Importance Measures of a Group of Components in a Reliability System

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Abstract

In this paper, we organize the ordered set theoretical works which have been done so far on extending the Birnbaum importance measure[4] for a single component in a binary-state system to a group of components in a multi-state system, and present stochastic bounds for criticality and Fussell-Vesely importance measure in the same situation, when the joint performance probability of the components, in other words, the probability on the product ordered set of the state spaces of the components is associated.

1 Introduction

An idea of an importance measure of a component in a system plays a crucial role for determining the maintenance priority of the components and has been examined so far, so much.

When components and a system composed of them are assumed to have binary-state spaces, i.e., possibly normal and failure states, the system is called a binary-state system, for which various ideas of importance measures are proposed, Birnbaum importance measure[4], criticality importance measure[5, 7], Fussell-Vesely importance measure[8, 28], risk reduction worth[6], availability importance measure[1, 9], and are applied to practical issues[26, 27], among which the concept of a critical state vector is the basis, and so a main problem in the examination of the importance measures for a multi-state system is how to generalize the concept of the critical state vector. These works about importance measures for binary-state systems are summarized by [10] and also refer to [3] for a total theory of binary-state systems.

The idea of the availability importance measure[1, 9] tries to define a joint importance measure of two components and is considered to relate to our importance measure of a group of components. However the relation is not cleared.

For binary-state systems, [14] presents a necessary condition for a state vector to be a critical vector in terms of minimal path and cut sets, and [11] shows the sufficiency of the necessary condition. Our work is a generalization of this if and only if condition to a group of components in a multi-state system, but the condition is somewhat modified and may be directly used for constructing an algorithm deriving critical state vectors.

The states of components and systems are not practically restricted to normal and failure, and have various intermediate states between normal and failure, so to say, deteriorating states, and so a theory of a multi-state system is required and has been examined so far from the ordered set theoretical point of view[12, 13, 15, 17, 18].

In this paper, we organize our works which have been done so far about importance measures of a group of components in a context of the multi-state system[19, 20, 21, 22, 23, 25] and present multi-state version of the criticality and Fussel-Vesely importance measures for a group of components and stochastic bounds for these measures.

In the sequel of this section, we use some symbols which are precisely described in the next section.

For a multi-state system (Ω_C, S, φ), $A \subset C$, an increasing subsets $U \subset \Omega_A$ and $V \subset S$, examinations of importance measures of a group of components start with the following definition of an A-U-V-critical state vector[23, 25] : a state vector $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$ satisfying the next condition is called an A-U-V-critical state vector and U is called a V-contributing set of $\boldsymbol{x}_{A^c}$.

$$\varphi(U_A, \boldsymbol{x}_{A^c}) \subseteqq V,\ \varphi(U_A^c, \boldsymbol{x}_{A^c}) \subseteqq V^c. \tag{1}$$

On the other hand, we may define a critical state vector as the following formula :

$$\exists \boldsymbol{a} \in \Omega_A,\ \exists \boldsymbol{b} \in \Omega_A,\ \varphi(\boldsymbol{a}_A, \boldsymbol{x}_{A^c}) \in V,\ \varphi(\boldsymbol{b}_A, \boldsymbol{x}_{A^c}) \in V^c. \tag{2}$$

$\boldsymbol{x}_{A^c} \in \Omega_{A^c}$ satisfying (2) is called an $(A;V)$-critical-state vector of the group A for V. The formula (2) is intuitively understandable and denotes that the state of the system is changed from V to V^c along with the change of the states of the components A from $\boldsymbol{a}$ to $\boldsymbol{b}$ at the state vector $\boldsymbol{x}_{A^c}$ of the components outside A, in other words, $\boldsymbol{x}_{A^c}$ may be interpreted as an environmental condition for the group A of the components to have the deciding vote for the state of the system. We may show an equivalent definition with the idea of a contributing set, which is finally shown to be uniquely the inverse set $\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)$.

We note here that for the definition of a critical state vector, we have two logical streams starting with a V-contributing set and a critical state vector itself which are shown to be equivalent in the section 3, where Birnbaum and other importance measures are defined. In the section 4, when the probability on the product ordered set of the state spaces of the components is associated[16], we give stochastic bounds for criticality and Fussell-Vesely importance measures for a group of components of a multi-state system. Stochastic upper bounds for Birnbaum importance measure are shown in [24, 25].

In [2] the mean of the Birnbaum importance measure along with time axis has been given as a stochastic dynamical importance measure for a binary-state system. This dynamical idea has been generalized to the multi-state case by [19, 20, 21, 25], but, is not mentioned in this paper because of the restriction of the number of the pages.

Following the original definition of the Birnbaum importance measure for a binary-state system[4], which is defined to be a difference between two kinds of conditional probabilities, we give an extended Birnbaum importance measure for a group of components by using conditional probabilities in the subsection 3.2. The detailed examination of this formulation is remained for the future work.

2 A Multi-State System

We first present a definition of a system, following [15, 17, 18].

Definition 2.1 *(A definition of a system) A multi-state system, which is sometimes simply called "a system φ" or "a system", is a triplet (Ω_C, S, φ) satisfying the following conditions:*

(1) C is a nonempty finite set, denoting the set of all the components of which the system consists.

(2) Ω_i $(i \in C)$ and S are ordered sets, not necessarily totally ordered sets, denoting the state space of the component i and the system, respectively.

In the context of the reliability theory, it is natural to assume the state spaces to have the maximum and minimum elements, each denoting the perfectly normal and failure states. In this paper, however, these special states are not assumed to exist.

(3) Ω_C is the product ordered set of Ω_i $(i \in C)$, i.e., $\Omega_C = \prod_{i \in C} \Omega_i$. An element $\boldsymbol{x} \in \Omega_C$ is written in detail as $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, where x_i denotes the state of the component i and an element of Ω_i.

(4) $\varphi : \Omega_C \to S$ is an increasing surjective mapping and called a structure function. ■

The symbol $\leqq$ is commonly used to denote all the "order" in this paper.

When $\Omega_i = \{0, 1\}$ $(i \in C)$, $S = \{0, 1\}$ and the order is defined as $0 < 1$, the system is called a binary-state system. In this case, each state space is a Boolean lattice.

Generally, for an ordered set W, $MI(W)$ and $MA(W)$ denote the sets of minimal and maximal elements of W, respectively.

For a subset $A \subseteqq C$, $\Omega_A = \prod_{i \in A} \Omega_i$ is the product ordered set of Ω_i $(i \in A)$. An element of Ω_A is written as $\boldsymbol{x}_A$. For $\boldsymbol{x}_A,\ \boldsymbol{y}_A \in \Omega_A$:

$$\begin{aligned}
\boldsymbol{x}_A \leqq \boldsymbol{y}_A &\iff \forall i \in A,\ x_i \leqq y_i\ , \\
\boldsymbol{x}_A = \boldsymbol{y}_A &\iff \forall i \in A,\ x_i = y_i\ , \\
\boldsymbol{x}_A < \boldsymbol{y}_A &\iff \forall i \in A,\ x_i \leqq y_i \text{ and } \exists j \in A\ x_j \neq y_j\ .
\end{aligned}$$

For example, for $C = \{1,2,3,4,5\}$ and $A = \{1,3,4\} \subseteqq C$, $\Omega_A = \Omega_1 \times \Omega_3 \times \Omega_4$ is the product ordered set of $\Omega_1,\ \Omega_3$ and Ω_4 and $\boldsymbol{x}_A = (x_1, x_3, x_4) \in \Omega_A$ is the combination of $x_1 \in \Omega_1,\ x_3 \in \Omega_3,\ x_4 \in \Omega_4$. As long as the index numbers are specified, it makes no sense how to arrange them. For example, (x_3, x_4, x_1) is the same to (x_1, x_3, x_4). For subsets A and $B \subseteqq C$, when $A \cap B = \phi$, $\boldsymbol{x}_{A \cup B} = (\boldsymbol{x}_A, \boldsymbol{x}_B)$. Generally, for subsets E and $F \subseteqq C$, $\boldsymbol{x}_{E \cup F} = (\boldsymbol{x}_{E \setminus F}, \boldsymbol{x}_{E \cap F}, \boldsymbol{x}_{F \setminus E})$.

Definition 2.2 *(A definition of partial structure function) For a non-empty subset $A \subseteqq C$ such that $A^c = C \setminus A \neq \phi$ and any fixed $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$, a mapping*

$$\varphi(\cdot_A, \boldsymbol{x}_{A^c}) : \Omega_A \to S \tag{3}$$

is defined as the following :

$$\boldsymbol{x}_A \in \Omega_A, \quad \varphi(\boldsymbol{x}_A, \boldsymbol{x}_{A^c}) \in S, \tag{4}$$

which is called $\boldsymbol{x}_{A^c}$-restricted structure function and is sometimes written as $\varphi_{\boldsymbol{x}_{A^c}}$. Furthermore, a system $(\Omega_A, S, \varphi(\cdot_A, \boldsymbol{x}_{A^c}))$ may be defined, which is not examined in this paper. ■

A state vector $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$ of the components of A^c denotes an (inner) operating environment for the components of A. The formula (3) signifies how the group of the components A contribute to the system's performance on the environment $\boldsymbol{x}_{A^c}$.

The following notations are used for an image and an inverse image with respect to $\varphi(\cdot_A, \boldsymbol{x}_{A^c})$:

$$\begin{aligned}
\text{for } U \subseteqq \Omega_A, \quad \varphi(U_A, \boldsymbol{x}_{A^c}) &= \{\varphi(\boldsymbol{x}_A, \boldsymbol{x}_{A^c}) \mid \boldsymbol{x}_A \in U\}\ , \\
\text{for } V \subseteqq S, \quad \varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V) &= \{\boldsymbol{x}_A \mid \varphi(\boldsymbol{x}_A, \boldsymbol{x}_{A^c}) \in V\} \\
&= \{\boldsymbol{x}_A \mid\ (\boldsymbol{x}_A, \boldsymbol{x}_{A^c}) \in \varphi^{-1}(V)\}, \\
&\qquad \text{i.e., the section of } \varphi^{-1}(V) \text{ at } \boldsymbol{x}_{A^c}.
\end{aligned}$$

The index A of U_A is intended to emphasise $U \subseteqq \Omega_A$, however, sometimes omitted when there is no confusion.

Definition 2.3 *(A definition of an increasing set) A subset X of an ordered set W is called an increasing set, when for every x and $y \in W$,*

$$x \in X \text{ and } x \leqq y \Longrightarrow y \in X$$

holds, and then, for an increasing set X, we have $X = \bigcup_{x \in MI(X)} [x, \to)$.

$Z \subseteqq W$ is called a decreasing set, when Z^c is an increasing set, i.e.,

$$y \in Z,\ x \leqq y \implies x \in Z. \tag{5}$$

and then, when Z is a decreasing set, we have $Z = \bigcup_{z \in MA(Z)} (\leftarrow, z]$. ■

It is easily proved that (5) and the increasing property of Z^c are equivalent, and so the proof is omitted.

Incidentally, since φ is increasing, for an increasing subset $V \subseteqq S$, $\varphi^{-1}(V) \subseteqq \Omega_C$ is an increasing set and then we have the following :

$$\varphi^{-1}(V) = \bigcup_{\boldsymbol{x} \in MI(\varphi^{-1}(V))} [\boldsymbol{x}, \to), \quad \varphi^{-1}(V^c) = \bigcup_{\boldsymbol{x} \in MA(\varphi^{-1}(V^c))} (\leftarrow, \boldsymbol{x}]. \tag{6}$$

where, for example, $[\boldsymbol{x}, \to) = \{\boldsymbol{z} | \boldsymbol{x} \leqq \boldsymbol{z}\}$, denoting an interval. The formulae of (6) tell us that an increasing mapping φ is determined by the minimal elements of inverse images of increasing subsets.

The restricted mapping $\varphi(\cdot_A, \boldsymbol{x}_{A^c})$ defined by the formula (4) is increasing, since φ is increasing. Then we have the following theorem:

Theorem 2.1 *(i) For an increasing subset $V \subseteqq S$, $\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)$ is an increasing subset of Ω_A.*
(ii) For a decreasing subset $V \subseteqq S$, $\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)$ is a decreasing subset of Ω_A. ■

For a group of components $A \subseteqq C$, an increasing subset $V \subseteqq S$ and $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$, we have the following theorem for a relation between $MI\left(\varphi^{-1}(V)\right)$ and $MI\left(\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)\right)$ of which proof is omitted:

Theorem 2.2 *We have the following equalities for $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$:*

$$\begin{aligned} MI\left(\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)\right) &= MI\left\{ \, \boldsymbol{m}_A \;\middle|\; \boldsymbol{m} \in MI\left(\varphi^{-1}(V)\right), \; \boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \, \right\}, \\ MA\left(\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V^c)\right) &= MA\left\{ \, \boldsymbol{M}_A \;\middle|\; \boldsymbol{M} \in MA\left(\varphi^{-1}(V^c)\right), \; \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c} \, \right\}. \end{aligned} \quad ■$$

3 A Critical State Vectors for a Birnbaum Importance Measure

3.1 A Critical State Vector

Definition 3.1 *Supposing $A \subset C$ to be a subset of components of a system (Ω_C, S, φ), for an increasing subset $V \subset S$ such that $V \neq \phi$ and $V^c \neq \phi$, a state vector $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$ stisfying the following condition is called a $(A; V)$-critical state vector :*

$$\exists \boldsymbol{a} \in \Omega_A \text{ and } \exists \boldsymbol{b} \in \Omega_A, \quad \varphi(\boldsymbol{a}_A, \boldsymbol{x}_{A^c}) \in V, \; \varphi(\boldsymbol{b}_A, \boldsymbol{x}_{A^c}) \in V^c, \tag{7}$$

which is a directly extended version of the critical state vector for binary-state systems[3]. The set of all the $(A; V)$-critical state vectors is written as $Cr(A; V)$. ■

Theorem 3.1 *For a system (Ω_C, S, φ), a state vector $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$ is an $(A; V)$-critical state vector if and only if*

$$\exists \text{ an increasing subset } U \subseteqq \Omega_A \text{ s.t. } U \neq \phi \text{ and } U^c \neq \phi, \quad \varphi(U_A, \boldsymbol{x}_{A^c}) \subseteqq V, \; \varphi(U_A^c, \boldsymbol{x}_{A^c}) \subseteqq V^c, \tag{8}$$

where $U^c = \Omega_A \backslash U$. U is called a contributing set at an $(A; V)$-critical state vector $\boldsymbol{x}_{A^c}$, and is actually $U = \varphi_{\boldsymbol{x}_{A^c}}^{-1}(V)$. Then, the contributing set at the critical state vector $\boldsymbol{x}_{A^c}$ is uniquely determined, if it exists.

Another necessary and sufficient condition is the pair of the following conditions:

$$\varphi_{\boldsymbol{x}_{A^c}}^{-1}(V) \neq \phi \quad \text{and} \quad \varphi_{\boldsymbol{x}_{A^c}}^{-1}(V^c) \neq \phi.$$

Proof : *Setting $U = \varphi_{\boldsymbol{x}_{A^c}}^{-1}(V)$, the theorem is clear.* ■

We use a notation $Cr(A, U; V)$ to denote the set of all the critical state vectors having a contribution set $U \subseteqq \Omega_A$, i.e.,

$$Cr(A, U; V) = \{ \, \boldsymbol{x}_{A^c} \mid U = \varphi_{\boldsymbol{x}_{A^c}}^{-1}(V) \, \}, \tag{9}$$

then we have clearly the set of all the $(A; V)$-critical state vectors as

$$Cr(A; V) = \bigcup_{U:\text{an increasing subset of } \Omega_A} Cr(A, U; V), \tag{10}$$

of which union is a disjoint union with respect to the increasing subset U.

In the sequel, we show a characterization of a critical state vector by $MI\left(\varphi^{-1}(V)\right)$ and $MA\left(\varphi^{-1}(V^c)\right)$, which plays a crucial role for constructing an algorithm to derive the critical state vectors.

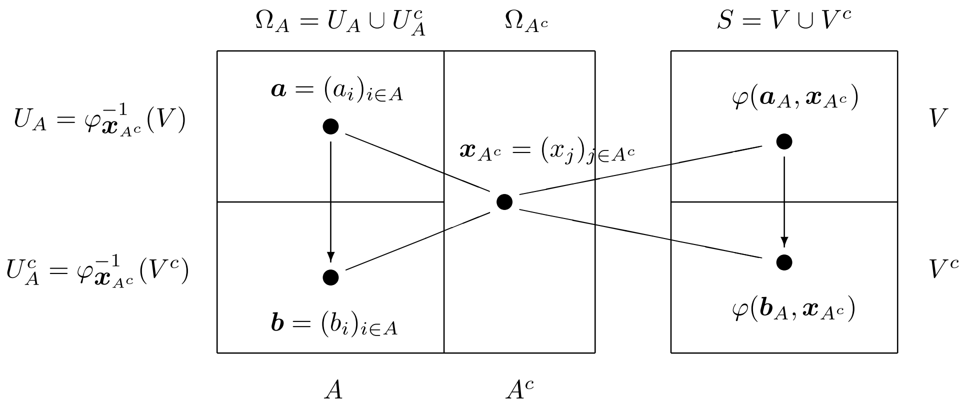

Figure 1: $(\cdot_i, \boldsymbol{x}_{A^c})$ is a (A, V)-critical state vector.

Following Theorem3.1, we examine the ordered set theoretical structure of $Cr(A;V)$. For $\boldsymbol{a} \in MI(U)$ and $\boldsymbol{b} \in MA(U^c)$, we have

$$\varphi(\boldsymbol{a}, \boldsymbol{x}_{A^c}) \in V, \quad \varphi(\boldsymbol{b}, \boldsymbol{x}_{A^c}) \in V^c, \tag{11}$$

and then by the property of a finite ordered set,

$$\exists \boldsymbol{m} \in MI(\varphi^{-1}(V)), \qquad \boldsymbol{m} \leqq (\boldsymbol{a}, \boldsymbol{x}_{A^c}), \tag{12}$$
$$\exists \boldsymbol{M} \in MA(\varphi^{-1}(V^c)), \qquad (\boldsymbol{b}, \boldsymbol{x}_{A^c}) \leqq \boldsymbol{M}. \tag{13}$$

Hence, the following inequalities hold :

$$\boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c}, \quad \boldsymbol{m}_A \leqq \boldsymbol{a}, \quad \boldsymbol{b} \leqq \boldsymbol{M}_A, \tag{14}$$

from which noting the following inequalities,

$$\boldsymbol{m} \leqq (\boldsymbol{m}_A, \boldsymbol{x}_{A^c}) \leqq (\boldsymbol{a}, \boldsymbol{x}_{A^c}), \quad \varphi(\boldsymbol{m}) \leqq \varphi(\boldsymbol{m}_A, \boldsymbol{x}_{A^c}) \leqq \varphi(\boldsymbol{a}, \boldsymbol{x}_{A^c}),$$

$\varphi(\boldsymbol{m}_A, \boldsymbol{x}_{A^c}) \in V$ holds, since V is an increasing set and $\varphi(\boldsymbol{m}) \in V$. Thus $\boldsymbol{m}_A \in U$, and then by the minimal property of $\boldsymbol{a}$, i.e., $\boldsymbol{a} \in MI(U)$, $\boldsymbol{m}_A = \boldsymbol{a}$ holds by the second inequality of (14). The equality, $\boldsymbol{M}_A = \boldsymbol{b}$, similarly holds. Arranging these examinations, we have the following theorem :

Theorem 3.2 *$\boldsymbol{x}_{A^c}$ is an $(A;V)$-critical state vector if and only if*

$$\forall \boldsymbol{a} \in MI\left(\varphi^{-1}_{\boldsymbol{x}_{A^c}}(V)\right) \text{ and } \forall \boldsymbol{b} \in MA\left(\varphi^{-1}_{\boldsymbol{x}_{A^c}}(V^c)\right), \tag{15}$$
$$\exists \boldsymbol{m} \in MI\left(\varphi^{-1}(V)\right) \text{ and } \exists \boldsymbol{M} \in MA\left(\varphi^{-1}(V^c)\right) \text{ such that } \boldsymbol{m}_{A^c} \leqq \boldsymbol{M}_{A^c}, \tag{16}$$
$$\boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c},\ \boldsymbol{m}_A = \boldsymbol{a},\ \boldsymbol{M}_A = \boldsymbol{b}. \tag{17}$$

For $\boldsymbol{m}$ and $\boldsymbol{M}$ of (16), $\boldsymbol{m}_A \not\leqq \boldsymbol{M}_A$ is derived from $\boldsymbol{m}_{A^c} \leqq \boldsymbol{M}_{A^c}$, and furthermore,

$$\boldsymbol{m}_A \in MI\left(\varphi^{-1}_{\boldsymbol{x}_{A^c}}(V)\right),\ \boldsymbol{M}_A \in MA\left(\varphi^{-1}_{\boldsymbol{x}_{A^c}}(V^c)\right),$$

of which belonging relations are derived from that $\boldsymbol{x}_{A^c}$ is an $(A;V)$-critical state vector, i.e., the essential relation is (17) from which we may construct a rough algorithm to derive $Cr(A;V)$. ■

Theorem 3.3 *$Cr(A;V)$ is given by using the following formulae:*

$$\mathcal{P}(A;V) = \left\{(\boldsymbol{m}, \boldsymbol{M}) \,\middle|\, \boldsymbol{m} \in MI\left(\varphi^{-1}(V)\right),\ \boldsymbol{M} \in MA\left(\varphi^{-1}(V^c)\right),\ \boldsymbol{m}_{A^c} \leqq \boldsymbol{M}_{A^c}\right\}, \tag{18}$$
$$Cr(A;V) = \bigcup_{(\boldsymbol{m},\ \boldsymbol{M}) \in \mathcal{P}(A;V)} [\boldsymbol{m}_{A^c}, \boldsymbol{M}_{A^c}]. \tag{19}$$

Proof : *From Theorem 3.2, $Cr(A;V)$ is clearly a subset of the right hand side of (19).*

Assuming that $\boldsymbol{x}_{A^c}$ is an element of the right hand side of (19), there exist $\boldsymbol{m} \in MI(\varphi^{-1}(1))$ and $\boldsymbol{M} \in MA(\varphi^{-1}(0))$ such that $\boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c}$. Then, $\boldsymbol{m} \leqq (\boldsymbol{m}_A, \boldsymbol{x}_{A^c})$ and $(\boldsymbol{M}_A, \boldsymbol{x}_{A^c}) \leqq \boldsymbol{M}$ hold and satisfy the condition of the Definition 3.1, we finally have $\boldsymbol{x}_{A^c} \in Cr(A;V)$. ■

Definition 3.2 *Supposing A to be a group of components of a system (Ω_C, S, φ) and $\boldsymbol{P}$ to be a probability on Ω_C, for an increasing subset $V \subsetneqq S$, $\boldsymbol{P}_{C\setminus A}(Cr(A;V))$ is called an $(A;V)$-importance measure of the group A with respect to V and a straight extention of the Birnbaum importance measure for a binary-state system to a multi-state and group case.* ■

From the above examination, $(A;V)$-importance measure for a system (Ω_C, S, φ) are given by the following procedure:

Step 1 : The calculation of $\mathcal{P}(A;V)$, the formula (18).

Step 2 : The calculation of $Cr(A;V)$, the formula (19).

Step 3 : The calculation of $\boldsymbol{P}_{C\setminus A}(Cr(A;V))$ by, for example, the inclusion and exclusion method.

It is, of course, an issue to develop effective calculation methods at each step of the above rough procedure.

Stochastic bounds for the Birnbaum importance measures of a component are introduced in [24].

Using an element $(\boldsymbol{m}, \boldsymbol{M})$ of $\mathcal{P}$ in a manner looser than the Birnbaum case, extended criticality and Fussell-Vesely importance measures are defined. In this paper, we present definitions of them for a group of components of a multi-state system and stochastic bounds for them.

3.2 An Alternative Definition of Birnbaum Importance Measure

Following the original definition of the Birnbaum importance measure for a binary-state system[4], which is defined to be a difference between two kinds of conditional expectations, we try to define $(A, U; V)$-importance measure for a group of components of a system (Ω_C, S, φ) by using conditional probabilities as follows:

$$Pr\{\varphi(\boldsymbol{X}_A, \boldsymbol{X}_{A^c}) \in V \mid \boldsymbol{X}_A \in U\} - Pr\{\varphi(\boldsymbol{X}_A, \boldsymbol{X}_{A^c}) \in V \mid \boldsymbol{X}_A \in U^c\}. \tag{20}$$

Pr is a probability on a probability space on which the random vector $(X_1, \cdots, X_n)$ is defined and each X_i is valued in Ω_i $(i \in C)$.

When the system is a binary-state system, the state spaces are $\{0, 1\}$ and a meaningful increasing subset is uniquely $\{1\}$. Then setting $A = \{i\}$ and $U = \{1\}$, we have the following formulation from (20)

$$\begin{aligned} & Pr\{\varphi(1_i, \boldsymbol{X}_{C\setminus\{i\}}) = 1 \mid X_i = 1\} - Pr\{\varphi(0_i, \boldsymbol{X}_{C\setminus\{i\}}) = 1 \mid X_i = 0\} \\ = \; & \mathbf{E}[\varphi(1_i, \boldsymbol{X}_{C\setminus\{i\}}) \mid X_i = 1] - \mathbf{E}[\varphi(0_i, \boldsymbol{X}_{C\setminus\{i\}}) \mid X_i = 0], \end{aligned}$$

which is the original Birnbaum importance measure[4] of a component, and also when X_i $(i \in C)$ are stochastically independent,

$$\text{the above formula} \;=\; Pr\{\boldsymbol{X}_{C\setminus\{i\}} \in Cr(\{i\}, \{1\}; \{1\})\}$$

The formula (20) implies the Definition 3.2 for a binary-state system, but not for a multi-state system, so to say, (20) and $\boldsymbol{P}(Cr(A, U; V))$ are not directly related with each other. The precise examination of this formulation (20) is remained for the future work.

4 Criticality and Fussel-Vesely Importance Measures

4.1 Critical State Vectors for the Criticality Importance Measure

Critical state vectors defining criticality importance measures are given as follows :

$$UCI(A;V) = \{\, \boldsymbol{x} \mid \exists(\boldsymbol{m}, \boldsymbol{M}) \in \mathcal{P}(A;V),\ \boldsymbol{m}_A \leqq \boldsymbol{x}_A,\ \boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c}\}, \tag{21}$$

$$LCI(A;V) = \{\, \boldsymbol{x} \mid \exists(\boldsymbol{m}, \boldsymbol{M}) \in \mathcal{P}(A;V),\ \boldsymbol{x}_A \leqq \boldsymbol{M}_A,\ \boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c}\}. \tag{22}$$

$UCI(A;V)$ and $LCI(A;V)$ may be written as the following formula :

$$\begin{aligned} UCI(A;V) &= \bigcup_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)} [\boldsymbol{m}_A, \rightarrow) \times [\boldsymbol{m}_{A^c}, \boldsymbol{M}_{A^c}], & (23) \\ LCI(A;V) &= \bigcup_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)} (\leftarrow, \boldsymbol{M}_A] \times [\boldsymbol{m}_{A^c}, \boldsymbol{M}_{A^c}]. & (24) \end{aligned}$$

4.2 Criticality Importance Measure

Supposing $\boldsymbol{P}$ to be a probability on Ω_C, two kinds of criticality importance measures are defined as the following conditional probabilities :

$$\begin{aligned} \boldsymbol{P}\{\ UCI(A;V) \mid \varphi^{-1}(V)\ \} &= \frac{\boldsymbol{P}\{\ UCI(A;V)\ \}}{\boldsymbol{P}\{\ \varphi^{-1}(V)\ \}}, & (25) \\ \boldsymbol{P}\{\ LCI(A;V) \mid \varphi^{-1}(V^c)\ \} &= \frac{\boldsymbol{P}\{\ LCI(A;V)\ \}}{\boldsymbol{P}\{\ \varphi^{-1}(V^c)\ \}}. & (26) \end{aligned}$$

Noting $UCI(A;V) \subseteqq \Omega_A \times Cr(A;V)$, we have the following relations among the Birnbaum and criticality importance measures :

$$\boldsymbol{P}(UCI(A;V)) \leqq \boldsymbol{P}_{A^c}(Cr(A;V)), \quad \boldsymbol{P}(LCI(A;V)) \leqq \boldsymbol{P}_{A^c}(Cr(A;V)) \ .$$

4.3 Stochastic Bounds for a Criticality Importance Measure

At the formula (23), setting

$$\begin{aligned} A_i &= [\boldsymbol{m}_A, \rightarrow) \subseteqq \Omega_A, \quad \text{an increasing subset,} \\ B_i &= [\boldsymbol{m}_{A^c}, \boldsymbol{M}_{A^c}] \subseteqq \Omega_{A^c}, \\ I_i &= [\boldsymbol{m}_{A^c}, \rightarrow) \subseteqq \Omega_{A^c}, \quad \text{an increasing subset,} \\ D_i &= (\leftarrow, \boldsymbol{M}_{A^c}] \subseteqq \Omega_{A^c}, \quad \text{a decreasing subset,} \end{aligned}$$

then

$$B_i = I_i \cap D_i, \quad \text{an intersection of an increasing and a decreasing set,}$$

$UCI(A;V)$ may be written as the following :

$$UCI(A;V) = \bigcup_i A_i \times B_i = \bigcup_i A_i \times (I_i \cap D_i) \qquad (27)$$

and $A_i \times B_i$ is written as

$$\begin{aligned} A_i \times B_i &= (A_i \times \Omega_{A^c}) \cap (\Omega_A \times B_i) \\ &= (A_i \times \Omega_{A^c}) \cap (\Omega_A \times (I_i \cap D_i)) \\ &= (A_i \times \Omega_{A^c}) \cap (\Omega_A \times I_i) \cap (\Omega_A \times D_i) & (28) \\ &= [\boldsymbol{m}, \rightarrow) \cap (\Omega_A \times D_i), & (29) \end{aligned}$$

and then (29) is also an intersection of an increasing and decreasing set.

When $\boldsymbol{P}$ on Ω_C is associated, taking the probability of (29), we have

$$\begin{aligned} \boldsymbol{P}(A_i \times B_i) &\leqq \boldsymbol{P}[\boldsymbol{m}, \rightarrow) \cdot \boldsymbol{P}(\Omega_A \times D_i) \\ &= \boldsymbol{P}[\boldsymbol{m}, \rightarrow) \cdot \boldsymbol{P}_{A^c}(D_i) \\ &= \boldsymbol{P}[\boldsymbol{m}, \rightarrow) \cdot \boldsymbol{P}_{A^c}(\leftarrow, \boldsymbol{M}_{A^c}] & (30) \end{aligned}$$

and then

$$\begin{aligned}\boldsymbol{P}(UCI(A;V)) &\leqq \sum_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)} \boldsymbol{P}\Big([\boldsymbol{m}_A,\rightarrow)\times[\boldsymbol{m}_{A^c},\boldsymbol{M}_{A^c}]\Big) \\ &\leqq \sum_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)} \boldsymbol{P}[\boldsymbol{m},\rightarrow)\cdot\boldsymbol{P}_{A^c}(\leftarrow,\boldsymbol{M}_{A^c}]. \qquad (31)\end{aligned}$$

When $\boldsymbol{P}$ on Ω_C is the product probability of $\boldsymbol{P}_i$ on $\Omega_i,\ (i\in C)$, i.e., the components are stochastically independent, from (28), we have

$$\boldsymbol{P}(A_i\times B_i) = \boldsymbol{P}_A(A_i)\cdot\boldsymbol{P}_{A^c}(B_i), \qquad (32)$$

furthermore, when $\boldsymbol{P}_i\ (i\in C)$ are associated,

$$\begin{aligned} &\leqq \boldsymbol{P}_A(A_i)\cdot\boldsymbol{P}_{A^c}(I_i)\cdot\boldsymbol{P}_{A^c}(D_i) \qquad (33)\\ &= \boldsymbol{P}[\boldsymbol{m},\rightarrow)\cdot\boldsymbol{P}_{A^c}(\leftarrow,\boldsymbol{M}_{A^c}], \qquad (34)\end{aligned}$$

which is the stochastic bound same to the (31). Anyway, the critical state vectors and the stochastic bounds are determined by $MI\left(\varphi^{-1}(V)\right)$ and $MA\left(\varphi^{-1}(V^c)\right)$.

We notice that every probability on a totally ordered set is automatically associated. And then, when the state spaces of the components are totally ordered sets, the probabilities used for the reliability theoretical examinations are associated.

4.4 A Note on an Exact Calculation of Criticality Importance Measure by the Inclusion and Exclusion Principle

We here give a note for an exact calculation of (28) by the inclusion and exclusion principle.

$$\begin{aligned}&[\boldsymbol{m}^1_A,\rightarrow)\times[\boldsymbol{m}^1_{A^c},\boldsymbol{M}^1_{A^c}]\bigcap[\boldsymbol{m}^2_A,\rightarrow)\times[\boldsymbol{m}^2_{A^c},\boldsymbol{M}^2_{A^c}] \qquad (35)\\ &= [\sup\{\boldsymbol{m}^1_A,\ \boldsymbol{m}^2_A\},\ \rightarrow)\times[\sup\{\boldsymbol{m}^1_{A^c},\ \boldsymbol{m}^2_{A^c}\},\ \inf\{\boldsymbol{M}^1_{A^c},\boldsymbol{M}^2_{A^c}\}], \qquad (36)\end{aligned}$$

then, we may consider a non-empty condition similar to the case of the Birnbaum importance measure[24].

4.5 Fussell-Vesely Importance Measure

Definition 4.1 *For a system (Ω_A,S,φ), a group of components $A\subset C$ and an increasing subset $V\subset S$, Fussel-Vesely critical state vectors are defined as the following four types :*

$$\begin{aligned}FVU(A;V) &= \{\ \boldsymbol{x}\mid\exists(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V),\ \boldsymbol{m}\leqq\boldsymbol{x}\ \} = \bigcup_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)}[\boldsymbol{m},\rightarrow), \qquad (37)\\ FVUA(A;V) &= \{\ (\cdot_A,\boldsymbol{x}_{A^c})\mid\exists(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V),\ \boldsymbol{m}\leqq\boldsymbol{x}\ \} = \bigcup_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)}[\boldsymbol{m}_{A^c},\rightarrow),\\ FVL(A;V) &= \{\ \boldsymbol{x}\mid\exists(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V),\ \boldsymbol{x}\leqq\boldsymbol{M}\ \},\\ FVLA(A;V) &= \{\ (\cdot_A,\boldsymbol{x}_{A^c})\mid\exists(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V),\ \boldsymbol{x}\leqq\boldsymbol{M}\ \},\end{aligned}$$

each element of which is respectively called as

Fussell-Vesely upper critical state vector,
Fussell-Vesely upper alternative critical state vector,
Fussell-Vesely lower critical state vector,
Fussell-Vesely lower alternative critical state vector.

The conditional probabilities of the above events given as the following are generically called as the Fussell-Vesely importance measure or precisely as group multi-state Fussell-Vesely importance measure.

$$\boldsymbol{P}\left(\ FVU(A;V)\mid\varphi^{-1}(V)\ \right) = \frac{\boldsymbol{P}(FVU(A;V))}{\boldsymbol{P}\left(\varphi^{-1}(V)\right)}, \tag{38}$$

$$\boldsymbol{P}_{A^c}\left(\ FVUA(A;V)\mid\{(\cdot_A,\boldsymbol{x}_{A^c})|\boldsymbol{x}\in\varphi^{-1}(V)\}\ \right) = \frac{\boldsymbol{P}_{A^c}(FVUA(A;V))}{\boldsymbol{P}_{A^c}\{(\cdot_A,\boldsymbol{x}_{A^c})|\boldsymbol{x}\in\varphi^{-1}(V)\}},$$

$$\boldsymbol{P}\left(\ FVL(A;V)\mid\varphi^{-1}(V^c)\ \right) = \frac{\boldsymbol{P}(FVL(A;V))}{\boldsymbol{P}\left(\varphi^{-1}(V^c)\right)},$$

$$\boldsymbol{P}_{A^c}\left(\ FVLA(A;V)\mid\{(\cdot_A,\boldsymbol{x}_{A^c})|\boldsymbol{x}\in\varphi^{-1}(V^c)\}\ \right) = \frac{\boldsymbol{P}_{A^c}(FVLA(A;V))}{\boldsymbol{P}_{A^c}\{(\cdot_A,\boldsymbol{x}_{A^c})|\boldsymbol{x}\in\varphi^{-1}(V^c)\}}. \qquad \blacksquare$$

Following the examination similar to the case of the criticality importance measure, we may have a stochastic bound for the $FVU(A;V)$-importance measure (38) as the following : supposing $\boldsymbol{P}$ to be associated, noticing that $[\boldsymbol{m},\rightarrow)^c$ is a decreasing set,

$$\boldsymbol{P}\left(\bigcup_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)}[\boldsymbol{m},\rightarrow)\right) = 1-\boldsymbol{P}\left(\bigcap_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)}[\boldsymbol{m},\rightarrow)^c\right) \leq 1-\prod_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)}\boldsymbol{P}([\boldsymbol{m},\rightarrow)^c),$$

$$\boldsymbol{P}\left(\ FVU(A;V)\mid\varphi^{-1}(V)\ \right) = \frac{\boldsymbol{P}(FVU(A;V))}{\boldsymbol{P}\left(\varphi^{-1}(V)\right)} \leqq \frac{1}{\boldsymbol{P}(\varphi^{-1}(V))}\left\{1-\prod_{(\boldsymbol{m},\boldsymbol{M})\in\mathcal{P}(A;V)}\boldsymbol{P}([\boldsymbol{m},\rightarrow)^c)\right\}.$$

We may have similar stochastic bounds for other Fussell-Vesely importance measures.

For a binary-state system, $FVU(A;V)$-importance measures are reduced to the usual Fusselle-Vesely importance measures.

5 Acknowledgment

In this paper we have organized the works about Birnbaum importance measure for a group of components in a multi-state system and have proposed definitions of the criticality and Fussell-Vesely importance measure for a group of components.

In this paper, we focused on defining the importance measures and giving the stochastic bounds. It is remained for the future work to explain a relationship among these importance measures, practical calculations of the importance measures for series and parallel multi-state systems and how they are calculated through a modular decomposition.

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