Ghost Effect for a Steady Boltzmann Equation in Bounded Domain

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Abstract

We briefly report on the results in [26] and [25], where we discuss and prove the mathematical validity of the so-called "ghost effect" for the steady Boltzmann equation in a general 3D bounded domain, with a finite variation of the tangential wall temperature.

1 Formulation and Introduction

1.1 Introduction to Ghost Effect

One of the important applications of the Boltzmann theory is to derive hydrodynamic (fluid) equations as $\varepsilon \to 0$, where ε is a dimensionless number called the Knudsen number (the ratio between the mean free path ℓ and the characteristic macroscopic length). There has been explosive literature on such hydrodynamic limits both from physical and mathematical standpoints. Almost all of the fluid equations obtained from such derivations are compatible with standard fluid theory, and the role of such limits can be viewed as further justification rather than new discovery of fluid equations from the Boltzmann theory.

In particular, the diffusive hydrodynamic limit of the Boltzmann equation in the low Mach number regime is described by the incompressible Navier-Stokes-Fourier equations under the extra assumption that the initial density and temperature profiles (and on the boundaries) differ from constants at most for terms of the order of the Knudsen number.

When the density and temperature do not satisfy the above mentioned assumptions, the limiting behavior of the Boltzmann equation deviates from the Navier-Stokes-Fourier equations. Such a discrepancy, called "ghost effect" [64], shows up in the macroscopic equations with the presence of some extra terms reminiscent of the limiting procedure such as some heat flow induced by the vanishingly small velocity field.

A rare exception occurs in the study of the fundamental hydrodynamic limit for a steady gas in a motionless bounded domain Ω. In his seminal paper [59], J. C. Maxwell introduces his kinetic formulation of boundary conditions to investigate such a fundamental problem in physics, and proposes a slip boundary condition for a fluid flow, by assuming thermal equilibria (local Maxwellian) for the gas. In the case of diffuse-reflection Maxwell BC (with the accommodation coefficient equal to 1), his now famous slip boundary condition takes the form of

$$u_\iota - u_w \simeq G\left(\frac{\partial u_\iota}{\partial n} - \frac{3\lambda}{2\rho_w T_w}\frac{\partial^2 T}{\partial\iota\partial n}\right) + \frac{3\lambda}{4\rho_w T_w}\frac{\partial T_w}{\partial\iota}. \tag{1.1}$$

Here (ρ, u, T) is the fluid density, velocity and temperature with (ρ_w, u_w, T_w) being its counterparts at the wall, n is the normal direction and ι the tangential direction, and λ the viscosity and G the slip coefficient proportional to the mean free path ε. The second term is of order ε [63]. This boundary condition also predicts and explains that a dilute gas slips from colder to hotter regions (the thermal transpiration or thermal creep phenomenon) at the boundary. Maxwell's work has inspired further research in physics and engineering ever since, including constructions of thermal pumps.

In the case of a motionless boundary $u_w = 0$ with Maxwell diffuse BC, Maxwell's kinetic formulation leads to the following stationary Boltzmann equation in a bounded three-dimensional C^3 domain $\Omega \ni x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$:

$$v\cdot\nabla_x\mathfrak{F} = \varepsilon^{-1}Q\big[\mathfrak{F},\mathfrak{F}\big], \quad \mathfrak{F}\big|_{v\cdot n<0} = M_w\int_{\mathfrak{u}\cdot n>0}\mathfrak{F}(\mathfrak{u})\,|\mathfrak{u}\cdot n|\,\mathrm{d}\mathfrak{u}, \tag{1.2}$$

and its $\varepsilon \to 0$ behavior is one of the most classical and basic hydrodynamic problems in the kinetic theory. Here $\mathfrak{F}$ is the phase space density, Q is the hard-sphere collision operator

$$Q[F,G] := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, v-\mathfrak{u})\Big(F(\mathfrak{u}_*)G(v_*) - F(\mathfrak{u})G(v)\Big)\mathrm{d}\omega\mathrm{d}\mathfrak{u}, \tag{1.3}$$

with $\omega \in \mathbb{S}^2$ a unit vector, $\mathfrak{u}_* := \mathfrak{u} + \omega\big((v-\mathfrak{u})\cdot\omega\big)$, $v_* := v - \omega\big((v-\mathfrak{u})\cdot\omega\big)$, and the hard-sphere collision kernel $q(\omega, v-\mathfrak{u}) := q_0\,|\omega\cdot(v-\mathfrak{u})|$ for a positive constant q_0.

Moreover, the wall Maxwellian in the diffuse-reflection boundary condition is

$$M_w(x_0, v) := \frac{1}{2\pi\big(T_w(x_0)\big)^2}\exp\left(-\frac{|v|^2}{2T_w(x_0)}\right) \tag{1.4}$$

for $x_0 \in \partial\Omega$ where n is the unit outward normal to the bounded domain Ω with a wall temperature

$$T_w = 1 + O\big(\,|\nabla T_w|_{L^\infty}\big) \tag{1.5}$$

satisfying

$$\int_{v\cdot n(x_0)>0} M_w(x_0, v)\,|v\cdot n(x_0)|\,\mathrm{d}v = 1. \tag{1.6}$$

In the case of $|\nabla T_w| = O(\varepsilon)$, an $L^6 - L^\infty$ framework has been developed in [23, 24] which leads to the validity of the Hilbert expansion

$$\mathfrak{F} \approx (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}} \bigg(1 + \varepsilon\Big(\rho_1 + T_1 \frac{|v|^2 - 3}{2}\Big)\bigg) \tag{1.7}$$

where $(2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}}$ is a global Maxwellian and T_1 satisfies the celebrated Fourier law for the infinitesimal temperature variation T_1 of order ε:

$$\Delta_x T_1 = 0 \tag{1.8}$$

with $u_1 = \mathbf{0}$. We note that in this case, Maxwell's slip condition (1.1) is trivial at the ε order.

In contrast, for the more natural and general case of $|\nabla T_w|_{L^\infty} = O(1)$, ε^0 order only results in $\nabla_x\big(\rho T\big) \equiv 0$. Even though $|\nabla T_w|_{L^\infty} = O(1)$ is within the regime of the compressible Euler limit, the vanishing of the velocity at ε^0 order requires further expansion to determine uniquely ρ and T. This process (see details in [26]) leads to that for any constant $P > 0$

$$\mathfrak{F} \approx \mu + \varepsilon\bigg\{\mu\bigg(\rho_1 + u_1 \cdot v + T_1 \frac{|v|^2 - 3T}{2}\bigg) - \mu^{\frac{1}{2}} \left(\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right)\bigg\} \tag{1.9}$$

where $\mu(x, v) := \dfrac{\rho(x)}{\big(2\pi T(x)\big)^{\frac{3}{2}}} \exp\left(-\dfrac{|v|^2}{2T(x)}\right)$,

$$\overline{\mathscr{A}} := v \cdot \big(|v|^2 - 5T\big)\, \mu^{\frac{1}{2}} \in \mathbb{R}^3, \quad \mathscr{A} := \mathcal{L}^{-1}\left[\overline{\mathscr{A}}\right] \in \mathbb{R}^3, \tag{1.10}$$

where $\mathcal{L}$ is defined in (1.28), and $(\rho, u_1, T, \mathfrak{p})$ is determined by a Navier-Stokes-Fourier system with "ghost" effect

$$\left\{\begin{array}{rcl} P &=& \rho T, \\ \rho(u_1 \cdot \nabla_x u_1) + \nabla_x \mathfrak{p} &=& \nabla_x \cdot \big(\tau^{(1)} - \tau^{(2)}\big), \\ \nabla_x \cdot (\rho u_1) &=& 0, \\ \nabla_x \cdot \left(\kappa \dfrac{\nabla_x T}{2T^2}\right) &=& 5P\big(\nabla_x \cdot u_1\big), \end{array}\right. \tag{1.11}$$

with the boundary condition

$$T\Big|_{\partial\Omega} = T_w, \quad u_1\Big|_{\partial\Omega} := \big(u_{1,\iota_1}, u_{1,\iota_2}, u_{1,n}\big)\Big|_{\partial\Omega} = \Big(\beta_0 \partial_{\iota_1} T_w, \beta_0 \partial_{\iota_2} T_w, 0\Big). \tag{1.12}$$

Here $\tau^{(1)} := \lambda\left(\nabla_x u_1 + (\nabla_x u_1)^t - \frac{2}{3}(\nabla_x \cdot u_1)\mathbf{1}\right)$, $\tau^{(2)} := \frac{\lambda^2}{P}\Big(K_1\big(\nabla_x^2 T - \frac{1}{3}\Delta_x T \mathbf{1}\big) + \frac{K_2}{T}\big(\nabla_x T \otimes \nabla_x T - \frac{1}{3}|\nabla_x T|^2 \mathbf{1}\big)\Big)$, K_1 and K_2 are positive constants, $\lambda[T] > 0$ is the viscosity coefficient, $\kappa[T] > 0$ is the heat conductivity, $(u_{1,\iota_1}, u_{1,\iota_2})$ are two tangential components and $u_{1,n}$ is the normal component of u_1, $\beta_0 = \beta_0[T_w]$ is a function of T_w. The system (1.11) has been discovered by various researches as early as in 1960's. We refer to [53, 54, 63, 64] for the stationary problem and to [19, 11, 9] for the time dependent one. In particular, Sone [63, 64] establishes a systematic Hilbert expansion in the bulk with careful and precise boundary layer corrections.

Sone's precise formula (1.12) confirms Maxwell's boundary condition (1.1) at $O(\varepsilon)$ since the first term in (1.1) are of the order $O(\varepsilon^2)$, and his β_0 formula matches the second term in (1.1):

$$\beta_0 \simeq \text{const} \times \frac{\sqrt{T_w}}{\rho_w T_w} \tag{1.13}$$

which can be solved from the kinetic boundary layer (the Milne problem) [64, 63, 26]. Sone's formula (1.12) serves as a benchmark for many numerical simulations.

Furthermore, Sone [63, 64], among others (see also [54]), makes an important and surprising observation that new limiting system (1.11) cannot be predicted by any classical fluid theory, which can be viewed as an exciting example of a new kinetic effect. Thanks to the mismatch of ε orders, the infinitesimal first-order velocity εu_1 acting like a "ghost" (a term introduced by Y. Sone [62]), has an $O(1)$ impact in determining zeroth order ρ and T, as long as the tangential temperature variation is of $O(1)$ in (1.12). Therefore, (1.11) is not compatible with any classical continuum fluid theory, in which all fluid quantities are determined at the same level of order of ε. In particular, $\tau^{(2)}$ is a new contribution different from the standard fluid theories. Subsequently, different types of ghost effects have been discovered, such as the ghost effect from curvature or from mixture of gases [65, 66]. In [30], a formal derivation of ghost-like equations is obtained starting from the Newtonian many particle system.

Due to fundamental mathematical difficulty created by $|\nabla T_w|_{L^\infty} = O(1)$, it has remained an intriguing outstanding question to rigorously derive (1.11). Particularly, the presence of new term $\mu^{\frac{1}{2}}\left(\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right)$ in (1.9) leads to a boundary layer correction of order ε, which can not be avoided in mathematical analysis. The main goal of this paper is to settle this open question in affirmative in a bounded domain under the assumption

$$|\nabla T_w|_{W^{3,\infty}} = o(1). \tag{1.14}$$

Throughout this paper, in the low Mach number regime, we always assume (1.14) and the physically important hard-sphere collision kernel, even though our analysis can be easily extended to kernels of hard potential with an angular cutoff.

1.2 Asymptotic Expansion and Remainder Equation

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ε. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a \lesssim b$ to denote $a \leq Cb$ and $a \gtrsim b$ to denote $a \geq Cb$. Also, we write $a \simeq b$ if both $a \lesssim b$ and $a \gtrsim b$ are valid.

Let $\langle \cdot\,, \cdot \rangle_v$ denote the inner product for $v \in \mathbb{R}^3$, $\langle \cdot\,, \cdot \rangle_x$ the inner product for $x \in \Omega$, and $\langle \cdot\,, \cdot \rangle$ the inner product for $(x, v) \in \Omega \times \mathbb{R}^3$. Also, let $\langle \cdot\,, \cdot \rangle_{\gamma_\pm}$ denote the inner product on $\gamma_\pm$ with measure $\mathrm{d}\gamma := |v \cdot n|\, \mathrm{d}v \mathrm{d}S_x$.

Denote the bulk and boundary norms

$$\|f\|_{L^r} := \left(\iint_{\Omega\times\mathbb{R}^3} |f(x,v)|^r \,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{r}}, \quad |f|_{L^r_{\gamma_\pm}} := \left(\int_{\gamma_\pm} |f(x,v)|^r |v\cdot n| \,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{r}}. \tag{1.15}$$

Define the weighted L^∞ norms for properly chosen $T_M>0$, $0\leq\varrho<\dfrac{1}{2}$ and $\vartheta\geq 0$

$$\|f\|_{L^\infty_{\varrho,\vartheta}} := \operatorname*{ess\,sup}_{(x,v)\in\Omega\times\mathbb{R}^3}\left(\langle v\rangle^{\vartheta}\,\mathrm{e}^{\varrho\frac{|v|^2}{2T_M}}\,|f(x,v)|\right), \quad |f|_{L^\infty_{\gamma_\pm,\varrho,\vartheta}} := \operatorname*{ess\,sup}_{(x,v)\in\gamma_\pm}\left(\langle v\rangle^{\vartheta}\,\mathrm{e}^{\varrho\frac{|v|^2}{2T_M}}\,|f(x,v)|\right). \tag{1.16}$$

Denote the ν-norm: $\|f\|_{L^2_\nu} := \left(\iint_{\Omega\times\mathbb{R}^3} \nu(x,v)\,|f(x,v)|^2\,\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}}$. Let $\|\cdot\|_{W^{k,p}}$ denote the usual Sobolev norm for $x\in\Omega$ and $|\cdot|_{W^{k,p}}$ for $x\in\partial\Omega$, and $\|\cdot\|_{W^{k,p}L^q}$ denote $W^{k,p}$ norm for $x\in\Omega$ and L^q norm for $v\in\mathbb{R}^3$. The similar notation also applies when we replace L^q by $L^\infty_{\varrho,\vartheta}$ or L^q_γ.

We also define

$$\kappa\mathbf{1} := \int_{\mathbb{R}^3}\left(\mathscr{A}\otimes\overline{\mathscr{A}}\right)\mathrm{d}v, \quad \boldsymbol{\sigma}\mathbf{1} := \int_{\mathbb{R}^3}\left(|v|^2-5T\right)\left(\mathscr{A}\otimes\overline{\mathscr{A}}\right)\mathrm{d}v, \quad \lambda := \frac{1}{T}\int_{\mathbb{R}^3}\mathscr{B}_{ij}\overline{\mathscr{B}}_{ij} \ \text{ for } \ i\neq j. \tag{1.17}$$

We follow the approach in [71] to define the geometric quantities and more details can be found in [26]. For smooth manifold $\partial\Omega$, there exists an orthogonal curvilinear coordinates system (ι_1,ι_2) such that the coordinate lines coincide with the principal directions at any $x_0\in\partial\Omega$ (at least locally). Assume $\partial\Omega$ is parameterized by $\mathbf{r}=\mathbf{r}(\iota_1,\iota_2)$. Let the vector length be $L_i=|\partial_{\iota_i}\mathbf{r}|$ and unit vector $\varsigma_i=L_i^{-1}\partial_{\iota_i}\mathbf{r}$. Based on sign of the flow direction $v\cdot n(x_0)$, we can divide the boundary $\gamma:=\left\{(x_0,v):\ x_0\in\partial\Omega, v\in\mathbb{R}^3\right\}$ into the incoming boundary γ_-, the outgoing boundary γ_+, and the grazing set γ_0. In particular, the boundary condition of (1.2) is only given on γ_-.

Consider the corresponding new coordinate system $\mathfrak{x}:=(\iota_1,\iota_2,\mathfrak{n})$, where $\mathfrak{n}$ denotes the normal distance to boundary surface $\partial\Omega$, i.e.

$$x=\mathbf{r}-\mathfrak{n}n. \tag{1.18}$$

Define the orthogonal velocity substitution for $\mathfrak{v}:=(v_\eta,v_\phi,v_\psi)$ as

$$-v\cdot n := v_\eta, \quad -v\cdot\varsigma_1 := v_\phi, \quad -v\cdot\varsigma_2 := v_\psi. \tag{1.19}$$

Finally, we define the scaled variable $\eta=\dfrac{\mathfrak{n}}{\varepsilon}$, which implies $\dfrac{\partial}{\partial\mathfrak{n}}=\dfrac{1}{\varepsilon}\dfrac{\partial}{\partial\eta}$.

We seek a solution in the form

$$\mathfrak{F}(x,v) = f + f^B + \varepsilon\mu^{\frac{1}{2}}R = \mu + \mu^{\frac{1}{2}}\left(\varepsilon f_1+\varepsilon^2 f_2\right) + \mu_w^{\frac{1}{2}}\left(\varepsilon f_1^B\right) + \varepsilon\mu^{\frac{1}{2}}R, \tag{1.20}$$

where the interior solution is

$$f(x,v) := \mu(x,v) + \mu^{\frac{1}{2}}(x,v)\Big(\varepsilon f_1(x,v) + \varepsilon^2 f_2(x,v)\Big), \tag{1.21}$$

and the new ε-cutoff (singularity in v_η) boundary layer is

$$f^B(\mathfrak{x},\mathfrak{v}) := \mu_w^{\frac{1}{2}}(\iota_1,\iota_2,\mathfrak{v})\Big(\varepsilon f_1^B(\mathfrak{x},\mathfrak{v})\Big). \tag{1.22}$$

The details about the construction of the asymptotic expansion can be found in [26], where f_1, f_2 and f_1^B are constructed precisely. $R(x,v)$ is the remainder with the minimum and natural ε pre-factor, $\mu(x,v)$ denotes a local Maxwellian and $\mu_w(\iota_1,\iota_2,\mathfrak{v}) = \mu(x_0,v)$ the boundary Maxwellian.

In the following, let o_T be a small constant depending on T_w satisfying

$$o_T = o(1) \to 0 \quad \text{as} \quad |\nabla T_w|_{W^{3,\infty}} \to 0. \tag{1.23}$$

In principle, while o_T is determined by ∇T_w a priori, we are free to choose $o(1)$ depending on the estimate.

In [26], we proved the following

Theorem 1.1. *Under the assumption* (1.14)*, for any given $P>0$, there exists a unique solution $(\rho,u_1,T;\mathfrak{p})$ (where $\mathfrak{p}$ has zero average) to the ghost-effect equation* (1.11) *and* (1.12) *satisfying for any $s\in[2,\infty)$*

$$\|u_1\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} + \|T-1\|_{W^{4,s}} \lesssim o_T. \tag{1.24}$$

Also, we can construct f_1, f_2 and f_1^B as in [25] and [26] such that

$$\|f_1\|_{W^{3,s}L^\infty_{\varrho,\vartheta}} + |f_1|_{W^{3-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim o_T, \quad \|f_2\|_{W^{2,s}L^\infty_{\varrho,\vartheta}} + |f_2|_{W^{2-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim o_T, \tag{1.25}$$

and for some $K_0>0$ and any $0<r\leq 3$, (uniform in ε-cutoff)

$$\left\|\mathrm{e}^{K_0\eta} f_1^B\right\|_{L^\infty_{\varrho,\vartheta}} + \left\|\mathrm{e}^{K_0\eta}\partial_{\iota_1}^r f_1^B\right\|_{L^\infty_{\varrho,\vartheta}} + \left\|\mathrm{e}^{K_0\eta}\partial_{\iota_2}^r f_1^B\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T. \tag{1.26}$$

Define the symmetrized version of Q

$$Q^*[F,G] := \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{S}^2} q(\omega,|\mathfrak{u}-v|)\Big(F(\mathfrak{u}_*)G(v_*) + F(v_*)G(\mathfrak{u}_*) - F(\mathfrak{u})G(v) - F(v)G(\mathfrak{u})\Big)\mathrm{d}\omega\mathrm{d}\mathfrak{u}. \tag{1.27}$$

Clearly, $Q[F,F] = Q^*[F,F]$. Denote the linearized Boltzmann operator $\mathcal{L}$

$$\mathcal{L}[f] := -2\mu^{-\frac{1}{2}} Q^*\big[\mu, \mu^{\frac{1}{2}} f\big], \tag{1.28}$$

Note that $\mathcal{L}$ is self-adjoint in $L^2_\nu(\mathbb{R}^3)$ satisfying the coercivity property

$$\int_{\mathbb{R}^3} f\mathcal{L}[f] \gtrsim \|(\mathbf{I}-\mathbf{P})[f]\|^2_{L^2_\nu}. \tag{1.29}$$

Denote $\mathcal{L}^{-1}:\mathcal{N}^\perp\to\mathcal{N}^\perp$ the quasi-inverse of $\mathcal{L}$. Also, denote the nonlinear Boltzmann operator Γ

$$\Gamma[f,g] := \mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g\right] \in \mathcal{N}^\perp. \tag{1.30}$$

Based on the analysis in [25, Section 4], in order to show the validity of (1.20), it suffices to consider the remainder equation for R:

$$\begin{cases} v\cdot\nabla_x\left(\mu^{\frac12}R\right)+\varepsilon^{-1}\mu^{\frac12}\mathcal{L}[R]=\mu^{\frac12}S \ \text{ in } \ \Omega\times\mathbb{R}^3, \\ R(x_0,v)=\mathcal{P}_\gamma[R](x_0,v)+h(x_0,v) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n(x_0)<0. \end{cases} \tag{1.31}$$

We do not give here the long expressions for the boundary and bulk sources h and S, that can be found in [25] and [26],

$$\mathcal{P}_\gamma[R](x_0,v):=m_w(x_0,v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu^{\frac12}(x_0,\mathfrak{u})R(x_0,\mathfrak{u})\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}, \tag{1.32}$$

with $m_w(x_0,v):=M_w\mu_w^{-\frac12}$ satisfying the normalization condition

$$\mu_w^{\frac12}(x_0,v)=m_w(x_0,v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu_w(x_0,\mathfrak{u})\left|\mathfrak{u}\cdot n(x_0)\right|\mathrm{d}\mathfrak{u}=\frac{P}{\big(2\pi T_w(x_0)\big)^{\frac12}}m_w(x_0,v). \tag{1.33}$$

Note that

$$v\cdot\nabla_x\left(\mu^{\frac12}R\right)=\mu^{\frac12}(v\cdot\nabla_xR)+\frac12\mu^{-\frac12}(v\cdot\nabla_x\mu)R=\mu^{\frac12}(v\cdot\nabla_xR)+\left(\overline{\mathscr{A}}\cdot\frac{\nabla_xT}{4T^2}\right)R, \tag{1.34}$$

where $\overline{\mathscr{A}}$ is defined in (1.10). Hence, we can rewrite (1.31) in the equivalent form

$$\begin{cases} v\cdot\nabla_xR+\left(\mu^{-\frac12}\overline{\mathscr{A}}\cdot\dfrac{\nabla_xT}{4T^2}\right)R+\varepsilon^{-1}\mathcal{L}[R]=S \ \text{ in } \ \Omega\times\mathbb{R}^3, \\ R(x_0,v)=\mathcal{P}_\gamma[R](x_0,v)+h(x_0,v) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n(x_0)<0. \end{cases} \tag{1.35}$$

1.3 Decomposition and Reformulation

1.3.1 $\mathscr{A}$-Hodge Decomposition and Local $\mathscr{A}$-Conservation Law

The presence of new contributions involving $\mathscr{A}\cdot\frac{\nabla_xT}{2T^2}$ in (1.9) and $\left(\mu^{-\frac12}\overline{\mathscr{A}}\cdot\frac{\nabla_xT}{4T^2}\right)R$ in (1.35) creates fundamental mathematical difficulties, and we define the following projection and $\mathscr{A}$-Hodge decomposition, as well as corresponding local $\mathscr{A}$-conservation law.

Note that the null space $\mathcal{N}$ of $\mathcal{L}$ is a five-dimensional space spanned by the orthogonal basis

$$\mu^{\frac12}\Big\{1,v,\left(|v|^2-3T\right)\Big\}. \tag{1.36}$$

We denote $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $L^2(\mathbb{R}^3)$. Define the kernel operator $\mathbf{P}$ as the orthogonal projection onto $\mathcal{N}$, and the non-kernel operator $\mathbf{I}-\mathbf{P}$. We decompose

$$R=\mathbf{P}[R]+(\mathbf{I}-\mathbf{P})[R]:=\mu^{\frac12}(v)\Big\{p_R(x)+v\cdot\mathbf{b}_R(x)+\Big(|v|^2-5T\Big)c_R(x)\Big\}+(\mathbf{I}-\mathbf{P})[R], \tag{1.37}$$

Note that we have chosen a different decomposition of $\mathbf{P}[R]$ in contrast with the classical $(a_R, \mathbf{b}_R, c_R)$ decomposition in previous work [24] ($p_R = a_R + 2Tc_R$). Moreover, we introduce a crucial further orthogonal split

$$(\mathbf{I}-\mathbf{P})[R] = \mathscr{A}\cdot\mathbf{d}_R(x) + (\mathbf{I}-\overline{\mathbf{P}})[R], \tag{1.38}$$

where $(\mathbf{I}-\overline{\mathbf{P}})[R]$ is the orthogonal complement to $\mathscr{A}\cdot\mathbf{d}_R(x)$ in $\mathcal{N}^\perp$ with respect to $(\,\cdot\,,\,\cdot\,)_{\mathcal{L}} := \Big\langle\,\cdot\,,\mathcal{L}[\,\cdot\,]\Big\rangle_v$

$$\Big(\mathscr{A},(\mathbf{I}-\overline{\mathbf{P}})[R]\Big)_{\mathcal{L}} = \Big\langle\overline{\mathscr{A}},(\mathbf{I}-\overline{\mathbf{P}})[R]\Big\rangle_v = 0. \tag{1.39}$$

From now on, when there is no confusion, we will omit the subscript R and simply write $p, \mathbf{b}, c, \mathbf{d}$. In summary, we decompose the remainder as

$$R = \Big(p + \mathbf{b}\cdot v + c\big(|v|^2 - 5T\big)\Big)\mu^{\frac{1}{2}} + \mathbf{d}\cdot\mathscr{A} + (\mathbf{I}-\overline{\mathbf{P}})[R], \tag{1.40}$$

and from (1.35), R satisfies

$$v\cdot\nabla_x R + \left(\mu^{-\frac{1}{2}}\overline{\mathscr{A}}\cdot\frac{\nabla_x T}{4T^2}\right)R + \varepsilon^{-1}\mathbf{d}\cdot\overline{\mathscr{A}} + \varepsilon^{-1}\mathcal{L}\Big[(\mathbf{I}-\overline{\mathbf{P}})[R]\Big] = S. \tag{1.41}$$

We can further define the $\mathscr{A}$-Hodge decomposition $\mathbf{d} = \nabla_x\xi + \mathbf{e}$ with the potential ξ solving the Poisson equation

$$\nabla_x\cdot(\kappa\nabla_x\xi) = \nabla_x\cdot(\kappa\mathbf{d}) \ \text{ in }\ \Omega, \quad \xi = 0 \ \text{ on }\ \partial\Omega. \tag{1.42}$$

We can directly compute

$$\nabla_x\cdot(\kappa\mathbf{e}) = \nabla_x\cdot(\kappa\mathbf{d}) - \nabla_x\cdot(\kappa\nabla_x\xi) = 0. \tag{1.43}$$

This implies the crucial orthogonality: for any $\eta(x)$ such that $\eta = 0$ on $\partial\Omega$, we have

$$\int_\Omega(\kappa\mathbf{e})\cdot\nabla_x\eta\,\mathrm{d}x = -\int_\Omega\nabla_x\cdot(\kappa\mathbf{e})\eta\,\mathrm{d}x = 0. \tag{1.44}$$

Thus, taking $\eta = \xi$, we know

$$\begin{aligned}\Big(\mathbf{d}\cdot\mathscr{A},\mathbf{d}\cdot\mathscr{A}\Big)_{\mathcal{L}} &= \Big\langle\mathbf{d}\cdot\mathscr{A},\mathcal{L}[\mathbf{d}\cdot\mathscr{A}]\Big\rangle = \Big\langle\mathbf{d}\cdot\mathscr{A},\mathbf{d}\cdot\overline{\mathscr{A}}\Big\rangle = \int_\Omega\kappa\left|\nabla_x\xi+\mathbf{e}\right|^2\mathrm{d}x \qquad (1.45)\\ &= \int_\Omega\kappa\left|\nabla_x\xi\right|^2\mathrm{d}x + \int_\Omega\kappa\left|\mathbf{e}\right|^2\mathrm{d}x + 2\int_\Omega\kappa\left(\nabla_x\xi\cdot\mathbf{e}\right)\mathrm{d}x\\ &= \int_\Omega\kappa\left|\nabla_x\xi\right|^2\mathrm{d}x + \int_\Omega\kappa\left|\mathbf{e}\right|^2\mathrm{d}x - 2\int_\Omega\nabla_x\cdot(\kappa\mathbf{e})\xi\,\mathrm{d}x = \int_\Omega\kappa\left|\nabla_x\xi\right|^2\mathrm{d}x + \int_\Omega\kappa\left|\mathbf{e}\right|^2\mathrm{d}x.\end{aligned}$$

We note the local conservation laws of mass, momentum and energy (with test functions $\mu^{\frac{1}{2}}, v\mu^{\frac{1}{2}}, |v|^2\mu^{\frac{1}{2}}$):

$$P\big(\nabla_x\cdot\mathbf{b}\big) = \Big\langle\mu^{\frac{1}{2}},S\Big\rangle_v, \tag{1.46}$$

$$P\nabla_x p + \nabla_x\cdot\int_{\mathbb{R}^3}(v\otimes v)\mu^{\frac{1}{2}}(\mathbf{I}-\overline{\mathbf{P}})[R] = \Big\langle v\mu^{\frac{1}{2}},S\Big\rangle_v, \tag{1.47}$$

$$5P\Big(\nabla_x\cdot(\mathbf{b}T)\Big) + \nabla_x\cdot(\kappa\mathbf{d}) = \Big\langle|v|^2\mu^{\frac{1}{2}},S\Big\rangle_v. \tag{1.48}$$

Notice that the construction of ξ in (2.7) leads to a natural kinetic equation for the combination $\nabla_x\big(c+\varepsilon^{-1}\xi\big)$ as

$$\begin{aligned} & v\cdot\nabla_x R+\varepsilon^{-1}\big(\nabla_x\xi+\mathbf{e}\big)\cdot\overline{\mathscr{A}} \\ \approx & \nabla_x\big(c+\varepsilon^{-1}\xi\big)\cdot\overline{\mathscr{A}}+v\cdot\nabla_x\Big(p\mu^{\frac{1}{2}}+\mathbf{b}\cdot v\mu^{\frac{1}{2}}+(\mathbf{I}-\mathbf{P})[R]\Big)+\varepsilon^{-1}\mathbf{e}\cdot\overline{\mathscr{A}}, \end{aligned} \tag{1.49}$$

along $\overline{\mathscr{A}}$ direction. This allows us to introduce the local $\mathscr{A}$-conservation law (with test function $\mathscr{A}$)

$$\begin{aligned} & \kappa\nabla_x\big(c+\varepsilon^{-1}\xi\big)+\frac{\nabla_x T}{2T^2}\big(\kappa p+\sigma c\big)+\varepsilon^{-1}\kappa\mathbf{e} \\ & +\big\langle v\cdot\nabla_x\big((\mathbf{I}-\overline{\mathbf{P}})[R]\big),\mathscr{A}\big\rangle_v+\left\langle\frac{\nabla_x T}{4T^2}(\mathbf{I}-\overline{\mathbf{P}})[R],\mathscr{A}\right\rangle_v=\langle S,\mathscr{A}\rangle_v\,. \end{aligned} \tag{1.50}$$

This new $\mathscr{A}$-Hodge decomposition and its $\mathscr{A}$-conservation plays the key role for us to circumvent the analytical difficulty.

1.3.2 Reformulation with Global Maxwellian

Due to the cubic velocity growth term in (1.35), we have to reformulate the remainder equation with a global Maxwellian in order to obtain L^∞ estimates. Considering $\|\nabla_x T\|_{L^\infty}\lesssim o_T$ for o_T defined in (1.23), choose constant $T_M: T_M<\min_{x\in\Omega}T<\max_{x\in\Omega}T<2T_M$ and $\max_{x\in\Omega}T-T_M=o_T$. Define a global Maxwellian

$$\mu_M:=\frac{P}{(2\pi)^{\frac{3}{2}}T_M^{\frac{5}{2}}}\exp\bigg(-\frac{|v|^2}{2T_M}\bigg). \tag{1.51}$$

We can rewrite (1.31) as

$$\begin{cases} v\cdot\nabla_x R_M+\varepsilon^{-1}\mathcal{L}_M[R]=S_M \ \text{ in } \ \Omega\times\mathbb{R}^3, \\ R_M(x_0,v)=\mathcal{P}_M[R_M](x_0,v)+h_M(x_0,v) \ \text{ for } \ x_0\in\partial\Omega \ \text{ and } \ v\cdot n(x_0)<0, \end{cases} \tag{1.52}$$

where $R_M=\mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}R$, $S_M=\mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}S$, $h_M=\mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}h$ and for $m_{M,w}:=\mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}(x_0,v)m_w(x_0,v)=M_w\mu_M^{-\frac{1}{2}}$

$$\mathcal{L}_M[R_M]:=-2\mu_M^{-\frac{1}{2}}Q^*\Big[\mu,\mu_M^{\frac{1}{2}}R_M\Big]:=\nu_M R_M-K_M[R_M], \tag{1.53}$$

$$\mathcal{P}_M[R_M](x_0,v):=m_{M,w}(x_0,v)\int_{\mathfrak{u}\cdot n(x_0)>0}\mu_M^{\frac{1}{2}}R_M(x_0,\mathfrak{u})\,|\mathfrak{u}\cdot n(x_0)|\,\mathrm{d}\mathfrak{u}. \tag{1.54}$$

1.3.3 Working Space

Denote the working space X via the norm

$$\begin{aligned}\|R\|_X :=&\varepsilon^{-1}\Big(\|p\|_{L^2} + \|\xi\|_{L^2} + \|\mathbf{e}\|_{L^2} + \left\|(\mathbf{I}-\overline{\mathbf{P}})[R]\right\|_{L^2_\nu} + \|\xi\|_{L^6}\Big) \\ &+ \varepsilon^{-\frac{1}{2}}\Big(\|\mathbf{b}\|_{L^2} + \|\xi\|_{H^2} + |(1-\mathcal{P}_\gamma)[R]|_{L^2_{\gamma_+}} + |\nabla_x\xi|_{L^2_{\partial\Omega}}\Big) \\ &+ \Big(\|c\|_{L^2} + \|p\|_{L^6} + \|\mathbf{b}\|_{L^6} + \|c\|_{L^6} + \|\xi\|_{W^{2,6}} \\ &+ \|\mathbf{e}\|_{L^6} + \left\|(\mathbf{I}-\overline{\mathbf{P}})[R]\right\|_{L^6} + |\mathcal{P}_\gamma[R]|_{L^2_\gamma} + \left|\mu^{\frac{1}{4}}(1-\mathcal{P}_\gamma)[R]\right|_{L^4_{\gamma_+}}\Big) \\ &+ \varepsilon^{\frac{1}{2}}\Big(\|R_M\|_{L^\infty_{\varrho,\vartheta}} + |R_M|_{L^\infty_{\gamma,\varrho,\vartheta}}\Big).\end{aligned} \tag{1.55}$$

1.4 Main Theorem

Theorem 1.2. *Assume that Ω is a bounded C^3 domain and (1.14) holds. Then for any given $P>0$, there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$, there exists a non negative solution $\mathfrak{F}$ to the equation (1.2) represented by (1.20) satisfying*

$$\int_\Omega p(x)\mathrm{d}x = 0, \tag{1.56}$$

and

$$\|R\|_X \lesssim o_T, \tag{1.57}$$

where the X norm is defined in (1.55). Such a solution is unique among all solutions satisfying (1.56) and (1.57). This further yields that in the expansion (1.20), $\mu+\varepsilon\mu\big(u_1\cdot v\big)$ is the leading-order terms in the sense of

$$\left\|\mu^{-\frac{1}{2}}\big[\mathfrak{F}-\mu\big]\right\|_{L^2_{x,v}} \lesssim \varepsilon \tag{1.58}$$

and

$$\left\|\int_{\mathbb{R}^3}\Big[\mathfrak{F}-\mu-\varepsilon\mu\big(u_1\cdot v\big)\Big]v\right\|_{L^2_x} \lesssim \varepsilon^{\frac{3}{2}}, \tag{1.59}$$

where (ρ,u_1,T) is determined by the ghost-effect equations (1.11) and (1.12).

Remark 1.3. *There is no restriction on ∇T_w except smallness condition (1.14). Moreover, our result is valid for all general smooth bounded domains (including non-convex domains), despite the presence of boundary-layer approximation.*

Remark 1.4. *We stress that the boundedness of $\|R\|_X$ only implies that $\varepsilon^{-\frac{1}{2}}\nabla_x\xi$ is bounded in L^2, not $\varepsilon^{-1}\nabla_x\xi$ is bounded in L^2. This is in contrast to the full $\varepsilon^{-1}\left\|(\boldsymbol{I}-\boldsymbol{P})[R]\right\|_{L^2_\nu}$ control in L^2 in [24],*

1.5 Literature Review

The hydrodynamic limit of the Boltzmann equation on the diffusive space-time scales has been the subject of many studies.

The Boltzmann solution has been solved close to the incompressible Navier–Stokes–Fourier system. Mathematical results were obtained, among others, in [10, 19, 34, 37, 39, 16, 31] for smooth solutions. In particular, the convergence of the renormalized solutions to the Navier-Stokes-Fourier system has been obtained by Golse and Saint-Raymond [32], see also contributions in [5, 6, 7, 8, 55, 58, 61, 49, 48]. We also refer to the survey [60]. In a pioneering study of the unsteady ghost effect for 1D geometry $-\infty < x < \infty$ [46], convergence is established for a temperature variation along the 1D normal direction via delicate analysis in a Sobolev space with a crucial sign condition for T'.

Much less is known about the hydrodynamic limits for steady Boltzmann solution, due to lack of basic L^1 and entropy estimates. Only the control of entropy production from $\int Q[\mathfrak{F},\mathfrak{F}]\ln(\mathfrak{F})$ is available analytically. Despite progress [67, 21, 20, 17, 1] on the control of entropy production in terms of $\mathfrak{F}-\mu_{\mathfrak{F}}$ (where $\mu_{\mathfrak{F}}$ is the local Maxwellian with the same mass, momentum and energy), its nonlinear nature prevents useful applications for Boltzmann solutions with large amplitudes so far (see an interesting progress in [4] and also [56, 18]). In [2], in the Rayleigh-Benard context existence results for small Knudsen and Mach numbers in two dimensional slab are obtained and the first bifurcation is studied. In [27, 28] the geometry is strictly one-dimensional and the Knudsen number is small, but not the Mach number. We refer to the recent review [29] for more details. In the above cases no slip boundary conditions are considered. Recently an interesting extension to the case of slip boundary condition has been obtained in [22]. Very few rigorous results, beyond the one-dimensional case, are available for the ghost effect. In [15, 14], the ghost effect for a one-dimensional mixture is established, and in [3] the ghost effect is studied for a one-dimensional problem with cylindrical symmetry. We remark that classical ghost effect from temperature variation (1.2) requires non-trivial tangential temperature variation which has a multi-dimensional nature.

2 Methodology

2.1 L^6-L^∞ Framework for Fourier Law

For a general 3D domain, an improved L^6-L^∞ framework is developed in [24], in which steady hydrodynamic limits to the celebrated Fourier law are established, along with their dynamical stability. For Boltzmann solutions close to a Maxwellian μ, the entropy production $\int Q[\mathfrak{F},\mathfrak{F}]\ln(\mathfrak{F})$ is approximated by the fundamental a priori bound for the microscopic part $\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu}$ associated with the linearized Boltzmann operator $\mathcal{L}$ around μ, where $\mu^{\frac{1}{2}}R\sim\mathfrak{F}-\mu$ instead of the natural difference with the local Maxwellian $\mathfrak{F}-\mu_{\mathfrak{F}}$. In such a setup, the fundamental analytic difficulty is to control the missing macroscopic part $\mathbf{P}[R]\sim\mu-\mu_{\mathfrak{F}}$ for the nonlinear closure.

In a series of papers [33, 36], an elliptic structure is discovered for $\mathbf{P}[R]$, and $\mathbf{P}[R]$ can be bounded via $(\mathbf{I}-\mathbf{P})[R]$ through the Boltzmann equation in high order Sobolev norms.

Unfortunately, it is well-known that Boltzmann solutions exhibit only limited regularity (or even singularity) in the presence of physical boundary conditions [51, 41] due to the characteristic nature of the grazing set in kinetic theory. To overcome this difficulty, a L^2-L^∞ framework is established in [35], in which $\|\mathbf{P}[R]\|_{L^2}$ is bounded via $\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2}$ to bootstrap weighted $\|R\|_{L^\infty}$ bound via double-Duhamel principle along the characteristics, thanks to the velocity mixing feature of K. An important new methodology is developed in [23], where $\|\mathbf{P}[R]\|_{L^2}$ is estimated quantitatively in terms of $\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2}$ via proper choices of test functions in the weak formulation with boundary effect, by solving dual Poisson equations of the form $\Delta_x\phi \sim a, \mathbf{b}, c$. Such a methodology entails a robust and flexible approach to grasp the ellipticity (positivity) estimates for $\mathbf{P}[R]$ in the presence of boundary conditions, reminiscing in spirit elliptic estimates in weak forms.

In recent papers [23, 24, 71], the steady solution to the Boltzmann equation close to Maxwellians was constructed, in 3D smooth domains, for a gas in contact with a boundary with a prescribed temperature profile modeled by the diffuse-reflection boundary condition. Also, the Navier-Stokes-Fourier limit was established in [24, 71] for the diffusive scaling of $|\nabla_x T| = o(1)\varepsilon$ based on an improved L^6-L^∞ framework. In particular, the proof in [24] relies on the L^2 energy/coercivity estimates combined with L^6 kernel estimates (by solving dual Poisson equations of $\Delta_x\phi \sim a\,|a|^4, \mathbf{b}\,|\mathbf{b}|^4, c\,|c|^4$)

$$\begin{aligned}&\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + \|\mathbf{P}[R]\|_{L^6} + \varepsilon^{-\frac12}\left|(1-\mathcal{P}_\gamma)[R]\right|_{L^2_{\gamma_+}} \\ \lesssim\, & o(1)\varepsilon^{\frac12}\|R\|_{L^\infty_{\varrho,\vartheta}} + o(1) \text{ boundary/source terms,}\end{aligned} \tag{2.1}$$

and the L^∞ estimates

$$\|R\|_{L^\infty_{\varrho,\vartheta}} \lesssim \varepsilon^{-\frac32}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} + \varepsilon^{-\frac12}\|\mathbf{P}[R]\|_{L^6} + o(1) \text{ boundary/source terms.} \tag{2.2}$$

The L^6 bound is crucial to control nonlinearity in 3D and to close the L^∞ estimates: weaker $\|\mathbf{P}[R]\|_{L^2}$ bound leads to extra ε^{-1} loss (compared with L^6 bound) as $\|R\|_{L^\infty_{\varrho,\vartheta}} \lesssim \varepsilon^{-\frac32}\|\mathbf{P}[R]\|_{L^2}$.

We also refer to the recent papers on the diffusive limit of the Boltzmann equation and related models [12, 13, 50, 47]. We list some recent developments along L^p-L^∞ framework [23, 35, 37, 39, 41, 40, 51, 52, 70].

2.2 New $L^2-L^6-L^\infty$ Framework

For hydrodynamic limits of (1.2), the basic energy estimate yields

$$\begin{aligned}\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})[R]\|^2_{L^2_\nu} =& -\varepsilon^{-1}\left\langle \mu^{-\frac12}\overline{\mathscr{A}}\cdot\frac{\nabla_x T}{4T^2}, R^2\right\rangle + \text{good terms} \\ =& -\varepsilon^{-1}5P\Big\langle\nabla_x T, \mathbf{b}c\Big\rangle + \text{good terms.}\end{aligned} \tag{2.3}$$

In the case of $\|\nabla_x T\|_{L^\infty} = o(1)\varepsilon$, we have

$$\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} \lesssim o(1)\|\mathbf{P}[R]\|_{L^6} + 1. \tag{2.4}$$

With a new L^6 gain for $\mathbf{P}[R]$ in [24], quantitative estimates in [23] lead to

$$\|\mathbf{P}[R]\|_{L^6} \lesssim \varepsilon^{-1} \|(\mathbf{I}-\mathbf{P})[R]\|_{L^2_\nu} \tag{2.5}$$

where c and $\mathbf{b}$ are estimated via test functions of the form $\nabla_x\phi \cdot \overline{\mathscr{A}}$ and $\nabla_x\psi : \overline{\mathscr{B}}$, and the convergence to the incompressible Navier-Stokes-Fourier system as $\varepsilon \to 0$ in [24] is established. We remark that thanks to the new L^6 estimate and $\|\nabla_x T\|_{L^\infty} = o(1)\varepsilon$, it is possible to avoid boundary layer approximation completely.

In the case of $\|\nabla_x T\|_{L^\infty} = o(1)$, however,

$$-\varepsilon^{-1}5P\Big\langle \nabla_x T, \mathbf{b}c\Big\rangle \approx o(1)\varepsilon^{-1}\|\mathbf{P}[R]\|_{L^6}^2 \tag{2.6}$$

which is impossible to close via (2.5) with a severe loss of ε^{-1}. We note that the term $-\varepsilon^{-1}5P\Big\langle \nabla_x T, \mathbf{b}c\Big\rangle$ presents a fundamental major difficulty, and the main mathematical achievement of our contribution is to develop a systematic methodology to overcome this loss of ε in the presence of boundary effects. Furthermore, (1.14) forces us to introduce boundary layer approximation at the leading order, in a stark contrast to the case $\|\nabla_x T\|_{L^\infty} = o(1)\varepsilon$. This presents another major technical challenge due to the well-known singularity at the grazing set: no mathematical theory is available for non-convex sets.

Reduced Energy Estimate of $(\mathbf{I}-\overline{\mathbf{P}})[R]+\mathbf{e}\cdot\mathscr{A}$ The first key idea is to use $\mathscr{A}$-Hodge decomposition to split

$$(\mathbf{I}-\mathbf{P})[R] = \mathbf{d}\cdot\mathscr{A} + (\mathbf{I}-\overline{\mathbf{P}})[R] = \nabla_x\xi\cdot\mathscr{A} + \mathbf{e}\cdot\mathscr{A} + (\mathbf{I}-\overline{\mathbf{P}})[R]. \tag{2.7}$$

A reduced energy estimate is then established as the building block of our analysis:

Proposition 2.1. *(Proof in [25, Section 5]) Let R be a solution to (1.31). Under the assumption (1.14), we have*

$$\varepsilon^{-\frac{1}{2}}\left|(1-\mathcal{P}_\gamma)[R]\right|_{L^2_{\gamma_+}} + \varepsilon^{-1}\|\mathbf{e}\|_{L^2} + \varepsilon^{-1}\left\|(\boldsymbol{I}-\overline{\boldsymbol{P}})[R]\right\|_{L^2_\nu} \lesssim o_T\|R\|_X + \|R\|_X^2 + o_T. \tag{2.8}$$

Remarkably, this new reduced energy estimate (2.8), in comparison to (2.4), sacrifices the control of $\nabla_x\xi\cdot\mathscr{A}$ in $(\mathbf{I}-\mathbf{P})[R]$ to cancel $-\varepsilon^{-1}5P\langle\nabla_x T, \mathbf{b}c\rangle$ via a string of delicate manipulations (ignoring boundary contributions):

$$\begin{aligned}
&\varepsilon^{-1}\Big\langle \nabla_x T, \mathbf{b}c\Big\rangle_x && (2.9)\\
\approx&-\varepsilon^{-1}\Big\langle \nabla_x\cdot\big(\kappa\mathbf{d}\big), c\Big\rangle_x && \big[\text{Mass and Energy Local Conservation Laws (1.46)(1.48)}\big]\\
=&-\varepsilon^{-1}\Big\langle \nabla_x\cdot\big(\kappa\nabla_x\xi\big), c\Big\rangle_x && \big[\mathscr{A}\text{-Hodge (1.43)}\big]\\
\approx&\,\varepsilon^{-1}\Big\langle \kappa\nabla_x\xi, \nabla_x c\Big\rangle_x && \big[\text{Integration by parts}\big]\\
\approx&-\varepsilon^{-1}\Big\langle \nabla_x\xi, \varepsilon^{-1}\kappa\nabla_x\xi\Big\rangle_x && \big[\text{Local }\mathscr{A}\text{-Conservation Law (1.50) and }\mathscr{A}\text{-Hodge (1.44)}\big]\\
=&-\varepsilon^{-2}\iint_{\Omega\times\mathbb{R}^3}\kappa\left|\nabla_x\xi\right|^2,
\end{aligned}$$

which is cancelled with its counterpart in $\varepsilon^{-2} \iint_{\Omega\times\mathbb{R}^3} R(\mathbf{I}-\mathbf{P})[R]$. We note that the ignored boundary contribution $\langle \nabla_x \xi, (1-\mathcal{P}_\gamma)[R]\rangle_{\gamma_+}$ is bounded by $o(1)\varepsilon \|R\|_X^2$ thanks to (1.55).

We remark that the a priori control of the full $(\mathbf{I}-\mathbf{P})[R]$, as entropy production, has been the starting point for PDE study for Boltzmann solutions near Maxwellians in the past. To our knowledge, this is the first time one needs to study fine structure within $(\mathbf{I}-\mathbf{P})[R]$ itself.

Estimate of p and $Z := c + \varepsilon^{-1}\xi$ We can control $p \approx O(\varepsilon)$ directly from local conservation laws (1.47) as in (2.5).

Proposition 2.2. *(Proof in [25, Section 6]) Let R be a solution to (1.31). Under the assumption (1.14) and (1.56), we have*

$$\varepsilon^{-1}\|p\|_{L^2} + \|p\|_{L^6} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{2.10}$$

Thanks to the local $\mathscr{A}$-conservation law (1.50), the L^6 norm of the natural combination

$$Z = c + \varepsilon^{-1}\xi \tag{2.11}$$

is bounded by the L^6 framework in [24].

Proposition 2.3. *(Proof in [25, Section 6]) Let R be a solution to (1.31). Under the assumption (1.14), for $\alpha \geq 1$, we have*

$$\|Z\|_{L^6} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{2.12}$$

In order to cope with the new ε-cutoff boundary layer interaction, we must further split Z as regular part Z^R and small singular part Z^S: $Z = Z^S + Z^R$ so that

Proposition 2.4. *(Proof in [25, Section 6]) Let R be a solution to (1.31). Under the assumption (1.14), we have $Z = Z^R + Z^S$ where*

$$\left\|Z^R\right\|_{H_0^1} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T, \tag{2.13}$$

$$\left\|Z^S\right\|_{L^2} \lesssim o_T \varepsilon^{\frac{1}{2}} \|R\|_X + \varepsilon^{\frac{1}{2}} \|R\|_X^2 + o_T \varepsilon. \tag{2.14}$$

Dual Stokes Problem and Improved Estimate $\|\mathbf{b}\|_{L^2} \lesssim \varepsilon^{\frac{1}{2}}$ We now obtain a surprising gain of $\varepsilon^{\frac{1}{2}}$ for $\mathbf{b}$, which is necessary to justify the ghost effect contribution $\varepsilon u_1 \cdot v \mu^{\frac{1}{2}}$ beyond the first order of ε.

Proposition 2.5. *(Proof in [25, Section 7]) Let R be a solution to (1.31). Under the assumption (1.14), we have*

$$\varepsilon^{-\frac{1}{2}}\|\boldsymbol{b}\|_{L^2} + \|\boldsymbol{b}\|_{L^6} \lesssim o_T \|R\|_X + \|R\|_X^2 + o_T. \tag{2.15}$$

Such a gain $\varepsilon^{\frac{1}{2}}$ is crucial to justify the ε order ghost effect εu_1 with a key removal of $\varepsilon^{-1}(\mathbf{I}-\overline{\mathbf{P}})[R]$ term via compensating with an extra local conservation law. Such a removal also plays the key role in the subsequent estimate for c.

Notice that the test function $\nabla_x\psi : \mathscr{B}$, where

$$\overline{\mathscr{B}} = \left(v\otimes v - \frac{|v|^2}{3}\mathbf{1}\right)\mu^{\frac{1}{2}} \in \mathbb{R}^{3\times 3}, \quad \mathscr{B} = \mathcal{L}^{-1}\left[\overline{\mathscr{B}}\right] \in \mathbb{R}^{3\times 3}, \tag{2.16}$$

yields

$$\begin{aligned}&\left\langle v\cdot\nabla_x R + \varepsilon^{-1}\mathbf{d}\cdot\overline{\mathscr{A}} + \varepsilon^{-1}\mathcal{L}\Big[(\mathbf{I}-\overline{\mathbf{P}})[R]\Big], \nabla_x\psi : \mathscr{B}\right\rangle \\ =&\left\langle v\cdot\nabla_x R + \varepsilon^{-1}\mathcal{L}\Big[(\mathbf{I}-\overline{\mathbf{P}})[R]\Big], \nabla_x\psi : \mathscr{B}\right\rangle \\ \approx&\left\langle |v|^2\mu^{\frac{1}{2}}\mathbf{b}, \nabla_x\Big(\nabla_x\psi : \mathscr{B}\Big)\right\rangle + \left\langle \varepsilon^{-1}(\mathbf{I}-\overline{\mathbf{P}})[R], \nabla_x\psi : \overline{\mathscr{B}}\right\rangle,\end{aligned} \tag{2.17}$$

and thus the combination of (2.17) and the local momentum conservation law (1.47)

$$\varepsilon^{-1}\left\langle(\mathbf{I}-\overline{\mathbf{P}})[R], \nabla_x\psi : \overline{\mathscr{B}}\right\rangle \approx -\varepsilon^{-1}\left\langle\nabla_x p, \psi\right\rangle \approx \varepsilon^{-1}\left\langle p, \nabla_x\cdot\psi\right\rangle. \tag{2.18}$$

Hence, the choice of the *new* test function $\nabla_x\psi : \mathscr{B} + \varepsilon^{-1}\psi\cdot v\mu^{\frac{1}{2}}$ with a smooth function $\psi(x)$ satisfying $\nabla_x\cdot\psi = 0$, $\psi\big|_{\partial\Omega} = 0$ exactly eliminates this most singular term for (1.31) and leads to

$$-\Big\langle\lambda\Delta_x\psi, \mathbf{b}\Big\rangle_x = \text{good terms}. \tag{2.19}$$

The key feature in (2.19) is the absence of the $O(1)$ term $\varepsilon^{-1}(\mathbf{I}-\overline{\mathbf{P}})[R]$, so that a gain of $\varepsilon^{\frac{1}{2}}$ is possible by constructing a solution to the dual Stokes problem for (ψ, q) (with an artificial pressure q)

$$-\lambda\Delta_x\psi + \nabla_x q \approx \mathbf{b}\,|\mathbf{b}|^{r-2} \ \text{ in }\ \Omega, \qquad \nabla_x\cdot\psi = 0 \ \text{ in }\ \Omega, \qquad \psi = 0 \ \text{ on }\ \partial\Omega. \tag{2.20}$$

Dual Stokes-Poisson Problem and Estimate of c The estimate of c is the most delicate part of the paper.

Proposition 2.6. *(Proof in [25, Section 8]) Let R be a solution to* (1.31)*. Under the assumption* (1.14)*, we have*

$$\|c\|_{L^6} \lesssim o_T\|R\|_X + \|R\|_X^2 + \|R\|_X^3 + o_T. \tag{2.21}$$

Because of (1.49), it seems impossible to split $Z = c + \varepsilon^{-1}\xi$ and to obtain c estimate independent of $\varepsilon^{-1}\xi$ via any test functions. The key new idea is to recombine $\varepsilon^{-1}\nabla_x\xi + \varepsilon^{-1}\mathbf{e} = \varepsilon^{-1}\mathbf{d}$ in the local $\mathscr{A}$-conservation law (1.50) with a *new* test function $\nabla_x\phi\cdot\mathscr{A}$:

$$\begin{aligned}&\left\langle v\cdot\nabla_x R + \varepsilon^{-1}\mathbf{d}\cdot\mathscr{A}, \nabla_x\phi\cdot\overline{\mathscr{A}}\right\rangle = \left\langle v\cdot\nabla_x R, \nabla_x\phi\cdot\mathscr{A}\right\rangle + \left\langle\varepsilon^{-1}\mathbf{d}\cdot\overline{\mathscr{A}}, \nabla_x\phi\cdot\mathscr{A}\right\rangle \\ \approx&\left\langle\overline{\mathscr{A}}\cdot\nabla_x c, \nabla_x\phi\cdot\mathscr{A}\right\rangle + \varepsilon^{-1}\left\langle\nabla_x\cdot(\kappa\mathbf{d}), \phi\right\rangle_x,\end{aligned} \tag{2.22}$$

and thus the combination of (2.22) and the mass/energy conservation (1.46)(1.48) yields

$$\varepsilon^{-1}\left\langle \nabla_x\cdot(\kappa\mathbf{d}),\phi\right\rangle_x \approx -\varepsilon^{-1}\Big\langle \nabla_x T,\mathbf{b}\phi\Big\rangle_x. \tag{2.23}$$

Hence, the choice of a *new pair* of test functions $\nabla_x\phi\cdot\mathscr{A}+\varepsilon^{-1}\phi\big(|v|^2-5T\big)\mu^{\frac12}$ where $\phi(x)$ is a smooth function satisfying $\phi\big|_{\partial\Omega}=0$ for (1.31) leads to

$$-\Big\langle \nabla_x\cdot(\kappa\nabla_x\phi),c\Big\rangle_x+\varepsilon^{-1}5P\Big\langle\phi,\nabla_x T\cdot\mathbf{b}\Big\rangle_x=\text{good terms}, \tag{2.24}$$

where the most singular term is in terms of $\mathbf{b}$. Then we can choose choose ψ in (2.19) coupled with (2.24) to kill $\varepsilon^{-1}5P\Big\langle\phi,\nabla_x T\cdot\mathbf{b}\Big\rangle_x$. To this end, we solve a *coupled* dual Stokes-Poisson system for the triple (ψ,q,ϕ)

$$-\lambda\Delta_x\psi+\nabla_x q\approx -5P\phi\nabla_x T \ \text{ in } \ \Omega,\qquad \nabla_x\cdot\psi=0 \ \text{ in } \ \Omega,\qquad \psi=0 \ \text{ on } \ \partial\Omega, \tag{2.25}$$

$$-\nabla_x\cdot\big(\kappa\nabla_x\phi\big)=c\,|c|^{r-2} \ \text{ in } \ \Omega,\qquad \phi=0 \ \text{ on } \ \partial\Omega. \tag{2.26}$$

Thanks to a precise cancellation, all but the boundary contribution

$$-\varepsilon^{-1}\Big\langle\nabla_x\psi:\mathscr{B},(1-\mathcal{P}_\gamma)[R]\Big\rangle_{\gamma+} \tag{2.27}$$

in $\varepsilon^{-1}\times$ (2.19) + (2.24) are under control. We note that $(1-\mathcal{P}_\gamma)[R]$ is of $O(\varepsilon^{\frac12})$ and $\nabla_x\psi:\mathscr{B}$ is of $O(1)$, so there is still a loss of $\varepsilon^{\frac12}$ here for closure.

ε-cutoff boundary layer g^B and Interior g Compensation Through an extensive effort, the original loss of ε^{-1} in $-\varepsilon^{-1}5P\left\langle\nabla_x T,\mathbf{b}c\right\rangle$ is now transferred to a boundary loss of $\varepsilon^{-\frac12}$ in $-\varepsilon^{-1}\Big\langle\nabla_x\psi:\mathscr{B},(1-\mathcal{P}_\gamma)[R]\Big\rangle_{\gamma+}$. Motivated by the gain of $\varepsilon^{\frac12}$ in L^2 for any boundary layers in the bulk, we carefully design an ε-cutoff boundary layer g^B and its interior counterpart g to compensate such a loss with

$$\big(g^B+g\big)\big|_{\gamma+}=-\big(\nabla_x\psi:\mathscr{B}\big)\big|_{\gamma+}. \tag{2.28}$$

Thanks to the fact that $\partial_n\psi_n\big|_{\partial\Omega}=0$ as well as the parity of $\mathscr{B}$, we can construct

$$g=\mu^{\frac12}(v)\big(v\cdot\mathfrak{B}\big),\quad \nabla_x\cdot\mathfrak{B}=0. \tag{2.29}$$

This crucial and precise structure and parity of g^B lead to two crucial cancellations, which ensure the final closure of estimates with no singular power of ε:

$$\varepsilon^{-1}\left\langle\mu^{\frac12}v\cdot\nabla_x\Big(\mu^{-\frac12}g\Big),R\right\rangle=\varepsilon^{-1}\left\langle\mu^{\frac12}v\cdot\nabla_x\big(\mathfrak{B}\cdot v\big),R\right\rangle=0, \tag{2.30}$$

$$\varepsilon^{-1}\left\langle g^B,\Gamma\Big[p\mu^{\frac12}+c\left(|v|^2-5T\right)\mu^{\frac12},p\mu^{\frac12}+c\left(|v|^2-5T\right)\mu^{\frac12}\Big]\right\rangle=0. \tag{2.31}$$

Even though boundary layer approximations have been established in kinetic theory for matching *given* boundary data, to our knowledge, this is the first time boundary layer construction is based on (unknown) ψ from (2.25) and (2.26) to estimate the remainder R itself.

New ε-Cutoff Boundary Layer Estimates for Non-Convex Domain

• **Hardy Inequality with ε Gain** One of the important challenges in the hydrodynamic limit of (1.2) is the necessary inclusion of ε-cutoff boundary layers f_1^B and g^B. In fact, the determination of the ghost-effect equations depends on f_1^B implicitly. Unfortunately, in [70, 42, 43], it is discovered that for non-flat domains the classical boundary layer theory in kinetic theory breaks down, due again to the characteristic and singular nature of the grazing set. Even though an alternative satisfactory theory has been established in convex domains [68, 69, 71], the non-convex case is completely open. In the analysis the most difficult contribution in $\varepsilon^{-1}\left\langle h^B, R\right\rangle$ (where h^B denotes a generic quantity related to ε-cutoff boundary layer) is treated as

$$\begin{aligned}
&\varepsilon^{-1}\left\langle h^B, c\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle \\
=&\varepsilon^{-1}\left\langle h^B, Z\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle-\varepsilon^{-1}\left\langle h^B, \varepsilon^{-1}\xi\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle \\
=&\varepsilon^{-1}\left\langle h^B, Z^R\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle+\varepsilon^{-1}\left\langle h^B, Z^S\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle \\
&-\varepsilon^{-2}\left\langle h^B, \xi\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle .
\end{aligned} \tag{2.32}$$

Thanks to the fact that $Z^R\in H_0^1$ and $\xi\in H_0^2$, we express

$$\frac{Z^R}{\mathfrak{n}}=\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}Z^R,\qquad \frac{\xi}{\mathfrak{n}}=\frac{1}{\mathfrak{n}}\int_0^{\mathfrak{n}}\partial_{\mathfrak{n}}\xi, \tag{2.33}$$

and apply Hardy's inequality [45, 57] along the normal $\mathfrak{n}$ direction to obtain

$$\left\|\frac{Z^R}{\mathfrak{n}}\right\|_{L^2_{\mathfrak{n}}}\lesssim\left\|\partial_{\mathfrak{n}}Z^R\right\|_{L^2_{\mathfrak{n}}},\qquad \left\|\frac{\xi}{\mathfrak{n}}\right\|_{L^2_{\mathfrak{n}}}\lesssim\left\|\partial_{\mathfrak{n}}\xi\right\|_{L^2_{\mathfrak{n}}}. \tag{2.34}$$

Note that $\mathfrak{n}=\varepsilon\eta$ in the ε-cutoff boundary layer scaling can be absorbed by h^B to produce extra ε in $\mathfrak{n}h^B=\varepsilon\eta h^B$:

$$\left|\varepsilon^{-2}\left\langle h^B, \xi\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\right\rangle\right|\lesssim\left(\varepsilon^{-\frac{1}{2}}\left\|\eta h^B\right\|_{L^2_{\mathfrak{n}}}\right)\left(\varepsilon^{-\frac{1}{2}}\left\|\partial_{\mathfrak{n}}\xi\right\|_{L^2_{\mathfrak{n}}}\right) \tag{2.35}$$

where $\varepsilon^{-\frac{1}{2}}\left\|\eta h^B\right\|_{L^2_\eta}$ is bounded thanks to the change-of-variable $\frac{\mathfrak{n}}{\varepsilon}=\eta$ which yields $\|\cdot\|_{L^2_{\mathfrak{n}}}\lesssim\varepsilon^{\frac{1}{2}}\|\cdot\|_{L^2_\eta}$, and $\varepsilon^{-\frac{1}{2}}\left\|\partial_{\mathfrak{n}}\xi\right\|_{L^2_{\mathfrak{n}}}$ is bounded via an interpolation of $\mathscr{A}$-Hodge decomposition (1.42) with $\varepsilon^{-1}\|\xi\|_{L^2}$.

• **BV Estimate to Bound** $\varepsilon^{-1}\langle\partial_{v_\eta}f_1^B, g^B\rangle$ It is well known from the boundary layer theory that the sharp pointwise bound near the grazing set holds $\left|\partial_{v_\eta}f_1^B\right|\approx|v_\eta|^{-1}$. With a cutoff $|v_\eta|\geq\varepsilon$, $\left\|\partial_{v_\eta}f_1^B\right\|_{L^1_{v_\eta}}\approx|\ln\varepsilon|$, which creates a fatal logarithmic loss for closure, even with ε in η integration. To overcome such a $|\ln\varepsilon|$ loss, we establish a new BV estimate for f_1^B in [25, Section 3], which amounts to a subtle but crucial gain in joint (η, v_η) integration with no loss of $|\ln\varepsilon|$ for the cutoff $\left\|\partial_{v_\eta}f_1^B\right\|_{L^1_{v_\eta}}\lesssim 1$. For example, we can bound

$$\varepsilon^{-1}\left\langle\partial_{v_\eta}f_1^B, g^B\right\rangle\lesssim\varepsilon^{-1}\left\|f_1^B\right\|_{\mathrm{BV}}\left\|g^B\right\|_{L^\infty}\lesssim\left\|g^B\right\|_{L^\infty}^2+\varepsilon^{-2}\left\|f_1^B\right\|_{\mathrm{BV}}^2\lesssim o_T\left\|c\right\|_{L^2}^2+o_T. \tag{2.36}$$

We design an ε-cutoff of the standard boundary-layer Milne solution to achieve this goal. Such a cutoff leads to a non-local commutator, which needs to be carefully controlled. Moreover, since it is extremely difficult to go beyond the first order boundary layer approximation in general domains, the presence of boundary layer dictates $\varepsilon\mu^{\frac{1}{2}}R$ in (1.20).

We can bootstrap L^2-L^6 estimates to L^∞ bound by the method developed in [23, 24, 39, 38].

In summary, we develop a systematic approach to study the steady Boltzmann equation near a *local Maxwellian* in a 3D bounded domain. Our analysis relies on elaborate and integrated schemes with several exact cancellations and sharp estimates with no room to spare. These new techniques have led to the final resolution of the diffusive limit of the neutron transport equation in [44], an important open question for one of the most classical problems in kinetic theory.

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