The stable Albanese homology of the IA-automorphism groups of free groups

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1 Introduction

The automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n of rank n is a fundamental object and appears in various fields such as group theory, representation theory, topology, geometric group theory, and functor homology. An important normal subgroup of $\mathrm{Aut}(F_n)$ is the inner automorphism group $\mathrm{Inn}(F_n)$ of F_n consisting of conjugations of elements of F_n. We will consider another normal subgroup IA_n of $\mathrm{Aut}(F_n)$, which is called the IA-automorphism group of F_n. The definition of IA_n is given as the kernel of the map between the automorphism groups induced by the abelianization map of F_n, that is, we have a short exact sequence of groups

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z}) \to 1. \tag{1}$$

If $n = 2$, then the inner automorphism group $\mathrm{Inn}(F_2)$ is equal to IA_2, and if $n \geq 3$, then $\mathrm{Inn}(F_n)$ is a proper subgroup of IA_n. Magnus [14] proved that IA_n is finitely generated and explicitly gave a finite set of beautiful generators. Krstić–McCool [11] proved that IA_3 is not finitely presentable. However, for $n \geq 4$, it is still open whether IA_n is finitely presentable or not.

Cohen–Pakianathan [4], Farb (unpublished) and Kawazumi [9] independently determined the first homology of IA_n by proving that the *Johnson homomorphism* for $\mathrm{Aut}(F_n)$ induces an isomorphism

$$H_1(\mathrm{IA}_n, \mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_{\mathbb{Z}}, \bigwedge^2 H_{\mathbb{Z}}), \quad H_{\mathbb{Z}} = H_1(F_n, \mathbb{Z}).$$

Bestvina–Bux–Margalit [1] proved that $H_2(\mathrm{IA}_3, \mathbb{Z})$ has infinite rank, but it is not known whether or not $H_2(\mathrm{IA}_n, \mathbb{Z})$ has finite rank for $n \geq 4$. As a $\mathrm{GL}(n, \mathbb{Z})$-representation, $H_2(\mathrm{IA}_n, \mathbb{Z})$ is finitely generated by Day–Putman [5].

By the short exact sequence (1), the homology of IA_n admits an action of $\mathrm{GL}(n, \mathbb{Z})$. We will study the $\mathrm{GL}(n, \mathbb{Z})$-representation structure of the rational homology of IA_n. Since the structure of the rational homology of IA_n is quite complicated for small n [18], we will consider the rational homology of IA_n for sufficiently large n with respect to the homological degree. Moreover, we will restrict our attention to the *Albanese homology* of IA_n, which is a quotient representation of the rational homology of IA_n. There had been several literatures on the notion of the Albanese homology of IA_n but the terminology was introduced in [2]. The Albanese homology $H_i^A(\mathrm{IA}_n, \mathbb{Q})$ of IA_n is defined as the image of the map between homology induced by the abelianization map of IA_n. Therefore, the Albanese homology of IA_n is an *algebraic* $\mathrm{GL}(n, \mathbb{Z})$*-representation*.

In [6], Habiro and the author studied the whole rational cohomology of IA_n and obtained a conjectural structure, which implies that the rational cohomology is generated by the Albanese (co)homology and the $\mathrm{GL}(n, \mathbb{Z})$-invariant part of the rational cohomology. We proved the conjecture under the assumption that the rational cohomology of IA_n is algebraic for sufficiently large n with respect to the cohomological degree. This is why we consider the Albanese homology of IA_n as an essential part of the whole rational homology of IA_n. In this report, we will exhibit our recent results on the Albanese homology of IA_n.

2 Preliminaries

Here, we will recall the previous results on the Albanese homology of IA_n. The second Albanese homology was determined by Pettet [16] and the third Albanese homology was determined in [7]. Moreover, in [7], the author detected a large subquotient $\mathrm{GL}(n, \mathbb{Z})$-representation of the Albanese homology of IA_n in higher degrees, which we will observe below.

Set $H = H(n) = H_1(F_n, \mathbb{Q})$. Let $U_* = \bigoplus_{i \geq 1} U_i$, $U_i = \mathrm{Hom}(H, \bigwedge^{i+1} H)$ for $i \geq 1$. Let $S^*(U_*)$ denote the graded-symmetric algebra of U_*. Let $W_* = \widetilde{S}^*(U_*)$ denote the *traceless part* of $S^*(U_*)$, which consists of elements that vanish under any *contraction maps* between distinct factors of $S^*(U_*)$. (See [7, Section 2] for details.) One of the main results of [7] is the following.

Theorem 1 ([7, Theorem 6.1])**.** *We have a morphism of graded* $\mathrm{GL}(n, \mathbb{Z})$*-representations*

$$F_* : H_*(U_1, \mathbb{Q}) \to S^*(U_*)$$

such that $F_*(H_*^A(\mathrm{IA}_n, \mathbb{Q})) \supset W_*$ *for* $n \geq 3*$.

Moreover, in [7], the author made the following conjecture on the structure of the stable Albanese homology of IA_n, which was known to hold for $i \leq 3$.

Conjecture 2 ([7, Conjecture 6.2])**.** *Let* $i \geq 1$*. For* $n \geq 3i$*, we have an isomorphism of* $\mathrm{GL}(n, \mathbb{Z})$*-representations*

$$F_i : H_i^A(\mathrm{IA}_n, \mathbb{Q}) \xrightarrow{\cong} W_i.$$

3 Main theorems

3.1 The stable Albanese homology of IA_n

In the appendix of [13], the author proved the statement of Conjecture 2 for $n \gg i$ by using Theorem 1 and [13, Proposition 6.3]. Furthermore, in [8], the author improved the stable range to prove Conjecture 2.

Theorem 3 ([8, Theorem 2.5])**.** *Conjecture 2 holds.*

As a corollary, the Albanese homology of IA_n is *representation stable* in $n \geq 3i$ in the sense of Church–Farb [3].

3.2 The relation between the Albanese homology of IA_n and the cohomology of $\mathrm{Aut}(F_n)$ with certain coefficients

The stability of the (co)homology of $\mathrm{Aut}(F_n)$ with coefficients in $H^{p,q} = H^{\otimes p} \otimes (H^*)^{\otimes q}$ was shown by Randal-Williams–Wahl [17]. Moreover, recently, the stable (co)homology was determined by Lindell [12] and the stable range was improved in [15]. On the other hand, it follows from Theorem 3 that the $\mathrm{GL}(n,\mathbb{Z})$-invariant part $H_A^*(\mathrm{IA}_n, H^{p,q})^{\mathrm{GL}(n,\mathbb{Z})} := [H_*^A(\mathrm{IA}_n,\mathbb{Q})^* \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}$ stabilizes. In [8], the author obtained the following relation between the stable cohomology $H^*(\mathrm{Aut}(F_n), H^{p,q})$ and $H_A^*(\mathrm{IA}_n, H^{p,q})^{\mathrm{GL}(n,\mathbb{Z})}$.

Theorem 4 ([8, Theorem 3.8])**.** *The inclusion map* $\mathrm{IA}_n \hookrightarrow \mathrm{Aut}(F_n)$ *induces an isomorphism of* $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$*-modules*

$$i^* : H^*(\mathrm{Aut}(F_n), H^{p,q}) \to H_A^*(\mathrm{IA}_n, H^{p,q})^{\mathrm{GL}(n,\mathbb{Z})}$$

for $n \geq \min(\max(3*+4, p+q), 2*+p+q+3)$.

In the proof of Theorem 4, we used the wheeled PROP structure of the stable cohomology of $\mathrm{Aut}(F_n)$ with coefficients in $H^{p,q}$ that was constructed in [10], and the 1-cocycle that was constructed in [9].

3.3 The stable Albanese homology of IO_n

We consider the analogue of IA_n to the outer automorphism group $\mathrm{Out}(F_n)$ of F_n. Since $\mathrm{Inn}(F_n)$ is a subgroup of IA_n, the group homomorphism $\mathrm{Aut}(F_n) \twoheadrightarrow \mathrm{GL}(n,\mathbb{Z})$ in the definition of IA_n induces the group homomorphism $\mathrm{Out}(F_n) \twoheadrightarrow \mathrm{GL}(n,\mathbb{Z})$. Let IO_n denote the kernel of the group homomorphism, that is, we have a short exact sequence of groups

$$1 \to \mathrm{IO}_n \to \mathrm{Out}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to 1.$$

The Albanese homology $H_*^A(\mathrm{IO}_n,\mathbb{Q})$ of IO_n is defined in a way similar to IA_n. The first Albanese homology of IO_n is isomorphic to the first homology of IO_n, which was determined in [9]. The second and the third Albanese homology of IO_n was determined in [16] and in [7], respectively. The author also made a conjectural structure of the Albanese homology of IO_n in [7] and proved the conjecture in [8].

Let $U_*^O = \bigoplus_{i\geq 1} U_i^O$, where $U_1^O = \mathrm{Hom}(H, \bigwedge^2 H)/H$, $U_i^O = U_i$ for $i \geq 2$. Let $S^*(U_*^O)$ denote the graded-symmetric algebra of U_*^O. Let $W_*^O = \widetilde{S}^*(U_*^O)$ denote the traceless part of $S^*(U_*^O)$. Then the structure of the Albanese homology of IO_n is the following.

Theorem 5 ([8, Theorem 3.3])**.** *Let* $i \geq 1$*. For* $n \geq 3i$*, we have an isomorphism of* $\mathrm{GL}(n,\mathbb{Z})$*-representations*

$$H_i^A(\mathrm{IO}_n,\mathbb{Q}) \cong W_i^O.$$

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