Algebro-geometric invariants reflected in counter-examples of the (integral) Hodge conjecture

南　範彦

Norihiko Minami

大和大学

Yamato University *

Abstract

I shall report my recent results which endows some algebro-geometric invariant interpretations for the descrepancies reflected in counter-examples of the (integral) Hodge conjecture, [1)]

1 Background

1.1 The cycle map and related maps

For a smooth projective complex variety X, we have the usual cycle map $cl^i_{Hdge}(X)$ and related maps, whose cokernels shall be denoted as follows:

$$\begin{array}{ccc} \mathrm{CH}^i(X) & \xrightarrow{cl^i_{Hdg}(X)} & \mathrm{Hdg}^{2i}(X,\mathbb{Z}) := H^{2i}(X,\mathbb{Z}) \cap H^{i,i}(X,\mathbb{C}) \\ \downarrow cl^i_{Tot}(X) & \searrow cl^i_{Betti}(X) & \downarrow \\ \left(MU^*(X) \otimes_{MU^*} \mathbb{Z}\right)^{2i} & \xrightarrow[Thom^{2i}(X)]{} & H^{2i}(X,\mathbb{Z}) \end{array}$$

$\Longrightarrow$

$$\begin{array}{ccccc} Z^{2i}_{Hdg}(X)\{tors\} & \hookrightarrow & Z^{2i}_{Hdg}(X); = \mathrm{Coker}\left(cl^i_{Hdg}(X)\right) & & \\ \| & & \downarrow & & \\ Z^{2i}_{Betti}(X)\{tors\} & \hookrightarrow & Z^{2i}_{Betti}(X); = \mathrm{Coker}\left(cl^i_{Betti}(X)\right) & \twoheadrightarrow & Z^{2i}_{Thom}(X); = \mathrm{Coker}\left(Thom^{2i}(X)\right) \end{array}$$

Of course, we have various conjectures concerning these cokernels:

- codimension i integral Hodge conjecture , which predicts $Z^{2i}_{Hdg}(X) = 0$. (This is well known to be false in general, ever since the famous Atiyah-Hirzebruch topological counter-example [AH62].)

*minami.norihiko@yamato-u.ac.jp

[1)] This report replaces the author's previous, unfortunately erroneous, report

Norihiko Minami, *Higher codimensional birational equivalences and counter-examples of the integral Hodge conjecture,* RIMS Kokyuroku,

which dealt with the same theme.

- codimension i weak integral Hodge conjecture , which predicts $Z^{2i}_{Hdg}(X)\{tors\} \left(\overset{\text{easy}}{=} Z^{2i}_{Betti}(X)\{tors\}\right)$ $= 0$. (This is also known to be false in general. Actually, all the known counter-examples of the integral Hodge conjecture are actually counter-examples of this weak conjecture.)

- codimension i Hodge conjecture predicts $Z^{2i}_{Hdg}(X) \otimes \mathbb{Q} = 0$, *the* famous conjecture.

1.2 The motivation of this work

The Quetion

Are there any algebro-geometric invariant interpretations of

$$Z^{2i}_{Hdg}(X), Z^{2i}_{Hdg}(X)\{tors\} \overset{\text{easy}}{=} Z^{2i}_{Betti}(X)\{tors\}, Z^{2i}_{Hdg}(X) \otimes \mathbb{Q}, \text{ and } Z^{2i}_{Thom}(X)?$$

In this paper, I shall announce my recent results on this question.

2 Known cases and a warning

2.1 The codimension $i = 1$ case

The codimension 1 integral Hodge conjecture is known to hold by Lefschetz and Hodge [KS53, p.876, Theorem 4]. The essense may be summarized by the following commutative diagram arising from the short exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_{X^{an}} \xrightarrow{\exp} \mathcal{O}^*_{X^{an}} \to 0$ of sheaves on the corresponding complex analytic manifold X^{an} :

$$\begin{array}{ccccc} H^1(X, \mathcal{O}^*_X) & = & CH^1(X) & & \\ \| \text{ Serre's GAGA} & & \downarrow cycl^1 & & \\ H^1(X^{an}, \mathcal{O}^*_{X^{an}}) & \longrightarrow & H^2(X^{an}, \mathbb{Z}) & \longrightarrow & H^2(X^{an}, \mathcal{O}_{X^{an}}) \end{array}$$

This means $Z^2(X) = 0$, and so not interesting from our point of view.
Warning: The above proof made use of analytic technique. Such is the case, the case $i = 1$ of the simarly defined Tate conjecture is still unknown, because of the lack of such analytic technique in its setting.

2.2 The condimension $i = \dim X - 1$ case (the so-called "curve case") and the codimension $i = 2$ case

- Thanks to the Hard Lefschetz theorem, we find the usual Hodge conjecture for the curve case follows from the codimension 1 case:

$$Z^2_{Hdge}(X) = 0 \quad \Longrightarrow \quad Z^2_{Hdge}(X)_{\mathbb{Q}} = 0 \quad \overset{\text{Hard Lefschetz}}{\Longrightarrow} \quad Z^{2(\dim X-1)}_{Hdge}(X)_{\mathbb{Q}} = 0$$

  We should be warned that there is NO integral Hard Lefschetz theorem. Actually, Kollár [K92] found a counter-example to the integral analogue for the curve case:

$$Z^{2(\dim X-1)}_{Hdge}(X)\{tors\} \neq 0 \quad (\text{with } \dim X = 3)$$

These facts imply whereas $Z^{2(\dim X-1)}_{Hdge}(X)_{\mathbb{Q}} = 0$ is not interesting. $Z^{2(\dim X-1)}_{Hdge}(X) \overset{Z^{2(\dim X-1)}_{Hdge}(X)_{\mathbb{Q}}=0}{=} Z^{2(\dim X-1)}_{Hdge}(X)\{tors\}$ is interesting from our point of view.

- The usual Hodge conjecture for the codimension $i = 2$ case $Z^4_{Hdge}(X) \otimes \mathbb{Q} \overset{?}{=} 0$ is still unsolved. However, many counter-examples of its integral analogue $Z^4_{Hdge}(X) \neq 0$ have been constructed ever since the Atiyah-Hirzebruch first counter-example [AH62], including the aforementioned counter-example of Kollár [K92].

Concerning these interesting invariants, Soulé-Voisin and Voisin proved the following important theorem:

Theorem 2.1. [SV05, p.113, Lemma 1][V07, Lemma 15] *$Z^4_{Hdg}(X)$ and $Z^{2\dim X-2}_{Hdg}(X)$ are birational invariant.*

WIth the weak factorization theorem at hand, the proof is an immedaite corollary of the blowup formula.

2.3 Warning

Whereas Theorem 2.1 might prompt readers to look after the birational invariance of other $Z^{2i}_{Hdg}(X)$ also. However, for general i, $Z^{2i}_{Hdg}(X)$ is not birational invariant. For instance, the integral Hodge conjecture of a smooth projective X, which is regularly embedded in $\mathbb{P}^N$, is equivalent to the integral Hodge conjecture of the rational smooth projective variety $\widetilde{\mathbb{P}^N}$, obtained as the embedded blowup of $\mathbb{P}^N$, centered at X.

$$\begin{array}{ccc} D & \hookrightarrow & \widetilde{\mathbb{P}^N} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}^N \end{array} \; : \qquad Z^{2*}_{Hdg}(X) = 0 \iff Z^{2*}_{Hdg}\left(\widetilde{\mathbb{P}^N}\right) = 0$$

3 The codimension $> c$ birational equivalence and Schreieder's work

In order to generalize Theorem 2.1 to other codimensions, the appropriate equivalence relation is not the birational equivalence, but the following hierarchical analogue:

Definition 3.1. *Let us say smooth projective equi-dimensional k-schemes X, Y are $\underline{\text{codimension} > c \text{ birational equivalent}}$ (or $\underline{\text{isomorphism in codimension } c}$), if there are closed subsets $Z_X \subset X$ and $Z_Y \subset Y$ s.t.*

- $\operatorname{codim}_X Z_X > c, \quad \operatorname{codim}_Y Z_Y > c.$
- $X \setminus Z_X \underset{f}{\overset{\sim}{\to}} Y \setminus Z_Y.$

(The case $c = 0$ is the usual birational equivalence.)

Now, Stefan Schreieder generalized the torsion analogue of Theorem 2.1 for smaller i's including the case of $i = 2$:

Theorem 3.2. [S23, Theorem 1.6.(i)] [S22, Corollary 6.12] $Z^{2i}_{Hdg}(X)\{tors\}$ *is codim* $> (i-2)$ *birational invariant.* □

(Schreieder's argument appears to be inapplicable to obtain similar results for $Z^{2i}_{Hdg}(X), Z^{2i}_{Hdg}(X)\otimes\mathbb{Q}$, and $Z^{2i}_{Thom}(X)$. Also not so useful when i is large, e.g. $Z^{2\dim X-2}_{Thom}(X)\{tors\}$.)

4 Main Theorem

Contrary to Schreieder's Theorem 3.2 , my main theorems, which I now state, are especially useful for large i, and, also cover $Z^{2i}_{Hdg}(X), Z^{2i}_{Hdg}(X)\otimes\mathbb{Q}$, and $Z^{2i}_{Thom}(X)$:

Theorem 4.1. (i) $Z^{2i}_{Hdg}(X)$ *is codimension* $> (\dim_{\mathbb{C}} X-1)-i$ *birational invariant.*
(When $i=\dim_{\mathbb{C}} X-1$, this recovers the birational invariance of $Z^{2\dim_{\mathbb{C}} X-2}_{Hdg}(X)$, which was first observed by Soulé-Voisin and Voisin.)

(ii) $Z^{2i}_{Hdg}(X)\otimes\mathbb{Q}$ *is codimension* $> (\dim X_{\mathbb{C}}-2)-i$ *birational invariant.*
(When $i=\dim_{\mathbb{C}} X-2$, this recovers the birational invariance of $Z^{2\dim_{\mathbb{C}} X-4}_{Hdg}(X)_{\mathbb{Q}}$, which should be well-known to experts as this too can be immediately observed from the blowup formula, under the weak factorization theorem.)

(iii) *Each p-primary component* $Z^{j}_{Thom}(X)_{(p)}$ *of the finite abelian group* $Z^{j}_{Thom}(X)$ *is codimension* $>$ $\left[\dim_{\mathbb{C}} X-\frac{(2p+1)+j}{2}\right]$ *birational invariant.*
(When $j=2\dim_{\mathbb{C}} X-2p, 2\dim_{\mathbb{C}} X-(2p+1)$, this recovers the birational invariance of $Z^{2\dim_{\mathbb{C}} X-2p}_{Thom}(X)_{(p)}$, $Z^{2\dim_{\mathbb{C}} X-(2p+1)}_{Thom}(X)_{(p)}$, which appear to be new.)

The above Theorem 4.1(iii) turns out to possese a "chromatic hierarchy."

Theorem 4.2. $Z^{j}_{Thom}(X)_{(p)}$ *admits a "chromatic" filtration of codim* $> \left[\dim_{\mathbb{C}} X-\frac{(2p+1)+j}{2}\right]$ *bir. inv.:*

$$\begin{aligned} Z^{j}_{Thom}(X)_{(p)} &= Z^{j}_{Thom}(X)_{(p)}\langle 0\rangle \supseteq Z^{j}_{Thom}(X)_{(p)}\langle 1\rangle \\ &\supseteq Z^{j}_{Thom}(X)_{(p)}\langle 2\rangle \supseteq Z^{j}_{Thom}(X)_{(p)}\langle 3\rangle \supseteq \cdots , \end{aligned}$$

arising from the "(cohomological) chromatic tower", given by the Johnson-Wilson spectra $BP\langle n\rangle$:

$$\begin{array}{ccc} MU^*(X)\otimes_{MU^*} H^* & \longrightarrow & H^*(X,\mathbb{Z}) \\ \uparrow\!\!\uparrow & & \downarrow \\ MU^*(X) \longrightarrow BP^*(X) \longrightarrow \cdots \longrightarrow BP\langle n+1\rangle^*(X) & \longrightarrow BP\langle n\rangle^*(X) \longrightarrow & \cdots BP\langle 0\rangle^*(X) = H^*(X,\mathbb{Z})_{(p)} \end{array}$$

by setting

$$Z^{j}_{Thom}(X)_{(p)}\langle n\rangle := \frac{\operatorname{Im}\left(BP\langle n\rangle^*(X)\to H^*(X,\mathbb{Z})_{(p)}\right)}{\operatorname{Im}\left(Thom^{2j}(X)_{(p)} : (MU^*(X)\otimes_{MU^*}\mathbb{Z})^{2i}\to H^*(X,\mathbb{Z})_{(p)}\right)}$$

My proof of the above theorems is some very careful analysis of the correspondences together with Hironaka.

In particular, my analysis also leads to some other higher codimensional birational invariants also.

The detail will appear elsewhere.

References

[AH62] M.F. Atiyah, F. Hirzebruch, *Analytic cycles on complex manifolds,* Topology 1 (1962), 25–45.

[BO74] Spencer Bloch, Arthur Ogus, *Gersten's conjecture and the homology of schemes,* Ann. Sci. École Norm. Sup. (4) 7 (1974), 181–201.

[BS83] Spencer J. Bloch, Vasudevan Srinivas, Remarks on correspondences and algebraic cycles, Amer. J. Math. 105 (1983), no. 5, 1235–1253.

[CTHK97] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, Bruno Kahn, *The Bloch-Ogus-Gabber theorem,* Algebraic K-theory (Toronto, ON, 1996), 31–94, Fields Inst. Commun., 16, Amer. Math. Soc., Providence, RI, 1997.

[CTV12] Jean-Louis Colliot-Thélène, Claire Voisin, *Cohomologie non ramifiée et conjecture de Hodge entière*, Duke Math. J. 161 (2012), no. 5, 735–801.

[KS53] K. Kodaira, D. C. Spencer, *Divisor class groups on algebraic varieties,* Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 872–877.

[K92] János Kollár, *Trento examples,* in Classification of irregular varieties, edited by E. Ballico, F. Catanese, C. Ciliberto, Lecture Notes in Math. 1515, Springer (1992).

[Q19] Gereon Quick, *Examples of non-algebraic classes in the Brown-Peterson tower,* Math. Z. 293 (2019), no. 1–2, 25–37.

[S22] Stefan Schreieder, *A moving lemma for cohomology with support,* **https://arxiv.org/pdf/2207.08297.pdf**

[S23] Stefan Schreieder, *Refined unramified cohomology of schemes,* Compos. Math. 159 (2023), no. 7, 1466–1530.

[SV05] C. Soulé, C. Voisin, *Torsion cohomology classes and algebraic cycles on complex projective manifolds,* Adv. Math. 198 (2005), no. 1, 107–127.

[T97] Burt Totaro, *Torsion algebraic cycles and complex cobordism,* J. Amer. Math. Soc. 10 (1997), no. 2, 467–493.

[V07] Claire Voisin, *Some aspects of the Hodge conjecture,* Jpn. J. Math. 2 (2007), no. 2, 261–296.

[V19] Claire Voisin, *Birational invariants and decomposition of the diagonal,* Birational geometry of hypersurfaces, 3–71, Lect. Notes Unione Mat. Ital., 26, Springer, Cham, 2019.