simple Lie group and K be its maximal rank subgroup such that $\mathrm{rank}(G) = \mathrm{rank}(K) = \dim T = n$. Then, $k(G/K) = 2, 3, n$ and the following holds:*

- *$k(G/K) = n$ if and only if (G, K) is one of the following types:*
  Type A_n: $(A_n, A_{n-1} \times T^1)$;
  Type B_n: (B_n, D_n).
- *$k(G/K) = 3(\neq n)$ if and only if (G, K) is one of the following types:*
  Type A_n: $(A_n, A_{i-1} \times A_{n-i} \times T^1)$;
  Type C_n: $(C_n, C_{i-1} \times C_{n-i+1})$;
  Type D_n: $(D_n, D_{n-1} \times T^1)$, $(D_n, D_{i-1} \times D_{n-i+1})$;
  Type F_4: (F_4, B_4);
  Type E_6: $(E_6, D_5 \times T^1)$, $(E_6, A_5 \times A_1)$, $(E_6, A_4 \times T^1 \times A_1)$;
  Type E_7: (E_7, A_7), $(E_7, D_6 \times A_1)$, $(E_7, E_6 \times T^1)$, $(E_7, D_5 \times T^2)$, $(E_7, A_5 \times A_1 \times T^1)$, $(E_7, A_4 \times A_1 \times T^2)$;
  Type E_8: (E_8, D_8).
- *$k(G/K) = 2$, otherwise.*

As a corollary of this theorem, we have the following characterization of the homogeneous space by using the independence of tangential weights:

Corollary 3.5. Assume that G is simple and of classical types, i.e., type A_n, B_n, C_n and D_n. Then, the following two facts hold:

- if the tangential representations on $eH \in G/H$ is more than 4-independent, then $G/H \simeq \mathbb{C}P^n$ or S^{2n};
- if the tangential representations on $eH \in G/H$ is 3-independent, then $G/H \simeq Gr(i, n; \mathbb{C}), \widetilde{Gr}(i-1, n; \mathbb{R}), Gr(i-1, n; \mathbb{H}), Q_{2n}$.

In [**Ku10**], we classify all homogeneous torus manifolds. Together with this, Theorem 3.4 gives the partial answer to Problem 1.2 for the case when a GKM manifold is a homogeneous space. In [**KS**], we will prove the main theorem by using the signed graphs.

Remark 3.6. In [**So23**], Solomadin constructs the GKM graph which is 4-independent but there is no extension. This may be regarded as the combinatorial counter-example for Problem 1.2. However, there is no GKM manifold whose GKM graph is the example in [**So23**]. Hence, Problem 1.2 is still open.

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