Modified calculations and visualizations for skipped, weighted, or right-upward typed Gibonacci sequences using related Pascal triangles and these matrices

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1. Introduction

The relation between Pascal's triangle [1,2], Newton's binomial coefficient [3], and Fibonacci sequence [4-6] are well known historically over the globe. The summations of right-upward diagonals show the mathematical beauty of Fibonacci sequence on Pascal's triangle [7,8]. In this paper, the same or similar summations on modified Pascal's triangles are gotten to create various sequences using the related Pascal matrices with several ideas [9-13]. To distinguish various sequences analytically, we would like to use skipped and weighted sequences instead of calling generalized sequences. If we use the initial conditions as 0 and 1, we would like to call the weighted Fibonacci sequence with the weights a and b in this paper. If we apply the other initial conditions of g_0 and g_1, we would like to name them the weighted Gibonacci sequences [4-6,14,15] with the weights a and b in the same way. We understand that Padovan sequence should be calculated to use the summations of knight moving diagonals on Pascal's triangle in the same concept [16,17]. We would like to emphasize that as 1 skipped sequence in this paper. If we consider n-Pell [18,19] or n-Jacobsthal sequences [20,21] as 1 skipped sequence such as Padovan sequence or Perrin sequence [22-24], it is effective to compute with related and modified Pascal triangles and these matrices to get that systematically. Based on this thinking, we extend models such as the work of the paper [17] and as k-skipped sequences using matrices forms. In addition to that, we can clarify the reverse of the sequence as the descending order of that using some mathematical techniques simply. Moreover, we deal with changing the initial conditions of Padovan or Perrin sequences in the same manner. we can also find some ratios such as super-golden ratio ($n = 1, j = 2$) [25], super-silver ratio ($n = 2, j = 2$) [26], and plastic ratio ($n = 1, j = 4$) [27] based on $x^{j+1} = n \cdot x^j - 1$ are much more useful than we thought to create k-upward sequences on modified Pascal's triangles [28].

2. Visualization for weighted Gibonacci sequence using modified Pascal matrices

Generally, if we use the Pascal matrix in Figure 1, it is well known to get the Fibonacci sequence [4-6]. On the other hand, if we modify the Pascal matrix to display the related sequences, it has not been systematized to distinguish types of various sequences. Now, we suppose the $(2l \times 2l)$ type of Pascal matrix [29-31] as follows.

$$
\exp \boldsymbol{A} = \begin{pmatrix}
1 &&&&&&&&&&& \\
-5 & 1 &&&&&&&&&& \\
10 & -4 & 1 &&&&&& \boldsymbol{0} &&& \\
-10 & 6 & -3 & 1 &&&&&&&& \\
5 & -4 & 3 & -2 & 1 &&&&&&& \\
-1 & 1 & -1 & 1 & -1 & 1 &&&&&& \\
&&&&&& 1 &&&&& \\
&&&&&& 1 & 1 &&&& \\
&&& \boldsymbol{0} &&& 1 & 2 & 1 &&& \\
&&&&&& 1 & 3 & 3 & 1 && \\
&&&&&& 1 & 4 & 6 & 4 & 1 & \\
&&&&&& 1 & 5 & 10 & 10 & 5 & 1
\end{pmatrix} = \prod_{j=0}^{\infty} \frac{1}{j!} \boldsymbol{A}^j
$$

$$
\text{based on } \boldsymbol{A} = \begin{pmatrix}
0 &&&&&&&&&&& \\
-5 & 0 &&&&&&&&&& \\
& -4 & 0 &&&&&& \boldsymbol{0} &&& \\
&& -3 & 0 &&&&&&&& \\
&&& -2 & 0 &&&&&&& \\
&&&& -1 & 0 &&&&&& \\
&&&&& 0 & 0 &&&&& \\
&&&&&& 1 & 0 &&&& \\
&&&&&&& 2 & 0 &&& \\
&& \boldsymbol{0} &&&&&& 3 & 0 && \\
&&&&&&&&& 4 & 0 & \\
&&&&&&&&&& 5 & 0
\end{pmatrix} \tag{2.1}
$$

If we estimate the summations of right upward diagonals, we can get the Fibonacci sequence. In the same way, if we use the matrix $\exp a \cdot \boldsymbol{A}$, we can get n-Pell sequence based on $n = a \in \mathbb{N}$. Moreover, if we apply the diagonal matrix [12]

$$\operatorname{diag}[b^h] = \begin{pmatrix} b^{-l} & 0 & & \cdots & & 0 \\ 0 & \ddots & \ddots & & & \\ & \ddots & b^{-1} & 0 & & \vdots \\ \vdots & & 0 & b^0 & \ddots & \\ & & & \ddots & \ddots & 0 \\ 0 & & \cdots & & 0 & b^{l-1} \end{pmatrix}, \qquad (h = -l, \cdots, -1, 0, 1, \cdots, l-1 \in \mathbb{Z}) \tag{2.2}$$

to the Pascal matrix such as $(\exp \boldsymbol{A}) diag[b^h]$ based on $b \in \mathbb{N}$, we can also calculate n-Jacobsthal sequence from right upward diagonals. Therefore, we can estimate the weighted Fibonacci sequence

$$F_{(a,b),0} = 0, \qquad F_{(a,b),1} = 1, \qquad F_{(a,b),j} = a \cdot F_{(a,b),j-1} + b \cdot F_{(a,b),j-2} \quad (j \geq 2). \tag{2.3}$$

using the summations of right upward diagonals of $(\exp a \cdot \boldsymbol{A})\operatorname{diag}[b^h]$. In the same manner, We think of weighted Gibonacci sequence included n-Pell or n-Jacosthal Lucas sequences and Mulatu sequence as

$$G_{(a,b),0} = g_0, \qquad G_{(a,b),1} = g_1, \qquad G_{(a,b),j} = a \cdot G_{(a,b),j-1} + b \cdot G_{(a,b),j-2} \quad (j \geq 2). \tag{2.4}$$

To get the Equation (2.4), we need to use the right upward diagonals of the matrices $\left(\exp(F_{(a,b),k+2} \cdot \boldsymbol{A})\right)\operatorname{diag}[(b \cdot F_{(a,b),k+1})^h]\boldsymbol{BG}$ using the band matrix [9-13]

$$\boldsymbol{B} = \begin{pmatrix} F_{(a,b),1} & \cdots & F_{(a,b),k+1} & 0 & \cdots & 0 \\ 0 & F_{(a,b),1} & \cdots & F_{(a,b),k+1} & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & F_{(a,b),1} & \ddots & F_{(a,b),k+1} \\ & & & & \ddots & \vdots \\ 0 & & \cdots & & 0 & F_{(a,b),1} \end{pmatrix}, \qquad \boldsymbol{G} = \begin{pmatrix} g_j & g_{j-1} & 0 & \cdots & & 0 \\ 0 & g_j & g_{j-1} & \ddots & & \vdots \\ & & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & g_{j-1} \\ 0 & & \cdots & & 0 & g_j \end{pmatrix}. \tag{2.5}$$

In addition to that, we formulate k-skipped weighted Gibonacci sequence [9-13]

$$\begin{gathered} G_{(a,b),0} = g_0, G_{(a,b),1} = g_1, G_{(a,b),2} = g_2 = a \cdot g_1 + b \cdot g_0, \cdots, G_{(a,b),k+2} = g_{k+2} = a \cdot g_{k+1} + b \cdot g_k, \\ G_{(a,b),j} = F_{(a,b),k+2} \cdot G_{(a,b),j-k-1} + b \cdot F_{(a,b),k+1} \cdot G_{(a,b),j-k-2} \quad (j \geq k+3). \end{gathered} \tag{2.6}$$

based on $k \in \mathbb{N}$ and illustrated in Figure 2. This is how $k(\geq 1)$ means the number of skips for k-skipped sequences. Based on this rule, we can create Lucas sequence, when $a = b = 1, g_0 = 2, g_{-1} = -1$ and put the various initial conditions of weighted Gibonacci sequence using Equation (2.5).

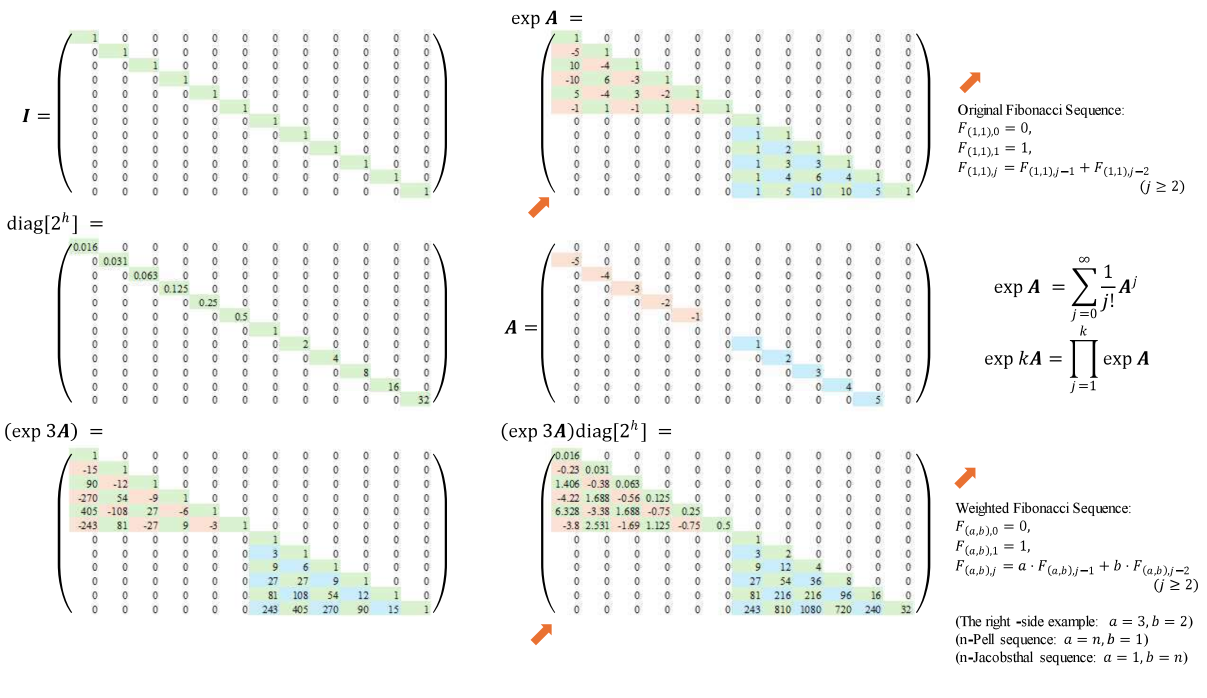

Figure 1 Exponential matrices for Pascal's triangle with negative orders and some modified Pascal matrices for various sequences.

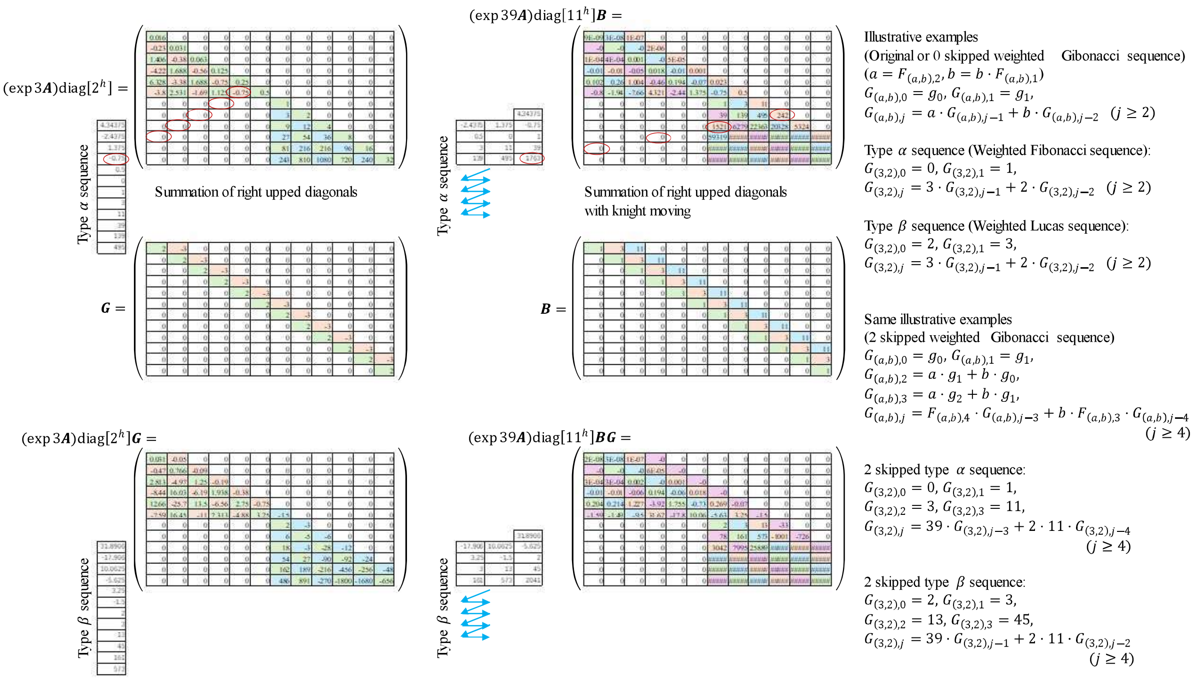

Figure 2 Concepts and illustrative examples of k skipped sequences from original or 0 skipped sequences.

Type 1

0	1	0	0	0	0	0	0	0
1	a	b	0	0	0	0	0	0
2	a^2	$2ab$	b^2	0	0	0	0	0
3	a^3	$3a^2b$	$3ab^2$	b^3	0	0	0	0
4	a^4	$4a^3b$	$6a^2b^2$	$4ab^3$	b^4	0	0	0
5	a^5	$5a^4b$	$10a^3b^2$	$10a^2b^3$	$5ab^4$	b^5	0	0
6	a^6	$6a^5b$	$15a^4b^2$	$20a^3b^3$	$15a^2b^4$	$6ab^5$	b^6	0
7	a^7	$7a^6b$	$21a^5b^2$	$35a^4b^3$	$35a^3b^4$	$21a^2b^5$	$7ab^6$	b^7

Type 2a

0	0	a	b	$\times b$
1	a	b	0	$\times a$
2	$a \cdot a$	$2ab$	b^2	
3	$a \cdot (a^2 + b)$	$2a^2b + b^2$	ab^2	
4	$a \cdot (a^3 + 2ab)$	$2a^3b + 3ab^2$	$b^2 \cdot (a^2 + b)$	
5	$a \cdot (a^4 + 3a^2b + b^2)$	$2a^4b + 5a^2b^2 + b^3$	$b^2 \cdot (a^3 + 2ab)$	

Type 3

0	0	0	a^2	$2ab$	b^2
1	0	a^2	$2ab$	b^2	0
2	a^2	$2ab$	b^2	0	0

Type 4

0	0	0	a^2	$2ab$	b^2
0	0	a	b	0	0
0	1	0	0	0	0

Type 2b

1	0	a^2	$2ab$	b^2	$\times b$
2	a^2	$2ab$	b^2	0	$\times a$
3	$a^2 \cdot a$	$3a^2b$	$3ab^2$	b^3	
4	$a^2 \cdot (a^2 + b)$	$3a^3b + 2ab^2$	$3a^2b^2 + b^3$	ab^3	
5	$a^2 \cdot (a^3 + 2ab)$	$3a^4b + 5a^2b^2$	$3a^3b^2 + 4ab^3$	$b^3 \cdot (a^2 + b)$	
6	$a^2 \cdot (a^4 + 3a^2b + b^2)$	$3a^5b + 8a^3b^2 + 2ab^3$	$3a^4b^2 + 7a^2b^3 + b^4$	$b^3 \cdot (a^3 + 2ab)$	

Insert the set from type 1 to 4, and so on into diagonals of the band metrices to change the initial conditions of weighted Gibonacci sequences.

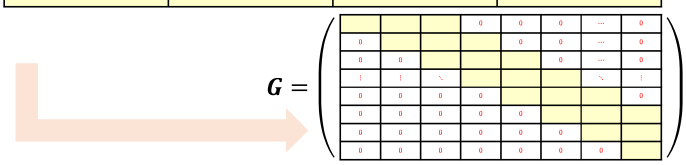

Figure 3 Illustrative examples for changing initial conditions to move smoothly using band matrices $\boldsymbol{G}$.

3. Changing initial conditions for weighted Gibonacci sequence using modified Pascal matrices

In section 2, we suggest that it is effective to get skipped weighted Gibonacci sequences using $\left(\exp(F_{(a,b),k+2} \cdot \boldsymbol{A})\right)\mathrm{diag}[(b \cdot F_{(a,b),k+1})^h]\boldsymbol{BG}$. In this section, we would like to consider how to change the initial conditions smoothly. We would like to devise illustrative diagrams of the changing conditions shown in Figure 3 concretely. If we apply the various types of diagrams effectively in Figure 3 to the modified pascal matrices, we can make the flows of sequences change naturally.

In figure 3, there are several categories for changing the initial conditions. We can confirm the expanded Pascal's triangle using the weights a and b on Type 1. On the left side of the Type 1 diagram, we can see the moving location numbers of the sequences. In the same way, on the left side of the Type 2 diagram, we can also admit the moving location numbers of the sequences. When we confirm the left side number 1 and 2 on Type 1 or 1 and 2 on Type 2 in Figure 3, we can admit those numbers as the moving location numbers. If we add the parts of left side number 0 times b and the left side number 1 times a equals to the left side number 2 using weighted digit shifts, we can get the solution as the parts of left side 2. This is how we can compute the changing initial conditions using Gibonacci sequences systematically. On the other hand, if we use the condition on Type 3, we can get the reverse moving locations of the sequence. Therefore, if we utilize Type 1 or Type 2 and Type 3 in Figure 3 at the same time effectively such as Type 4 in Figure 3, we can change the initial conditions arbitrary or systematically.

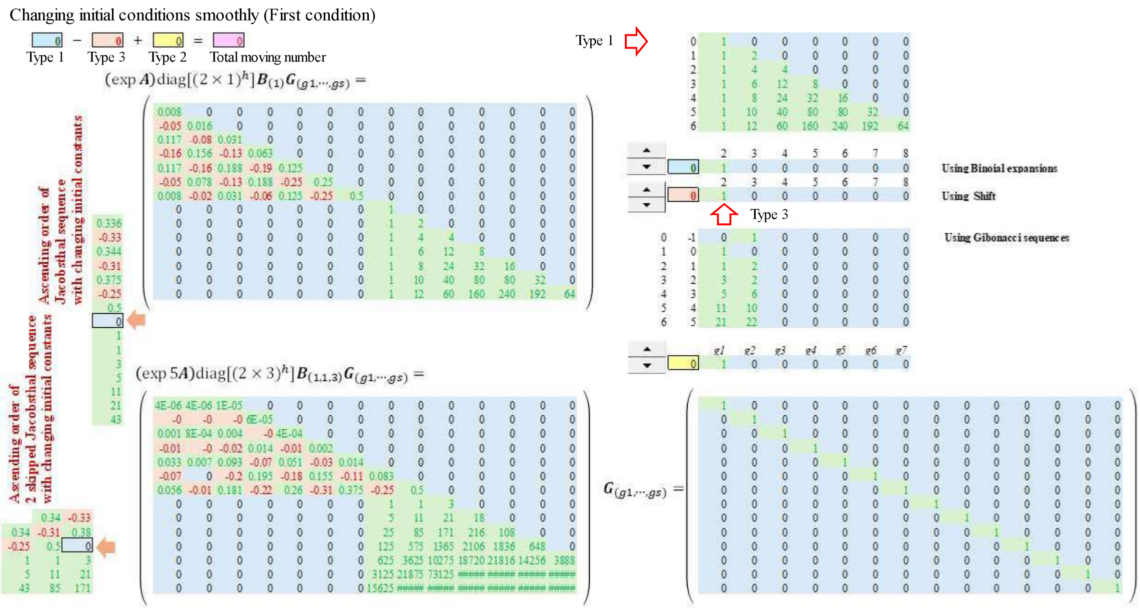

Figure 4 Illustrative examples for firstly initial condition before moving the location of Jacobsthal sequence.

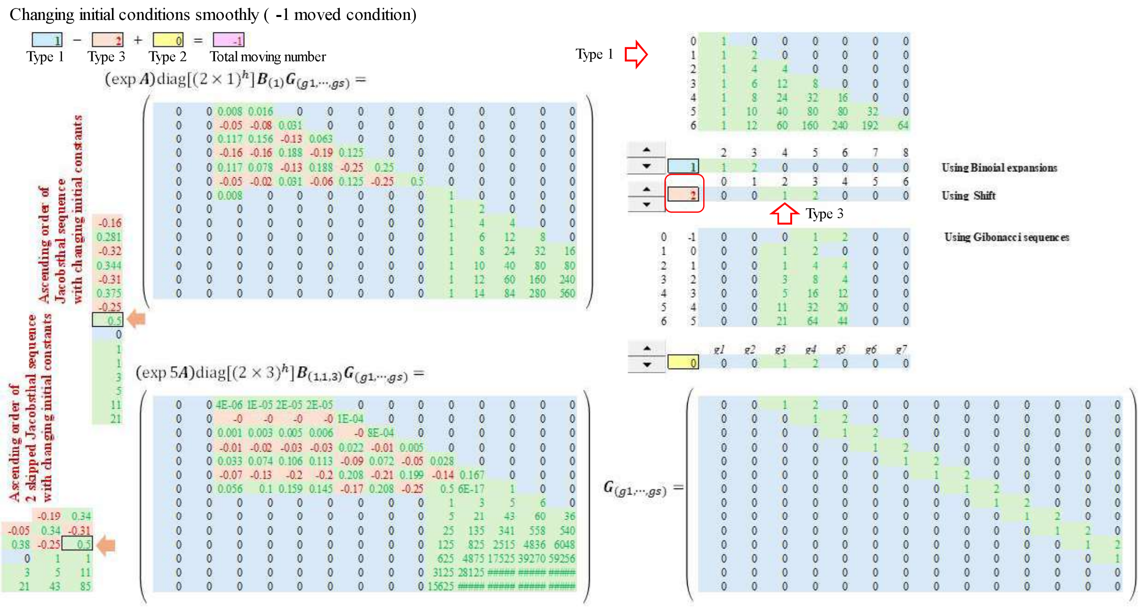

Figure 5 Illustrative examples for changing the initial condition after moving the location of Jacobsthal sequence.

Based on the rules of Figure 3, we can display illustrative examples before moving the condition in Figure 4 and after locating in Figure 5 using Jacobsthal sequence visually. In Figure 5, we confirm that Type1 and Type 3 give the moving points of the sequences precisely.

4. Visualizations for descending orders for skipped weighted Gibonacci sequence using modified Pascal matrices

In section 2, we mentioned that it is effective to create skipped weighted Gibonacci sequences using $\left(\exp(F_{(a,b),k+2}\cdot \boldsymbol{A})\right)\mathrm{diag}[(b\cdot F_{(a,b),k+1})^{h}]\boldsymbol{BG}$. In this section, we would like to consider visualizations of reverse orders of the sequences. As shown in Figure 6, if we simply estimate the matrices forms such as $\left(\exp(-F_{(a,b),k+2}\cdot \boldsymbol{A})\right)^{\mathrm{T}}\mathrm{diag}[(b\cdot F_{(a,b),k+1})^{-h}]\boldsymbol{BG}/(b\cdot F_{(a,b),k+1})$ to create reverse order of $\left(\exp(F_{(a,b),k+2}\cdot \boldsymbol{A})\right)\mathrm{diag}[(b\cdot F_{(a,b),k+1})^{h}]\boldsymbol{BG}$, we can get the descending orders of the sequences above mentioned with some revisions of size.

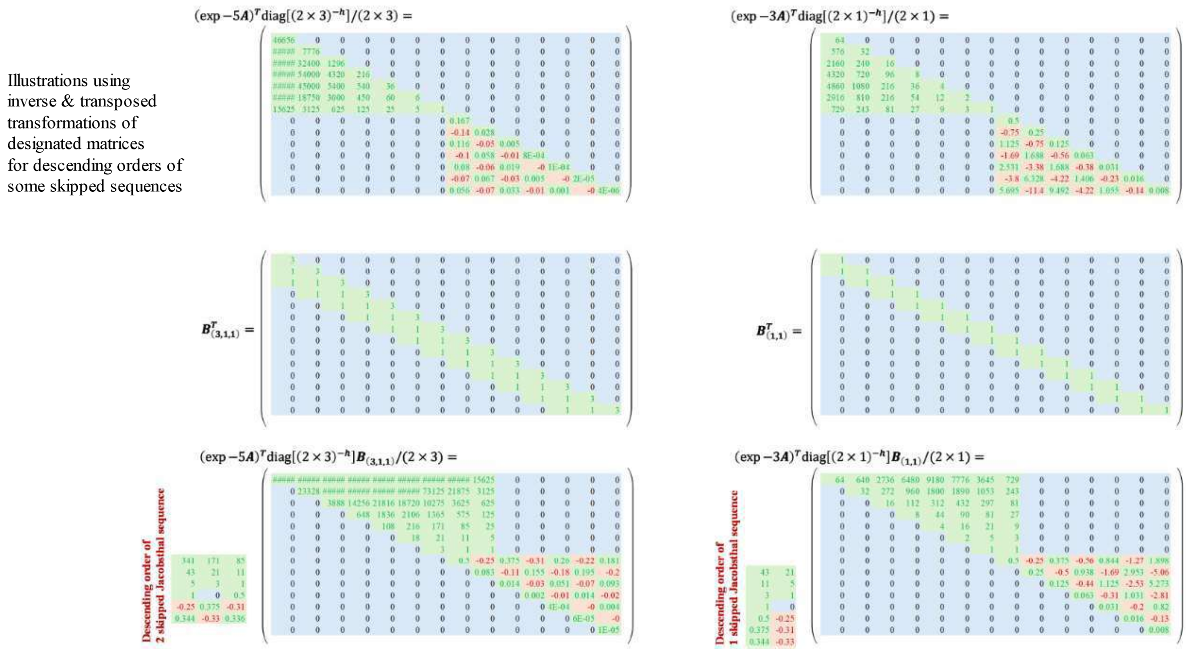

Figure 6 Illustrative examples of descending orders for 1 and 2 skipped Jacobthal sequences from matrices forms.

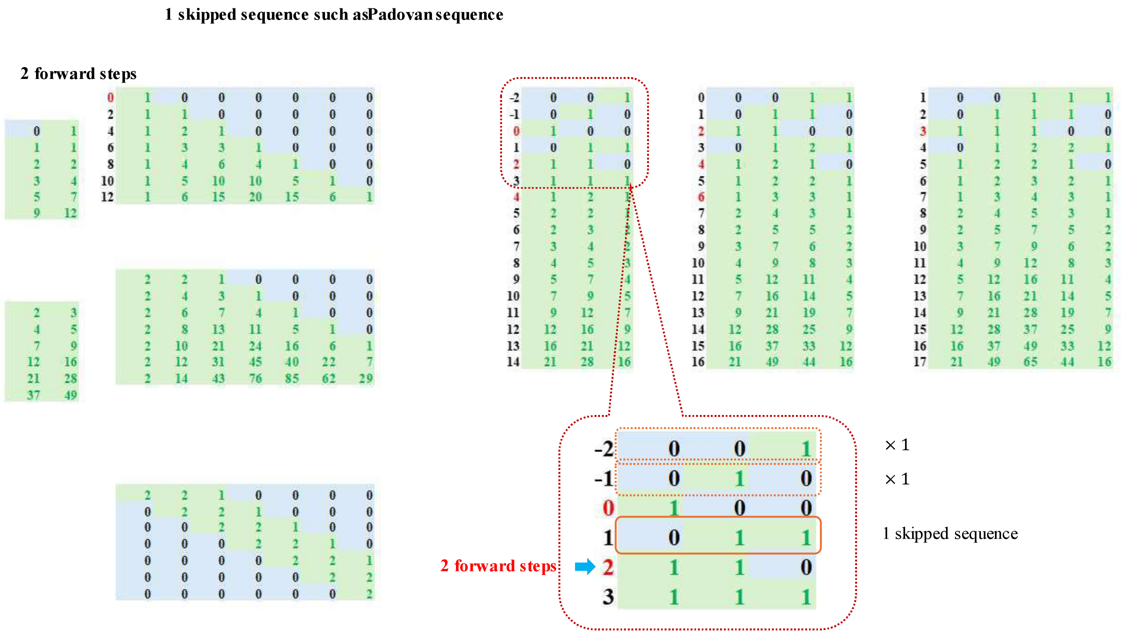

Figure 7 Concepts for changing the initial conditions of Padovan sequence using band matrices $\boldsymbol{G}$.

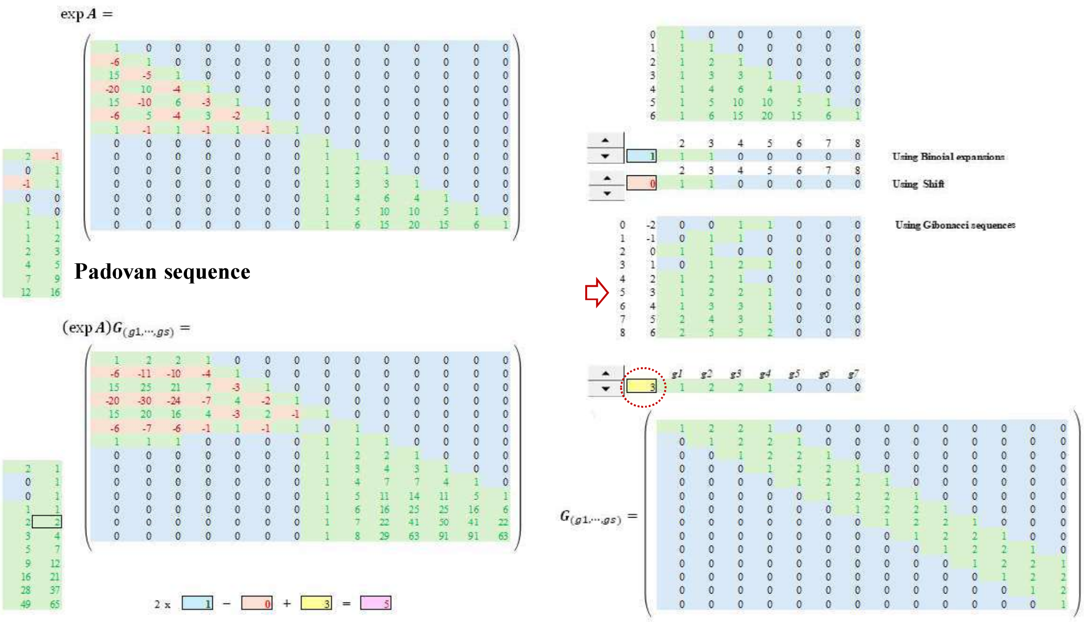

Figure 8 Illustrative examples for changing the initial conditions of Padovan sequence applying band matrices $\boldsymbol{G}$.

If you would like to use part of negative order of the sequences on modified Pascal's triangles systematically, I am happy if you search for my ideas [12,13] that we suggested some solving techniques before this presentation.

5. Visualizations and changing initial conditions for original 1 skipped sequence using modified Pascal matrices

In this section, we would like to propose how to create the initial conditions for some 1 skipped sequences such as Padovan sequence [22]

$$P_{(1,1),0}^{(1,1,1)} = 1, P_{(1,1),2}^{(1,1,1)} = 1, P_{(1,1),2}^{(1,1,1)} = 1,$$
$$P_{(1,1),j}^{(1,1,1)} = P_{(1,1),j-2}^{(1,1,1)} + P_{(1,1),j-3}^{(1,1,1)} \quad (j \geq 3) \tag{5.1}$$

or Perrin sequence [23,24]

$$P_{(1,1),0}^{(3,0,2)} = 3, P_{(1,1),2}^{(3,0,2)} = 0, P_{(1,1),2}^{(3,0,2)} = 2,$$
$$P_{(1,1),j}^{(3,0,2)} = P_{(1,1),j-2}^{(3,0,2)} + P_{(1,1),j-3}^{(3,0,2)} \quad (j \geq 3) \tag{5.2}$$

using proper matrices $\boldsymbol{G}$ in the same way. In Figure 7, we can confirm that the principles of 1 skipped sequence according to the moving numbers of the left side of the diagrams should be gotten systematically. If we apply one of them into the band matrix $\boldsymbol{G}$ properly, we can change the initial conditions precisely based on these numbers. Therefore, we can understand that the layers of Pascal's triangle show 2 forward steps theoretically. That is why 1 skipped sequence should be gotten from Pascal's triangle and related calculations.

In Figure 8, we can illustrate the visualized examples for changing the initial conditions of Padovan sequence based on the rules of Figure 7 in practice. In Figure 9, we can display the visualized examples for changing the initial conditions of Perrin sequence based on the same concept of Figure 7.

6. Visualizations for k-upward typed sequence using modified Pascal matrices

In this section, we would like to suggest how to create upward typed sequences [28] shown in Figure 10 as follows.

$$P_{(n,1),0}^{(1,\cdots,1)} = 1, \cdots, P_{(n,1),k}^{(1,\cdots,1)} = 1,$$
$$P_{(n,1),j}^{(1,\cdots,1)} = n \cdot P_{(n,1),j-1}^{(1,\cdots,1)} + P_{(n,1),j-k}^{(1,\cdots,1)} \quad (j \geq k+1, k = 1,2,\cdots) \tag{6.1}$$

should be related to $x^{k+1} = n \cdot x^k - 1$ mathematically. Based on Figures 3 and 7, we can also get the changing initial conditions of these sequences in Equation (6.1) and move that systematically. However, we should take care of the upward steps k to reverse locations in Figure 11. Frome the characterizations of these figures, we can understand the changing initial conditions have same concepts shown in Figures 3 and 7 with some local rules.

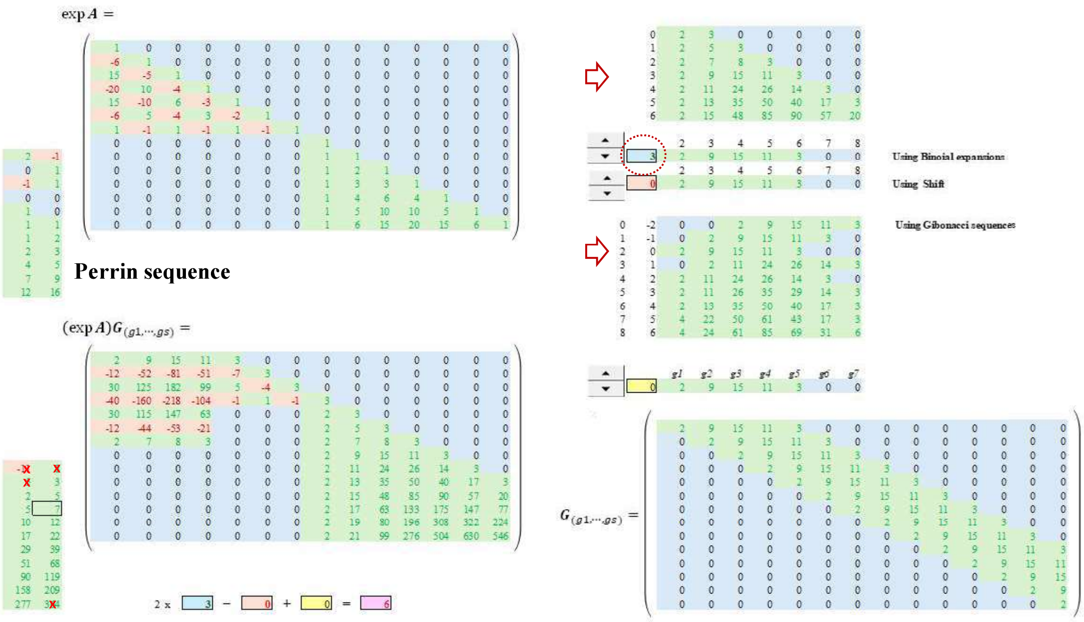

Figure 9 Illustrative examples for changing the initial conditions of Perrin sequence applying band matrices $\boldsymbol{G}$.

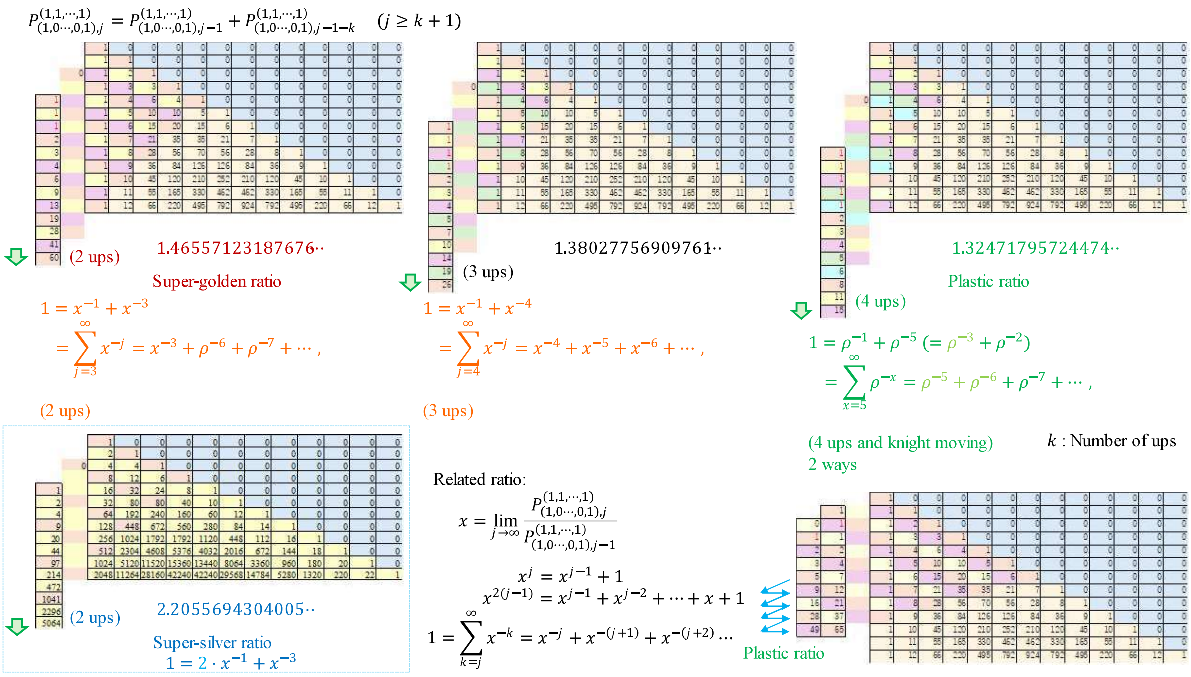

Figure 10 Illustrative examples of 2, 3, and 4 upward typed sequences on related Pascal's triangles and these ratios.

It is known that the ratio x are called super-golden ratio ($k = 2, n = 1$) [25], super-silver ratio ($k = 2, n = 2$) [26], plastic ratio ($k = 4, n = 1$) [27] shown in Figure 10 and so on. We can especially find that the plastic ratio on Pascal's triangle is gotten by using 2 ways. One is mentioned in Figure 8 and knight moving on Pascal's triangle as you can see. The other is 4 upward typed sequence on Pancal's triangle shown in Figure 10 in this section. Therefore, the plastic ratio should be one of special ratios on Pascal's triangle. Based on this concept shown in Figure 10, we can also change the initial conditions to move the sequences smoothly in Figure 11. We can confirm that

$$x = \sum_{j=k}^{\infty} x^{-j} \quad (\because x^{k+1} = x^k + 1) \tag{6.2}$$

if the ratio $x^{k+1} = x^k + 1$. From the geometrical viewpoint, the ratio x is much more useful to create regular hexagon spirals as the extended ratios shown in Figures 12 and 13 [32,33]. Moreover, if we describe the designated sequence as one skipped sequence, the first terms shown as 1, 2, and 3 are equal to the orders of single, double, and triple types of hexagon spirals or equilateral triangle spirals coincidentally. The relation of golden ratio and regular hexagons be confirmed on the website [34].

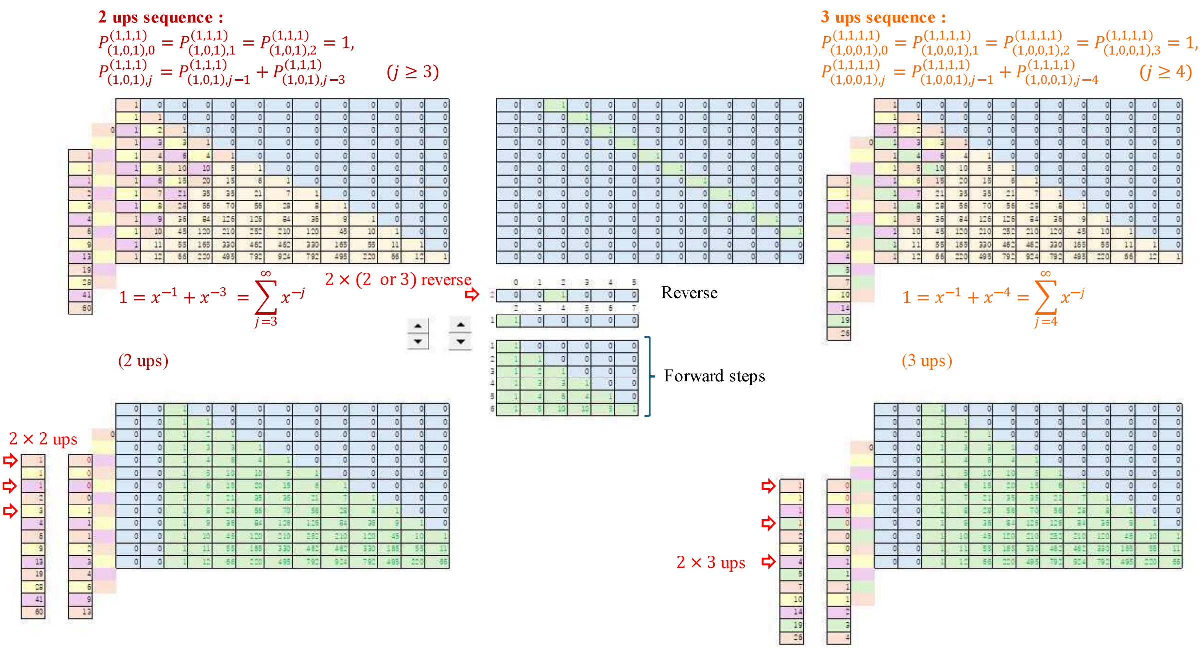

Figure 11 Illustrative examples for changing the initial conditions of 2 and 3 upward typed sequences using band matrices.

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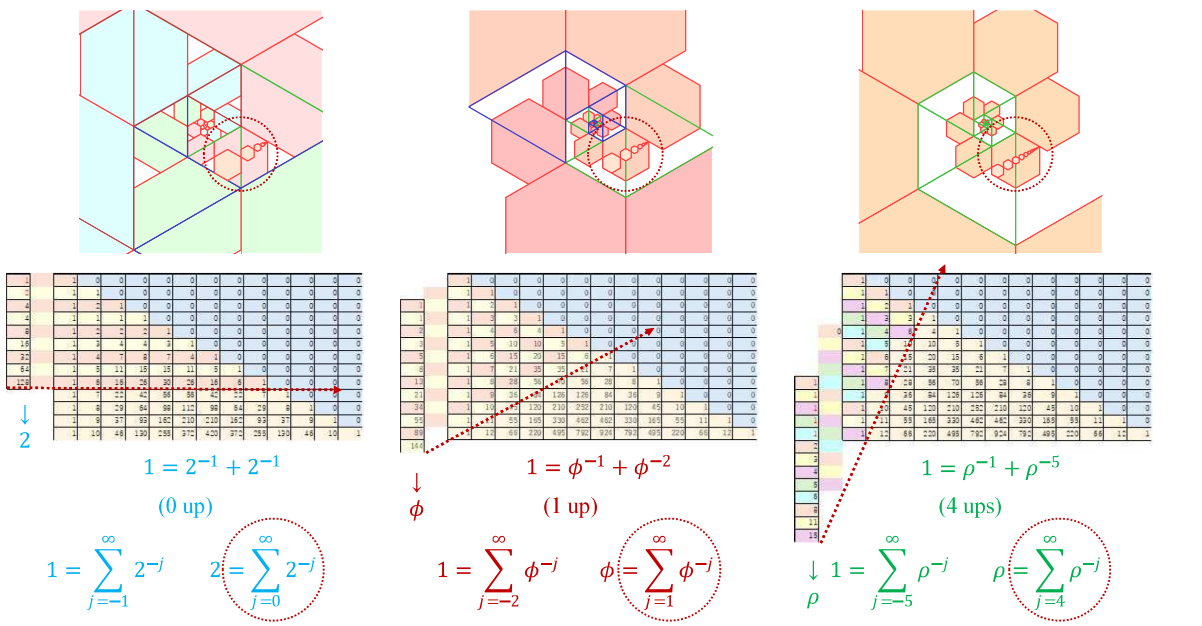

Figure 12 Illustrative examples for right upward typed sequences of 0, 1, and 4 upward typed sequences on Pascal's triangle and related single, double, and triple regular hexagon spirals and these series (Part 1) [32].

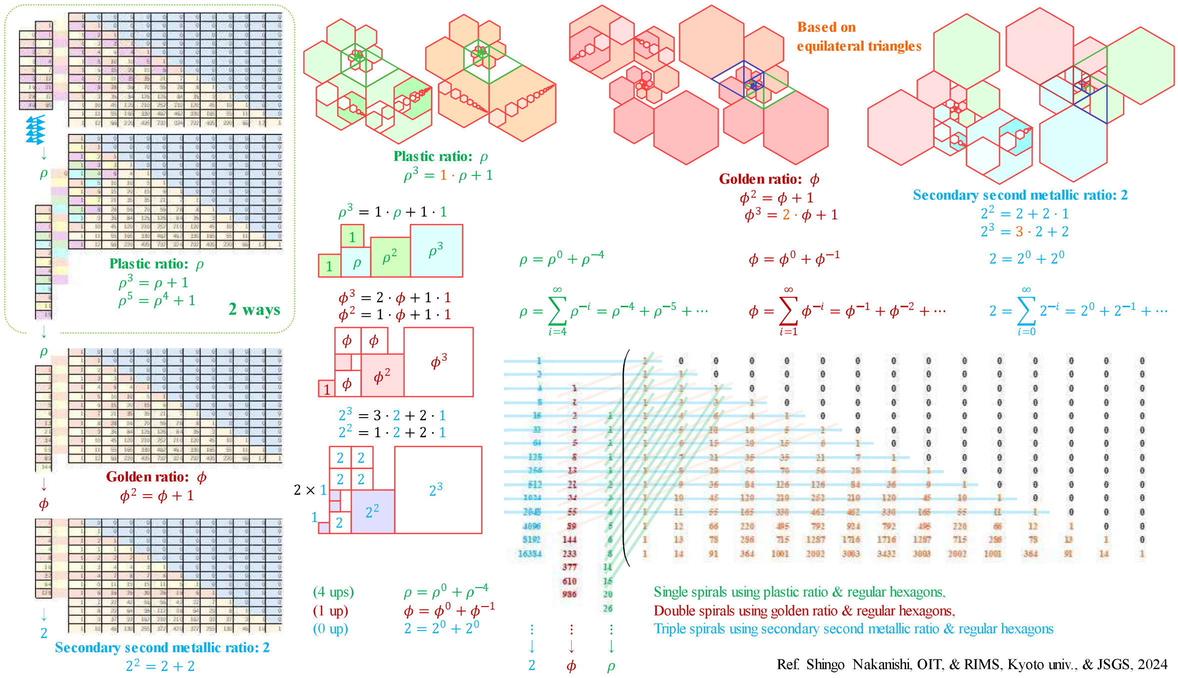

Figure 13 Illustrative examples for right upward typed sequences of 0, 1, and 4 upward typed sequences on Pascal's triangle and related single, double, and triple regular hexagon spirals and these series (Part 2) [32].

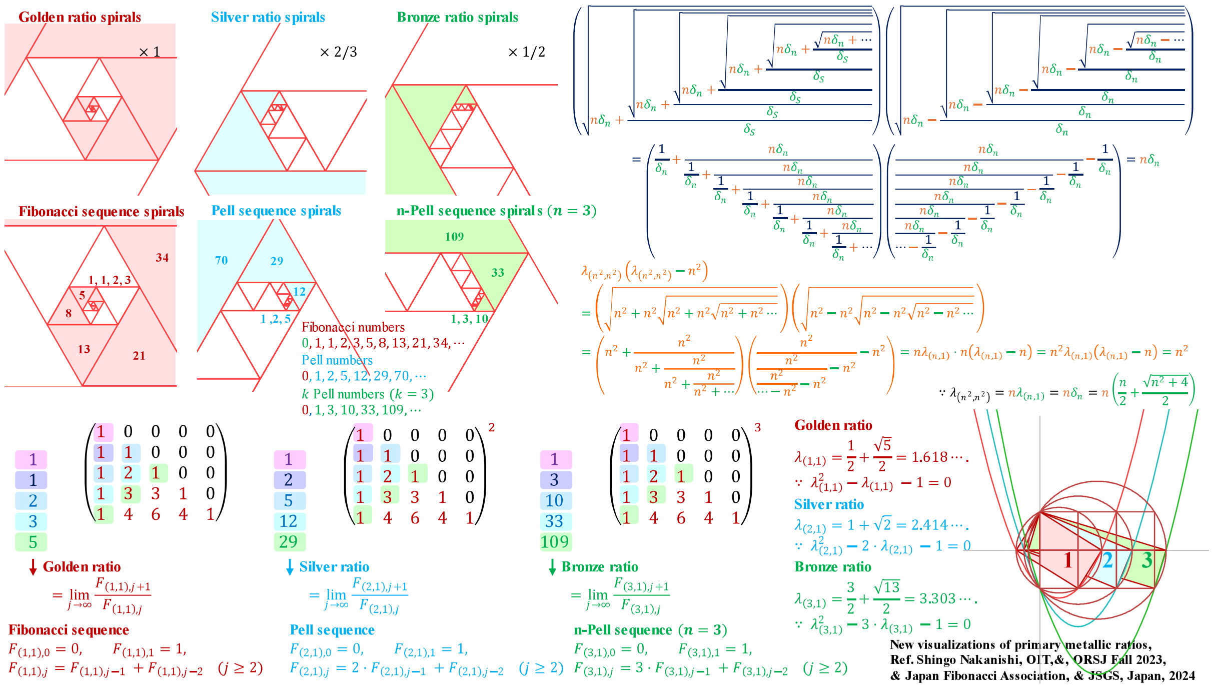

Figure 14 Illustrative examples for double spirals of equilateral triangles as golden ratio, silver ratio, bronze ratio of primary metallic ratios and these characterizations on modified Pascal's triangles [32,33,35].

In Figure 14, we can illustrate the new visualization for primary metallic ratios such as golden ratio, silver ratio, and bronze ratio related Fibonacci sequence, Pell sequence, and 3-Pell sequence respectively using double spirals of equilateral triangles and these visualizations on modified Pascal's triangles [35]. From this figure, we can understand the meaning of geometric characterizations of primary metallic ratios clearly. In addition to that, we can also visualize the extended continued fractions and nested radicals for primary metallic ratios as new concepts in Figure 14.

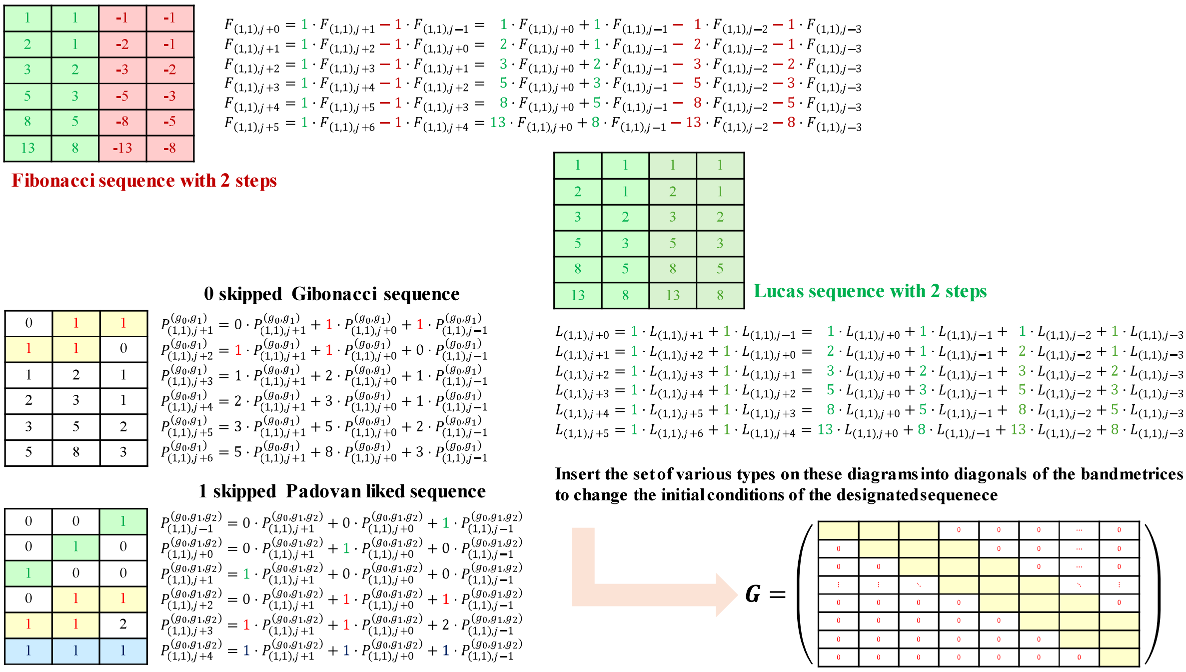

Figure 15 Illustrative examples for the set (1,1), (1,1,1), and (1,1,1,1) insert into the diagonals of the band matrices $\boldsymbol{G}$.

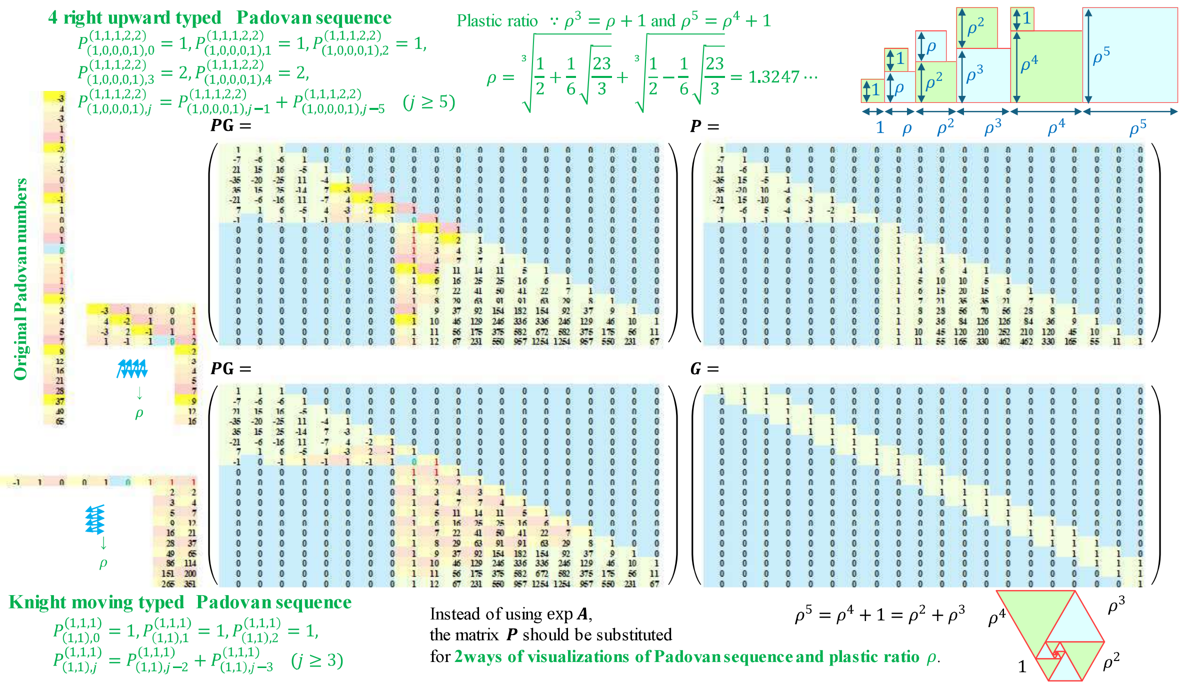

Figure 16 2 ways of visualizations for Padovan sequence and plastic ratio on modified Pascal's triangles.

In this paper, we investigate the visualizations and these changing initial conditions of various types of sequences using the proper designated band matrices. We would like to discuss some special cases to clarify the principles precisely. That is, we can show that the cases of Fibonacci or Lucas sequences with 2 steps [14] in Figure 15. To use the set of one of these combinations brings us the beauty of smoothly changing initial conditions as you can understand. In the same way, we also display the various Gibonacci sequences, and 1 skipped Padovan liked sequences. These sets such as $(1,1)$, $(1,1,1)$, and $(1,1,1,1)$ to insert into the diagonals of the matrices $\boldsymbol{G}$ in Figure 15 show the strict characterizations concretely. If you are interested in the 2 steps of Gibonacci sequence, I am happy to confirm the RIMS paper by IWAMOTO and KIMURA [14].

Finally, we would like to display the 2 ways of visualizations of Padavan sequence and plastic ratio on the related modified Pascal's triangles at the same time in Figure 16 to understand my ideas [13,36,37] throughout this study clearly. I would be so happy if you imagine two types of beautiful plastic ratios like Fibonacci and Lucas sequences on modified Pascal's triangles in Figure 17 [4-8] as one of my original works with thinkings of great mathematical history.

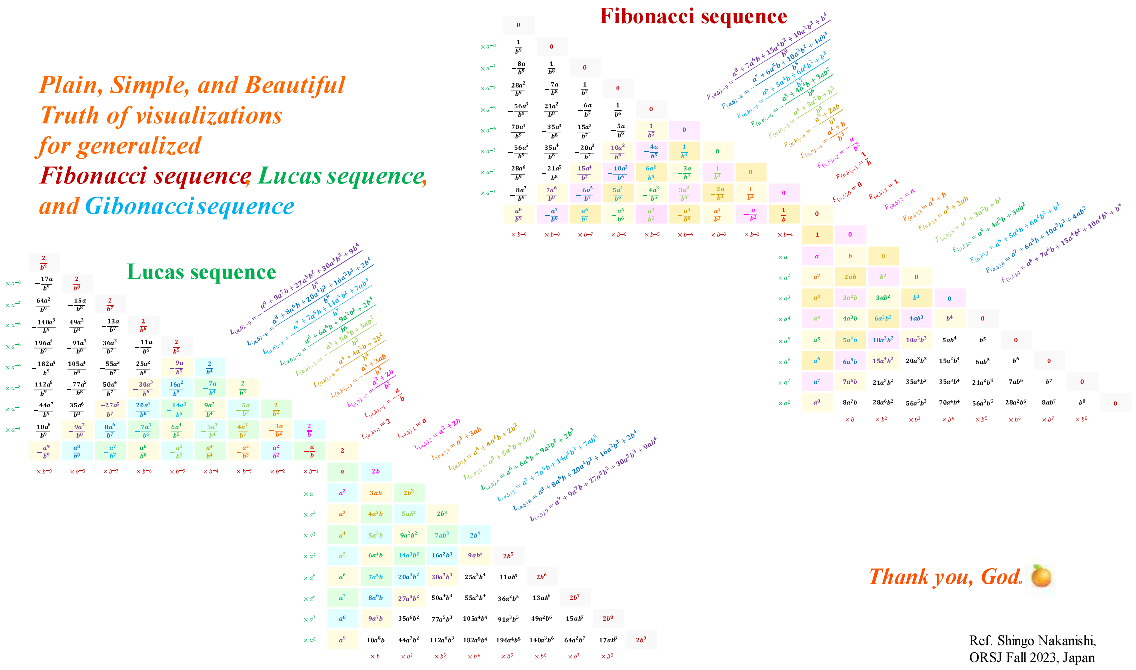

Figure 17 Artistic Illustrative examples for weighted Fibonacci sequence and Lucas sequence on modified Pascal's triangles [9,37].

7. Conclusions

In this paper, we dealt with the visualizations for various types of sequences on modified Pascal's triangles or using the related Pacal matrices systematically. In addition to that, we can devise the changing initial conditions to move the sequences on Pascal's triangle naturally, smoothly and visually.

First, we can show that the skipped or weighted sequences instead of using generalizations of Fibonacci or Gibonacci sequences to confirm the characterizations visually on modified Pascal's triangles. To distinguish weighted and skipped sequences make us investigate the characterizations on modified Pascal's triangles by changing initial conditions effectively. At the same time, we understand that the principles of changing initial conditions connect to the original characterizations of Gibonacci sequences mathematically. These ideas come from the addition theorems of Gibonacci sequences.

Second, we can treat original 1-skipped sequences such as Padovan or Perrin sequences in the same way. Therefore, we can find that the principles also mean the original's 1-skipped sequences to change the initial conditions systematically.

Third, we can clarify that the increasing ratios of the right upward typed sequences on Pascal's triangle are equal to x, which means $x^{k+1} = x^k + 1$, and extended ratios of related regular hexagon spirals such as single, double, and triple regular hexagon spirals with equilateral triangle spirals mathematically. Moreover, we can confirm that the number of right upwards is equal to k on Pascal's triangle. These preparations and changing conditions for visualizations are much more useful to introduce matrices forms such as Pascal matrices and the related several band matrices easily in this paper than we thought.

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References

[1] T. M. Green and C. L. "Hamberg: Pascal's triangle, 2nd edition", CreateSpace Independent Publishing Platform, (2012).
[2] T. M. Green, "The Simplex, duplex and Pascal's triangles: relatives of Pascal's triangle, with excursions into hyperspace. CreateSpace Independent Publishing Platform, (2015).
[3] "Binomial coefficient", Wikipedia, https://en.wikipedia.org/wiki/Binomial_coefficient, (last accessed on January 2, 2025).
[4] T. Koshy, "Fibonacci and Lucas numbers with applications", Volume 1 (2nd Edition), Wiley, (2014).
[5] T. Koshy: Fibonacci and Lucas numbers with applications, Volume 2 (2nd Edition), Wiley, (2018).
[6] S. Nakamura, "フィボナッチ数の小宇宙 ―フィボナッチ数，リュカ数，黄金分割―", 日本評論社, (2008) (in Japanese).
[7] G. B. Meisner and R. Araujo, "The golden ratio: the divine beauty of mathematics", Race Point Publishing, (2018).
[8] H. Hosoya, "ピタゴラスの三角形とその数理", 共立出版, (2011) (in Japanese).
[9] S. Nakanishi, "一般化されたフィボナッチ数列およびリュカ数列の修正パスカルの三角形による幾何学的可視化", Abstract of the Annual Fall Meeting of ORSJ, (2023), 144-145, (in Japanese).
[10] S. Nakanishi, "細矢の三角形の拡張図ならびに黄金比やプラスチック比を参考とするいくつかの数列の修正版パスカルの三角形と螺旋図による可視化", Abstract of the Annual Fall Meeting of ORSJ, (2023), 148-149, (in Japanese).
[11] S. Nakanishi, "修正パスカルの三角形の活用を含む貴金属比と関連する数列の数理とその可視化", Proceedings of the Annual Meeting of Japan Society for Graphic Science, (2023) 63-68 (in Japanese)
[12] S. Nakanishi, "修正パスカル三角形によるスキップ型フィボナッチ数列とリュカ数列の演算行列", Abstract of the Annual Spring Meeting of ORSJ, (2024), 174-175, (in Japanese).
[13] S. Nakanishi, "重み付き Gibonacci 数列のためのシフト演算", Abstract of the Annual Fall Meeting of ORSJ, (2024), 180-181, (in Japanese).
[14] S. Iwamoto and Y. Kimura: Gibonacci Optimization - duality -. (Mathematical Decision Making Under Uncertainty and Related, Topics), RIMS-Kôkyûroku, Kyoto University, 2242, (2023), 1-13.
[15] A. T. Benjamin and J. J. Quinn, "Proofs that really count: the art of combinatorial proof", The Mathematical Association of America, (2003).
[16] E. Wilson: "Scales of Mt. Meru", (1993), https://www.anaphoria.com/meruone.pdf (last accessed on January 11, 2024).
[17] A. Giuseppina, L. N´emeth, and G. Vincenzi, "Generalized Pascal's triangles and associated k-Padovan-like sequences", Mathematics and Computers in Simulation, 192, (2022), 278-290.
[18] H. H. Gulec and N. Taskara, "On the (s, t)-Pell and (s, t)-Pell–Lucas sequences and their matrix representations", Applied Mathematics Letters, 25, (2012), 1554-1559.
[19] A. Panwar, K. Sisodiya, and G.P.S. Rathore, "On the products of k-Pell number and k-Pell Lucas number", IOSR Journal of Mathematics, 13, 5-III, (2017), 85-87
[20] S. Uygun, "The (s t)-Jacobsthal and (s t)-Jacobsthal Lucas sequences", Applied Mathematical Sciences, 9, 70, (2015), 3467- 3476.
[21] S. Uygun and H. Eldogan, "k-Jacobsthal and k-Jacobsthal Lucas matrix sequences", International Mathematical Forum, 11, 3, (2016), 145 - 154.
[22] "Padovan sequence", https://en.wikipedia.org/wiki/Padovan sequence, (last accessed on January 2, 2025).
[23] "Perrin number", https://en.wikipedia.org/wiki/Perrin number, (last accessed on January 2, 2025).
[24] M. Yasuda, "Increasing product and Perrin sequence", https://www.math.s.chiba-u.ac.jp/~yasuda/ippansug/fibo/Perrin14U.pdf, (2016), (in Japanese) (last accessed on January 2, 2025).
[25] "Super-golden ratio", Wikipedia, https://en.wikipedia.org/wiki/Supergolden_ratio, (last accessed on January 2, 2025).

[26] “Super-silver ratio”, Wikipedia, https://en.wikipedia.org/wiki/Supersilver_ratio, (last accessed on January 2, 2025).
[27] “Plastic ratio”, Wikipedia, https://en.wikipedia.org/wiki/Plastic_ratio, (last accessed on January 2, 2025).
[28] T. M. Green, “Recurrent sequences and Pascal triangle”, Mathematics Magazine, 14-1, (1968), 13-21.
[29] “Pascal’s triangle”, Wikipedia, https://en.wikipedia.org/wiki/Pascal’s_triangle, (last accessed on January 2, 2025).
[30] G. S. Call and D. J. Velleman, “Pascal’s matrices”, The American Mathematical Monthly, 100, 4, (1993) 372-376.
[31] A. Edelman and G. Strang, “Pascal matrices”, The American Mathematical Monthly, 111, 3, (2004), 189-197.
[32] S. Nakanishi, “図余りや図足らずを認めた正三角形と正六角形の螺旋図”, Proceedings of the Annual Meeting of Japan Society for Graphic Science, (2024) 161-166, (in Japanese).
[33] S. Nakanisihi, “Spirals based on Equilateral Triangles about Various Related Padovan and Weighted Gibonacci Sequences”, Annual meeting of Japan Fibonacci association, https://www.oit.ac.jp/center/~nakanishi/study/equilateral_triangle_spirals_using_metallic_ratios_and%20related_sequences2024-08-23.pdf, (last accessed on January 4, 2025), (2024) (in Japanese).
[34] P. B. Krieg, “Hexagons and the Golden Ratio”, Playful bookbinding and paper works, https://bookzoompa.wordpress.com/2018/10/10/hexagons-and-the-golden-ratio/, (2018), (last accessed on January 5, 2025)
[35] S. Nakanishi, “黄金比とフィボナッチ数列とリュカ数列，白銀比とペル数列とペル・リュカ数列を基準とする図形による幾何学的可視化”, Abstract of the Annual Spring Meeting of ORSJ, (2023), 142-143, (in Japanese).
[36] S. Nakanishi, “図余りと図足らずを許した正三角形を連ねた螺旋図　ーパドヴァン数列とペラン数列に習ってー”, Kansai blanch annual meeting of the Japan Society for Graphic Science, 6 pages, (2024) (in Japanese).
[37] S. Nakanishi, “Modified Pascal’s triangles and matrices for flexibly changing initial conditions about weighted skipped Fibonacci, Lucas, Pell, Jacobsthal, or related sequences”, Handout paper of Evolution of Mathematical Decision-Making in Uncertain Environments of RIMS Workshop, Kyoto university, https://www.oit.ac.jp/center/~nakanishi/study/rims-handoout-2024-11-12.pdf, 26 pages, (2024) (last accessed on January 8, 2025).

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