Geometric effects on $W^{1,p}$ regularity of the stationary Boltzmann equation

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1 Introduction

We consider the incoming boundary value problem of the stationary Boltzmann equation

$$\begin{cases} v \cdot \nabla_x F = Q(F, F) & \text{in } \Omega \times \mathbb{R}^3, \\ F = G & \text{on } \in \Gamma^-, \end{cases} \tag{1}$$

and its linearized problem

$$\begin{cases} v \cdot \nabla_x f = Lf & \text{in } \Omega \times \mathbb{R}^3, \\ f = g & \text{on } \Gamma^-. \end{cases} \tag{2}$$

The solution F, which is called the velocity distribution function, describes the density of rarefied gas molecular at the position x with velocity v. Here, Ω is a bounded domain in $\mathbb{R}^3$ with C^1 boundary. The incoming boundary Γ^- is defined by

$$\Gamma^- := \{(x, v) \in \partial\Omega \times \mathbb{R}^3 \mid n(x) \cdot v < 0\},$$

where $n(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$. The nonlinear collision operator Q reads

$$Q(F_1, F_2) := \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} [F_1(v')F_2(v'_*) - F_1(v)F_2(v_*)] B(|v - v_*|, \theta)\, d\theta d\phi dv_*$$

where

$$v' := v + ((v_* - v) \cdot \omega)\omega,\ v'_* := v_* - ((v_* - v) \cdot \omega)\omega,$$
$$\omega := \cos\theta \frac{v_* - v}{|v_* - v|} + (\sin\theta\cos\phi)e_2 + (\sin\theta\sin\phi)e_3,$$

ϕ and θ are real values satisfying $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi/2$, and the vectors e_2, e_3 are chosen so that the pair $\{\frac{v_* - v}{|v_* - v|}, e_2, e_3\}$ forms an orthonormal basis. Roughly speaking, the collision operator Q describes the interaction of two particles due to the collision. For the cross section B, we assume that

$$B(|v-v_*|,\theta)=B_0|v-v_*|^\gamma\cos\theta\sin\theta, \tag{3}$$

where B_0 is a positive constant and $0\leq\gamma\leq1$. The range of γ covers the hard sphere model, the hard potential model, and the Maxwellian molecular gases. We note that the cross section of the form (3) is regarded as an angular cutoff potential.

Let $M(v):=\pi^{-\frac{3}{2}}e^{-|v|^2}$ be the standard Maxwellian. It is known that $v\cdot\nabla_xM=Q(M,M)=0$. Based on this observation, we consider the fluctuation of the solution to the problem (1) from the Maxwellian:

$$F=M+M^{\frac{1}{2}}f,\quad G=M+M^{\frac{1}{2}}g.$$

Then, the problem (1) is reduced to the following one.

$$\begin{cases}v\cdot\nabla_xf=Lf+\Gamma(f,f) & \text{in }\Omega\times\mathbb{R}^3,\\ f=g & \text{on }\Gamma^-,\end{cases} \tag{4}$$

where

$$\begin{aligned}Lh:=&M^{-\frac{1}{2}}\left(Q(M,M^{\frac{1}{2}}h)+Q(M^{\frac{1}{2}}h,M))\right),\\ \Gamma(h_1,h_2):=&M^{-\frac{1}{2}}Q(M^{\frac{1}{2}}h_1,M^{\frac{1}{2}}h_2).\end{aligned}$$

In this article, we are interested in $W^{1,p}$ regularity of solutions to boundary value problems (2) and (4). For the case where the domain is non-convex, it is known in the time-dependent case that the grazing set creates discontinuity of the solution due to the nature of characteristic lines [10] (or [9] with the diffuse reflection boundary condition), and it is expected that it is true for the stationary case. Thus, in this article, we assume that the domain is convex.

In this direction, H. Chen and Kim [1] showed the $W^{1,p}_x$ regularity of the solution to the nonlinear Boltzmann equation with the diffuse reflection boundary condition in bounded uniformly convex domains with C^3 boundaries. On the other hand, in this article, we discuss $W^{1,p}$ regularity with respect to the both x and v variables in general bounded convex domains with C^2 boundaries. Instead, we assume the smallness of the domain.

We first introduce the regularity result on the linearized problem (2). For the linearized collision operator L, we assume that it is decomposed into two parts:

$$Lh(x,v)=-\nu(v)h(x,v)+Kh(x,v),$$

where

$$Kh(x,v):=\int_{\mathbb{R}^3}k(v,v_*)h(x,v_*)\,dv_*,$$

and that the function ν and k satisfy the following estimates.

Assumption A.

$$\begin{aligned}
&\nu_0(1+|v|)^\gamma \le \nu(v) \le \nu_1(1+|v|)^\gamma,\\
&|k(v,v_*)| \lesssim \frac{1}{|v-v_*|(1+|v|+|v_*|)^{1-\gamma}} E_\rho(v,v_*),\\
&|\nabla_v k(v,v_*)| \lesssim \frac{1+|v|}{|v-v_*|^2(1+|v|+|v_*|)^{1-\gamma}} E_\rho(v,v_*),\\
&|\nabla_v \nu(v)| \lesssim (1+|v|)^{\gamma-1},
\end{aligned}$$

where $\nu_0 > 0$, $0 < \rho < 1$, $0 \le \gamma \le 1$, and

$$E_\rho(v,v_*) := e^{-\frac{1-\rho}{4}\left(|v-v_*|^2+\left(\frac{|v|^2-|v_*|^2}{|v-v_*|}\right)^2\right)}.$$

Here and in what follows, we denote $f \lesssim g$ if there exists a constant $C \ge 0$ such that $f \le Cg$. We remark that the decomposition of L and **Assumption A** holds if we adopt the cross section B of the form (3).

We introduce function spaces in order to describe our result. For $1 \le p < \infty$ and $\alpha \ge 0$, let

$$L^p_\alpha(\Omega\times\mathbb{R}^3) := \{f \mid \|f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|^p_{L^p_\alpha(\Omega\times\mathbb{R}^3)} := \int_{\Omega\times\mathbb{R}^3} |f(x,v)|^p e^{p\alpha|v|^2}\,dxdv.$$

Also, for $1 \le p < \infty$ and $\alpha \ge 0$, let

$$W^{1,p}_\alpha(\Omega\times\mathbb{R}^3) := \{f \mid \|f\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} := \|f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \|\nabla_x f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \|\nabla_v f\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}.$$

We remark that $W^{1,p}_0(\Omega\times\mathbb{R}^3)$ is the standard Sobolev space $W^{1,p}(\Omega\times\mathbb{R}^3)$.

Theorem 1.1 (Chen, Hsia, K., Su [5])**.** *Suppose the linearized collision operator L satisfies* **Assumption A***. Let Ω be a bounded convex domain with C^2 boundary. Then, the following statements hold.*

(i) For any given $1 \le p < 2$ and $0 \le \alpha < (1-\rho)/2$, there exists $\epsilon = \epsilon(p,\alpha) > 0$ such that: for any Ω with $\mathrm{diam}(\Omega) < \epsilon$, the boundary value problem (2) has a unique solution $f \in W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)$ if and only if $Jg \in W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)$, where

$$\begin{aligned}
Jg(x,v) :=& e^{-\nu(v)\tau(x,v)} g(q(x,v),v),\\
\tau(x,v) :=& \inf\{s > 0 \mid x - sv \in \Omega^c\},\\
q(x,v) :=& x - \tau(x,v)v.
\end{aligned}$$

(ii) If we further assume that $\partial\Omega$ is of positive Gaussian curvature, namely, the Gaussian curvature of $\partial\Omega$ is uniformly positive, then the range of p in (i) *can be extended to $1 \le p < 3$.*

(iii) Upper bounds of p in (i) *and* (ii) *are optimal.*

Remark 1.2. After uploading our manuscript [5], we noticed that the parameter $0 < \rho < 1$ is arbitrary though the implicit constant in **Assumption A** depends on it. Thus, we may take $0 \le \alpha < 1/2$ first in the statement (i) of Theorem 1.1 and next take $0 < \rho < 1$ such that $\alpha < (1-\rho)/2$.

Remark 1.3. $Jg \in W^{1,p}_{\alpha}(\Omega \times \mathbb{R}^3)$ if g satisfies

$$
\begin{aligned}
&|g(X,v)| \lesssim e^{-\alpha|v|^2} \text{ for all } (X,v) \in \Gamma^-,\\
&|g(X,v) - g(Y,v)| \lesssim |X-Y| e^{-\alpha|v|^2} \text{ for all } (X,v), (Y,v) \in \Gamma^-,\\
&|\nabla_v g(X,v)| \lesssim e^{-\alpha|v|^2} \text{ for all } (X,v) \in \Gamma^-.
\end{aligned}
$$

For the case $p = 2$ with the positive Gaussian curvature condition, see [2].

We next establish a pointwise estimate for the solution (and its first derivatives) to the nonlinear boundary value problem (4) on bounded convex domains with C^2 boundaries of positive Gaussian curvature. The $W^{1,p}$ estimate of the solution follows from pointwise estimates of the derivatives.

We list notations and function spaces for the nonlinear problem as follows:

- $|f|_{\infty,\alpha} := \operatorname*{ess\,sup}_{(x,v)\in\Omega\times\mathbb{R}^3} e^{\alpha|v|^2}|f(x,v)|$.
- $w(x,v) := \frac{|v|}{|v|+1} N(x,v)$.
- $N(x,v) := \left| n(q(x,v)) \cdot \frac{v}{|v|} \right|$.
- $|f|_{\infty,\alpha,w} := |wf|_{\infty,\alpha}$.
- $\|f\|_{\infty,\alpha} := |f|_{\infty,\alpha} + |\nabla_x f|_{\infty,\alpha,w} + |\nabla_v f|_{\infty,\alpha,w}$.
- $L^\infty_\alpha := \{f \mid |f|_{\infty,\alpha} < \infty\}$.
- $\hat{L}^\infty_\alpha := \{f \mid ||f||_{\infty,\alpha} < \infty\}$.

We remark that $\hat{L}^\infty_\alpha \subset W^{1,p}(\Omega \times \mathbb{R}^3)$ for $\alpha > 0$ and $1 \le p < 3$.

If the domain Ω is bounded convex domain with C^2 boundary of positive Gaussian curvature, then the boundary $\partial\Omega$ satisfies both uniform circumscribed and interior sphere conditions.

Definition 1.4. Given $\Omega \subset \mathbb{R}^3$, we say that the boundary of Ω satisfies the uniform circumscribed sphere condition if there exists a positive constant R such that for any $x \in \partial\Omega$ there exists a ball B_R with radius R such that

$$x \in \partial B_R,\ \bar{\Omega} \subset \bar{B}_R.$$

The constant R is called the uniform circumscribed radius.

Definition 1.5. Given $\Omega \subset \mathbb{R}^3$, we say that the boundary of Ω satisfies the uniform interior sphere condition if there exists a positive constant r such that for any $x \in \partial\Omega$ there exists a ball B_r with radius r such that

$$x \in \partial B_r, \;\; \bar{B}_r \subseteq \overline{\Omega}.$$

The constant r is called the uniform interior radius.

Theorem 1.6 (Chen, Hsia, K., Su [6]). *Given $0 \leq \alpha < (1-\rho)/2$, where ρ is the constant in **Assumption A**, there exists a positive constant δ such that: For any bounded convex domain Ω with C^2 boundary of positive Gaussian curvature with uniform circumscribed and interior radii R and r respectively, if*

$$\max\left\{\operatorname{diam}(\Omega), (Rr)^{\frac{1}{2}}\left(1+\frac{R}{r}\right), \|Jg\|_{\infty,\alpha}\right\} < \delta,$$

then the boundary value problem (4) admits a solution in $\hat{L}^\infty_\alpha$.

The organization of the rest part of this article is the following. In Section 2, we introduce some estimates as preliminary. With these estimates, we give a proof of Theorem 1.1 in Section 3. Following the same idea, we give a sketch of proof of Theorem 1.6 in Section 4. The conclusion of this article is given in Section 5. The presentation was based on joint works [5, 6] with Prof. I-Kun Chen, Prof. Chun-Hsiung Hsia, and Mr. Jhe-Kuan Su in National Taiwan University.

2 Preliminary

In this section, we introduce some estimates as preliminary.

Lemma 2.1. *Let Ω be a bounded convex domain. Then, for $v \neq 0$, $a > 0$ and $b \geq 0$, we have*

$$\int_0^{\tau(x,v)} t^b e^{-at}\, dt \lesssim \min\left\{1, \frac{\operatorname{diam}(\Omega)}{|v|}\right\}$$

for all $x \in \Omega$.

Proof. We notice that

$$t^b e^{-at} \leq \left(\sup_{t>0} t^b e^{-\frac{a}{2}t}\right) e^{-\frac{a}{2}t} \lesssim e^{-\frac{a}{2}t}.$$

By the direct integration, we have

$$\int_0^{\tau(x,v)} e^{-\frac{a}{2}t}\, dt = \frac{2}{a}\left(1 - e^{-\frac{a}{2}\tau(x,v)}\right) \leq \frac{2}{a}.$$

On the other hand, since $e^{-at/2} \leq 1$, we have

$$\int_0^{\tau(x,v)} e^{-\frac{a}{2}t}\, dt \leq \tau(x,v) \leq \frac{\mathrm{diam}(\Omega)}{|v|}.$$

The last estimate follows from the definition of the function $\tau(x,v)$. This completes the proof. □

In the following sections, we use the following identity.

Lemma 2.2 ([2], Lemma 2.5)**.** *Let Ω be a bounded convex domain. Then, for a nonnegative measurable function h on $\Omega \times \mathbb{R}^3 \times [0,\infty)$, we have*

$$\int_{\mathbb{R}^3}\int_{\Omega}\int_0^{\tau(x,v)} h(x,v,s)\, dsdxdv = \int_{\mathbb{R}^3}\int_{\Omega}\int_0^{\tau(y,-u)} h(y+tu,u,t)\, dtdydu.$$

In the exponent of the function E_ρ, we see that

$$\begin{aligned}
&-\frac{1-\rho}{4}\left(|v-v_*|^2 + \left(\frac{|v|^2-|v_*|^2}{|v-v_*|}\right)^2\right)\\
=&a|v|^2 - \alpha_{1,a,\rho}|v-v_*|^2 - (1-\rho)\left(\frac{(v-v_*)\cdot v}{|v-v_*|} - \alpha_{2,a,\rho}|v-v_*|\right)^2 - a|v_*|^2\\
=&-a|v|^2 - \alpha_{1,a,\rho}|v-v_*|^2 - (1-\rho)\left(\frac{(v-v_*)\cdot v_*}{|v-v_*|} + \alpha_{2,a,\rho}|v-v_*|\right)^2 + a|v_*|^2
\end{aligned}$$

for all $v, v_* \in \mathbb{R}^3$, where

$$\alpha_{1,a,\rho} := \frac{(1-\rho+2a)(1-\rho-2a)}{4(1-\rho)}, \quad \alpha_{2,a,\rho} := \frac{1-\rho-2a}{2(1-\rho)}.$$

Thanks to the factorization, we may prove the following estimates.

Lemma 2.3 ([5], Corollary 2.14)**.** *Let μ_1 and μ_2 be two real numbers such that $\mu_1 \geq 0$, $\mu_2 > 0$ and $\mu_1+\mu_2 < 3$. Assume $0 < \rho < 1$. Then, for any $-\mu_2(1-\rho)/2 < a < \mu_2(1-\rho)/2$, we have*

$$\int_{\mathbb{R}^3} \frac{1}{|v|^{\mu_1}}|k(v,v_*)|^{\mu_2} e^{a|v|^2}\, dv \lesssim e^{a|v_*|^2}$$

and

$$\int_{\mathbb{R}^3} \frac{1}{|v_*|^{\mu_1}}|k(v,v_*)|^{\mu_2} e^{a|v_*|^2}\, dv_* \lesssim e^{a|v|^2}.$$

Lemma 2.4 ([5], Corollary 2.15)**.** *Let μ_1 and μ_2 be two real numbers such that $\mu_1 \geq 0$, $\mu_2 > 0$ and $\mu_1 + 2\mu_2 < 3$. Also, let $0 < \rho < 1$. Then, for any a satisfying $-\mu_2(1-\rho)/2 < a < \mu_2(1-\rho)/2$, we have*

$$\int_{\mathbb{R}^3} \frac{1}{|v|^{\mu_1}}|\nabla_v k(v,v^*)|^{\mu_2} e^{a|v|^2}\, dv \lesssim (1+|v^*|)^{\mu_2-1} e^{a|v^*|^2}.$$

Lemma 2.5 ([5], Corollary 2.16). *Let $0 < \mu < 3/2$ and $0 < \rho < 1$. Then, for any $-\mu(1-\rho)/2 < a < \mu(1-\rho)/2$, we have*

$$\int_{\mathbb{R}^3} |\nabla_v k(v, v^*)|^\mu e^{a|v|^2} \, dv \lesssim (1 + |v^*|)^{-1} e^{a|v^*|^2}.$$

3 $W^{1,p}$ regularity for the linearized problem

In this section, we give a sketch of a proof of Theorem 1.1 following [5].

We integrate the equation of (2) along the characteristic line to obtain

$$f = S_\Omega K f + Jg,$$

where

$$S_\Omega h(x, v) := \int_0^{\tau(x,v)} e^{-\nu(v)s} h(x - sv, v) \, ds.$$

The formal solution to the above integral equation reads:

$$f = \sum_{i=0}^{\infty} (S_\Omega K)^i Jg. \tag{5}$$

We make use of the smallness of the domain in order to apply the contraction mapping theorem in $W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$. To this aim, we prove the following estimate.

Lemma 3.1. *Given $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ with $1 \leq p < 2$ and $0 \leq \alpha < (1-\rho)/2$, where ρ is the constant in* **Assumption A**, *we have*

$$\|S_\Omega K h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \operatorname{diam}(\Omega)^{\frac{1}{p}} \|h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \operatorname{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)},$$

where $\|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}$ is defined as

$$\|h\|^p_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)} := \int_{\partial\Omega\times\mathbb{R}^3} |h(z, v)|^p e^{p\alpha|v|^2} \, d\Sigma(z) dv,$$

and $d\Sigma$ denotes the surface measure on $\partial\Omega$.

We decompose the estimates into three parts; L^p norms of $S_\Omega K h$, $\nabla_x S_\Omega K h$ and $\nabla_v S_\Omega K h$. For the first part, we obtain the following estimate.

Lemma 3.2. *Let $1 \leq p < \infty$ and $0 \leq \alpha < (1-\rho)/2$, where ρ is the constant in* **Assumption A**. *Then, for any $h \in L^p_\alpha(\Omega \times \mathbb{R}^3)$, we have*

$$\|S_\Omega K h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \operatorname{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}.$$

Proof. When $p = 1$, applying Lemma 2.2, Lemma 2.1 and Lemma 2.3, we have

$$\int_{\Omega\times\mathbb{R}^3} |S_\Omega K h(x, v)| e^{\alpha|v|^2} \, dxdv$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3}\int_\Omega \left| \int_0^{\tau(x,v)} e^{-\nu(v)s} Kh(x-sv,v)ds \right| e^{\alpha|v|^2}\, dxdv \\
&\le \int_{\mathbb{R}^3}\int_\Omega \int_0^{\tau(x,v)} e^{-\nu_0 s}\, |Kh(x-sv,v)|\; ds\, e^{\alpha|v|^2}\, dxdv \\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega \frac{\mathrm{diam}(\Omega)}{|u|} \int_{\mathbb{R}^3} |k(u,v_*)||h(y,v_*)|\, dv_*\, e^{\alpha|u|^2}\, dydu \\
&= \mathrm{diam}(\Omega) \int_{\mathbb{R}^3}\int_\Omega \left(\int_{\mathbb{R}^3} \frac{1}{|u|} |k(u,v_*)| e^{\alpha|u|^2}\, du \right) |h(y,v_*)|\, dydv_* \\
&\lesssim \mathrm{diam}(\Omega) \int_{\mathbb{R}^3}\int_\Omega |h(y,v_*)| e^{\alpha|v_*|^2}\, dydv_*.
\end{aligned}
$$

For $1 < p < \infty$, by the Hölder inequality and Lemma 2.1, we have

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} |S_\Omega Kh(x,v)|^p e^{p\alpha|v|^2}\, dxdv \\
&= \int_{\mathbb{R}^3}\int_\Omega \left| \int_0^{\tau(x,v)} e^{-\nu(v)s} Kh(x-sv,v)ds \right|^p e^{p\alpha|v|^2}\, dxdv \\
&\le \int_{\mathbb{R}^3}\int_\Omega \left(\int_0^{\tau(x,v)} e^{-\nu_0 s}\, ds \right)^{\frac{p}{p'}} \left(\int_0^{\tau(x,v)} e^{-\nu_0 s}\, |Kh(x-sv,v)|^p\; ds \right) e^{p\alpha|v|^2}\, dxdv \\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega \int_0^{\tau(x,v)} e^{-\nu_0 s}\, |Kh(x-sv,v)|^p\; ds\, e^{p\alpha|v|^2}\, dxdv,
\end{aligned}
$$

where p' is the Hölder conjugate of p.

In the same way as for the case $p = 1$, by choosing a parameter α_1 carefully, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_\Omega \int_0^{\tau(x,v)} e^{-\nu_0 s}\, |Kh(x-sv,v)|^p\; ds\, e^{p\alpha|v|^2}\, dxdv \\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega \frac{\mathrm{diam}(\Omega)}{|u|} \left| \int_{\mathbb{R}^3} k(u,v_*)h(y,v_*)\, dv_* \right|^p e^{p\alpha|u|^2}\, dydu \\
&\le \int_{\mathbb{R}^3}\int_\Omega \frac{1}{|u|} \left(\int_{\mathbb{R}^3} |k(u,v_*)| e^{-p'\alpha_1|v_*|^2}\, dv_* \right)^{\frac{p}{p'}} \\
&\quad \times \left(\int_{R^3} |k(u,v_*)||h(y,v_*)|^p e^{p\alpha_1|v_*|^2}\, dv_* \right) e^{p\alpha|u|^2} dydu \\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega \left(\int_{\mathbb{R}^3} \frac{1}{|u|} |k(u,v_*)| e^{p(\alpha-\alpha_1)|u|^2}\, du \right) |h(y,v_*)|^p\, e^{p\alpha_1|v_*|^2} dydv_* \\
&\lesssim \int_{\mathbb{R}^3} e^{p(\alpha-\alpha_1)|v_*|^2} \int_\Omega |h(y,v_*)|^p\, e^{p\alpha_1|v_*|^2} dydv_* \\
&= \int_{\mathbb{R}^3}\int_\Omega |h(y,v_*)|^p\, e^{p\alpha|v_*|^2} dydv_*.
\end{aligned}
$$

Summarizing the above estimates, we obtain

$$\int_{\Omega\times\mathbb{R}^3} |S_\Omega K h(x,v)|^p \, e^{p\alpha|v|^2} \, dxdv \lesssim \mathrm{diam}(\Omega) \int_{\Omega\times\mathbb{R}^3} |h(x,v)|^p \, e^{p\alpha|v|^2} \, dxdv$$

for $1 \leq p < \infty$. This completes the proof. □

For the x derivative, we have the following estimate.

Lemma 3.3. *Let $1 \leq p < 2$ and $0 \leq \alpha < (1-\rho)/2$, where ρ is the constant in* **Assumption A**. *Then, for $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$, we have*

$$\|\nabla_x S_\Omega K h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}.$$

Proof. Observe that

$$\nabla_x S_\Omega K h = S_\Omega K \nabla_x h + S_{\Omega,x} K h,$$

where

$$S_{\Omega,x} h(x,v) := (\nabla_x \tau(x,v)) e^{-\nu(v)\tau(x,v)} h(q(x,v),v).$$

By Lemma 3.2, we have

$$\int_{\Omega\times\mathbb{R}^3} |S_\Omega K \nabla_x h(x,v)|^p e^{p\alpha|v|^2} \, dxdv \lesssim \mathrm{diam}(\Omega) \int_{\Omega\times\mathbb{R}^3} |\nabla_x h(x,v)|^p e^{p\alpha|v|^2} \, dxdv.$$

Thus, we focus on the estimate for the second term. In other words, we prove that

$$\|S_{\Omega,x} K h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}.$$

It is known in [7] that

$$\nabla_x \tau(x,v) = \frac{-n(q(x,v))}{N(x,v)|v|}.$$

Now we perform the change of variables $z = q(x,v)$ and $s = \tau(x,v)$. Noting that $z \in \Gamma^-_v$, where

$$\Gamma^-_v := \{z \in \partial\Omega \mid n(z)\cdot v < 0\},$$

we have

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} |(\nabla_x \tau(x,v)) e^{-\nu(v)\tau(x,v)} K h(q(x,v),v)|^p e^{p\alpha|v|^2} \, dxdv \\
&\leq \int_{\mathbb{R}^3}\int_\Omega \frac{1}{N(x,v)^p|v|^p} e^{-p\nu_0\tau(x,v)} |Kh(q(x,v),v)|^p e^{p\alpha|v|^2} \, dxdv \\
&= \int_{\mathbb{R}^3}\int_{\Gamma^-_v}\int_0^{\tau(z,-v)} \frac{1}{N(z,v)^p|v|^p} e^{-p\nu_0 s} |Kh(z,v)|^p N(z,v)|v| e^{p\alpha|v|^2} \, ds d\Sigma(z) dv \qquad (6) \\
&\lesssim \mathrm{diam}(\Omega) \int_{\mathbb{R}^3}\int_{\Gamma^-_v} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-1}} |Kh(z,v)|^p \, e^{p\alpha|v|^2} d\Sigma(z) dv \\
&= \mathrm{diam}(\Omega) \int_{\partial\Omega}\int_{\Gamma^-_z} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-1}} |Kh(z,v)|^p \, e^{p\alpha|v|^2} dv d\Sigma(z).
\end{aligned}$$

Here,

$$\Gamma_z^- := \{v \in \mathbb{R}^3 \mid n(z) \cdot v < 0\},$$

and we have used Lemma 2.1.

In the case $p = 1$, we have

$$\begin{aligned}
&\int_{\partial\Omega} \int_{\Gamma_z^-} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-1}} |Kh(z,v)|^p \, e^{p\alpha|v|^2} dv d\Sigma(z) \\
&\leq \int_{\mathbb{R}^3} \int_{\partial\Omega} \left(\int_{\mathbb{R}^3} \frac{1}{|v|} |k(v,v_*)| e^{\alpha|v|^2} \, dv \right) |h(z,v_*)| \, d\Sigma(z) dv_* \\
&\lesssim \int_{\partial\Omega \times \mathbb{R}^3} |h(z,v_*)| e^{\alpha|v_*|^2} \, d\Sigma(z) dv_*.
\end{aligned}$$

For $1 < p < 2$, we fix the variable z and decompose the velocity v into two components: $v = v_n + v_t$, where $v_n := (v \cdot n(z))n(z)$ and $v_t := v - v_n$. Then, we have

$$\begin{aligned}
&\int_{\Gamma_z^-} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-1}} |Kh(z,v)|^p \, e^{p\alpha|v|^2} dv \\
&\lesssim \int_{\Gamma_z^-} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-1}} \left(\int_{\mathbb{R}^3} |k(v,v_*)| |h(z,v_*)|^p e^{p\alpha_1|v_*|} \, dv_* \right) e^{p(\alpha-\alpha_1)|v|^2} dv \\
&= \int_{\mathbb{R}^3} \left(\int_{\Gamma_z^-} \frac{1}{|v|} \frac{1}{|v_n|^{p-1}} |k(v,v_*)| e^{p(\alpha-\alpha_1)|v|^2} \, dv \right) |h(z,v_*)|^p \, e^{p\alpha_1|v_*|^2} dv_* \\
&\lesssim \int_{\mathbb{R}^3} \left(\int_{\Gamma_z^-} \frac{1}{|v|} \frac{1}{|v_n|^{p-1}} |v_t - v_{*,t}|^{-1} e^{-\alpha_2|v_t - v_{*,t}|^2} \, dv \right) |h(z,v_*)|^p \, e^{p\alpha|v_*|^2} dv_*,
\end{aligned}$$

where α_1 and α_2 are some positive constants. By the rotation, we simply denote $v_n = r(1,0,0)$ for $r > 0$ and $v_t = (0, v_b)$ for $v_b \in \mathbb{R}^2$. Then, we further have

$$\begin{aligned}
&\int_{\Gamma_z^-} \frac{1}{|v|} \frac{1}{|v_n|^{p-1}} |v_t - v_{*,t}|^{-1} e^{-\alpha_2|v_t - v_{*,t}|^2} \, dv \\
&= \int_{\mathbb{R}^2} \left(\int_0^\infty \frac{1}{\sqrt{r^2 + |v_b|^2}} \frac{1}{r^{p-1}} \, dr \right) |v_b - v_{*,b}|^{-1} e^{-\alpha_2|v_b - v_{*,b}|^2} \, dv_b \\
&= \int_{\mathbb{R}^2} \left(\int_0^\infty \frac{1}{\sqrt{r^2 + 1}} \frac{1}{r^{p-1}} \, dr \right) |v_b|^{1-p} |v_b - v_{*,b}|^{-1} e^{-\alpha_2|v_b - v_{*,b}|^2} \, dv_b.
\end{aligned}$$

Since $0 < p - 1 < 1$, we have

$$\int_0^\infty \frac{1}{\sqrt{r^2 + 1}} \frac{1}{r^{p-1}} \, dr \lesssim 1$$

and

$$\int_{\mathbb{R}^2} |v_b|^{1-p} |v_b - v_{*,b}|^{-1} e^{-\alpha_2|v_b - v_{*,b}|^2} \, dv_b \lesssim 1.$$

Therefore, we have

$$\int_{\Gamma_z^-} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-1}} |Kh(z,v)|^p \, e^{p\alpha|v|^2} dv \lesssim \int_{\mathbb{R}^3} |h(z,v_*)|^p \, e^{p\alpha|v_*|^2} dv_*.$$

The conclusion is obtained by integrating the above estimate with respect to $z \in \partial\Omega$. $\square$

For the v derivative, we have the following estimates.

Lemma 3.4. *Let $1 \le p < 2$ and $0 \le \alpha < (1-\rho)/2$, where ρ is the constant in* **Assumption A**. *Then, for $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$, we have*

$$\|\nabla_v S_\Omega K h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + \mathrm{diam}(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}.$$

Proof. By the straightforward computation, we obtain

$$\nabla_v S_\Omega K h = S_{\Omega,v} K h - (\nabla_v \nu) S_{\Omega,s} K h + S_\Omega K_v h - S_{\Omega,s} K \nabla_x h,$$

where

$$\begin{aligned} S_{\Omega,v} h(x,v) :=& (\nabla_v \tau(x,v)) e^{-\nu(v)\tau(x,v)} h(q(x,v),v), \\ S_{\Omega,s} h(x,v) :=& \int_0^{\tau(x,v)} s e^{-\nu(v)s} h(x-sv,v)\, ds, \\ K_v h(x,v) :=& \int_{\mathbb{R}^3} \nabla_v k(v,v_*) h(x,v_*)\, dv_*. \end{aligned}$$

Notice that, in [7], we see that

$$|\nabla_v \tau(x,v)| \le \frac{|x-q(x,v)||n(q(x,v))|}{|v|^2 N(x,v)} = |\nabla_x \tau(x,v)| \tau(x,v).$$

Since

$$\tau(x,v)^p e^{-\frac{p}{2}\nu(v)\tau(x,v)} \le \sup_{t>0} t^p e^{-\frac{p}{2}\nu_0 t} \lesssim 1,$$

we have

$$\begin{aligned} \|S_{\Omega,v} K h\|^p_{L^p_\alpha(\Omega\times\mathbb{R}^3)} &= \int_{\Omega\times\mathbb{R}^3} |(\nabla_v \tau(x,v)) e^{-\nu(v)\tau(x,v)} K h(q(x,v),v)|^p e^{p\alpha|v|^2}\, dxdv \\ &\lesssim \int_{\Omega\times\mathbb{R}^3} |\nabla_x \tau(x,v)|^p e^{-\frac{p}{2}\nu_0 \tau(x,v)} |K h(q(x,v),v)|^p e^{p\alpha|v|^2}\, dxdv. \end{aligned}$$

We can give an estimate for the above inequality in the same way as in the proof of Lemma 3.3 to obtain

$$\|S_{\Omega,v} K h\|^p_{L^p_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega) \|h\|^p_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}$$

for $1 \le p < 2$ and $0 \le \alpha < (1-\rho)/2$.

For the second and the fourth term, we have

$$\begin{aligned} \|S_{\Omega,s} K h\|_{L^1(\Omega\times\mathbb{R}^3)} &= \int_{\Omega\times\mathbb{R}^3} \left| \int_0^{\tau(x,v)} s e^{-\nu(v)s} \nabla_v \nu(v) K h(x-sv,v) ds \right| e^{\alpha|v|^2}\, dxdv \\ &\lesssim \int_{\mathbb{R}^3} \int_\Omega \left(\int_0^{\tau(x,v)} e^{-\frac{\nu_0}{2}s} |K h(x-sv,v)|\, ds \right) e^{\alpha|v|^2}\, dxdv \end{aligned}$$

$$\lesssim \mathrm{diam}(\Omega)\|h\|_{L^1_\alpha(\Omega\times\mathbb{R}^3)},$$

and

$$\begin{aligned}
&\|S_{\Omega,s}Kh\|^p_{L^p(\Omega\times\mathbb{R}^3)}\\
&=\int_{\Omega\times\mathbb{R}^3}\left|\int_0^{\tau(x,v)} se^{-\nu(v)s}Kh(x-sv,v)ds\right|^p e^{p\alpha|v|^2}\,dxdv\\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega\left(\int_0^{\tau(x,v)} s^{p'}e^{-\nu_0 s}\,ds\right)^{\frac{p}{p'}}\left(\int_0^{\tau(x,v)} e^{-\nu_0 s}\,|Kh(x-sv,v)|^p\,ds\right)e^{p\alpha|v|^2}\,dxdv
\end{aligned}$$

for $1<p<2$. By Lemma 2.1 and the proof of Lemma 3.2, we have

$$\begin{aligned}
\|S_{\Omega,s}Kh\|^p_{L^p(\Omega\times\mathbb{R}^3)} &\lesssim \int_{\mathbb{R}^3}\int_\Omega\int_0^{\tau(x,v)} e^{-\nu_0 s}\,|Kh(x-sv,v)|^p\,e^{p\alpha|v|^2}\,ds\,dxdv\\
&\lesssim \mathrm{diam}(\Omega)\|h\|^p_{L^p_\alpha(\Omega\times\mathbb{R}^3)}.
\end{aligned}$$

Recalling **Assumption A** for $\nabla_v\nu$, we have

$$\|(\nabla_v\nu)S_{\Omega,s}Kh\|_{L^p(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}}\|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}$$

and

$$\|S_{\Omega,s}K\nabla_x h\|_{L^p(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{p}}\|\nabla_x h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}$$

for $1\le p<2$ and $0\le\alpha<(1-\rho)/2$.

For the third term, by Lemma 2.2, Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{aligned}
\|S_\Omega K_v h\|_{L^1_\alpha(\Omega\times\mathbb{R}^3)} &= \int_{\Omega\times\mathbb{R}^3}\left|\int_0^{\tau(x,v)} e^{-\nu(v)s}\int_{\mathbb{R}^3}\nabla_v k(v,v_*)h(x-sv,v_*)dv_*ds\right| e^{\alpha|v|^2}\,dxdv\\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega\left|\int_{\mathbb{R}^3}\nabla_v k(u,v_*)h(y,v_*)dv_*\right| e^{\alpha|u|^2}\,dydu\\
&\le \int_{\mathbb{R}^3}\int_\Omega\left(\int_{\mathbb{R}^3}|\nabla_v k(u,v_*)|e^{\alpha|u|^2}\,du\right)|h(y,v_*)|\,dudv_*\\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega |h(y,v_*)|e^{\alpha|v_*|^2}\,dv_*\,dy
\end{aligned}$$

and

$$\begin{aligned}
\|S_\Omega K_v h\|^p_{L^p_\alpha(\Omega\times\mathbb{R}^3)} &= \int_{\Omega\times\mathbb{R}^3}\left|\int_0^{\tau(x,v)} e^{-\nu(v)s}\int_{\mathbb{R}^3}\nabla_v k(v,v_*)h(x-sv,v_*)dv_*ds\right|^p e^{p\alpha|v|^2}\,dxdv\\
&\lesssim \int_{\mathbb{R}^3}\int_\Omega\left|\int_{\mathbb{R}^3}\nabla_v k(u,v_*)h(y,v_*)dv_*\right|^p e^{p\alpha|u|^2}\,dydu\\
&\le \int_{\mathbb{R}^3}\int_\Omega\left(\int_{\mathbb{R}^3}|\nabla_v k(u,v_*)|e^{-p'\alpha_1|v_*|^2}\,dv_*\right)^{\frac{p}{p'}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_{\mathbb{R}^3} |\nabla_v k(u, v_*)| |h(y, v_*)|^p \, e^{p\alpha_1 |v_*|^2} dv_* \right) e^{p\alpha |u|^2} \, dy du \\
&\lesssim \int_{\mathbb{R}^3} \int_{\Omega} \left(\int_{\mathbb{R}^3} |\nabla_v k(u, v_*)| e^{p(\alpha - \alpha_1)|u|^2} \, du \right) |h(y, v_*)|^p \, e^{p\alpha_1 |v_*|^2} dv_* \, dy \\
&\lesssim \int_{\mathbb{R}^3} \int_{\Omega} |h(y, v_*)|^p \, e^{p\alpha |v_*|^2} dv_* \, dy,
\end{aligned}
$$

where α_1 is a suitably chosen positive constant. Thus, we have

$$
\|S_\Omega K_v h\|_{L^p_\alpha(\Omega \times \mathbb{R}^3)} \lesssim \|h\|_{L^p_\alpha(\Omega \times \mathbb{R}^3)}
$$

for $1 \leq p < 2$ and $0 \leq \alpha < (1 - \rho)/2$.

The proof of Lemma 3.4 is complete. □

Lemma 3.1 follows from Lemma 3.2, Lemma 3.3 and Lemma 3.4.

In order to control the L^p norm on the boundary in the estimate of Lemma 3.1, we introduce two trace lemmas.

Lemma 3.5 ([8])**.** *Let Ω be a bounded domain with Lipschitz boundary. Also, let $1 \leq p < \infty$ and $\alpha \geq 0$. Then, there exists a positive constant $C_1(\Omega)$ such that*

$$
\|h\|_{L^p_\alpha(\partial\Omega \times \mathbb{R}^3)} \leq C_1(\Omega) \left(\delta^{\frac{p-1}{p}} \|\nabla_x h\|_{L^p_\alpha(\Omega \times \mathbb{R}^3)} + \delta^{-\frac{1}{p}} \|h\|_{L^p_\alpha(\Omega \times \mathbb{R}^3)} \right)
$$

for all $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ and $0 < \delta < 1$.

We remark that the constant $C_1(\Omega)$ in Lemma 3.5 can be large when $\mathrm{diam}(\Omega)$ is small.

Lemma 3.5 is not enough to obtain the desired estimate when $p = 1$. Thus, we need to introduce another trace estimate.

Lemma 3.6 ([11])**.** *Let Ω be a bounded domain with C^2 boundary, and let $\alpha \geq 0$. Then, for any $\delta > 0$, there exists a positive constant $C_\delta(\Omega)$ such that*

$$
\|h\|_{L^1_\alpha(\partial\Omega \times \mathbb{R}^3)} \leq (1 + \delta) \|\nabla_x h\|_{L^1_\alpha(\Omega \times \mathbb{R}^3)} + C_\delta(\Omega) \|h\|_{L^1_\alpha(\Omega \times \mathbb{R}^3)}
$$

for all $h \in W^{1,1}_\alpha(\Omega \times \mathbb{R}^3)$.

Combining the above lemmas and taking δ and $\mathrm{diam}(\Omega)$ sufficiently small, we obtain

$$
\|(S_\Omega K)^i J g\|_{W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)} \leq \frac{1}{2} \left\| (S_\Omega K)^{i-1} J g \right\|_{W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)} + C_2(\Omega) \|(S_\Omega K)^{i-1} J g\|_{L^p_\alpha(\Omega \times \mathbb{R}^3)}
$$

for all $i \geq 1$, where $C_2(\Omega)$ is a positive constant which may depend on Ω. Summing it from $i = 1$ to n, we obtain

$$
\frac{1}{2} \sum_{i=0}^{n} \|(S_\Omega K)^i J g\|_{W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)}
$$

$$\leq \sum_{i=1}^{n} \left(\|(S_\Omega K)^i Jg\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} - \frac{1}{2}\|(S_\Omega K)^{i-1} Jg\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \right) + \|Jg\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)}$$

$$\leq \|Jg\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + C_2(\Omega) \sum_{i=1}^{n} \|(S_\Omega K)^i Jg\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)}.$$

With the help of Lemma 3.2, the above estimate converges as $n \to \infty$. Thus, the series (5) converges in $W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ for fixed $1 \leq p < 2$ and $0 \leq \alpha < (1-\rho)/2$, which implies the existence of the $W^{1,p}_\alpha$ solution assuming $Jg \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$. On the other hand, if there exists a $W^{1,p}_\alpha$ solution to the integral equation, then we have $Jg = f - S_\Omega K f \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$. This proves the first statement of Theorem 1.1.

For the case $2 \leq p < 3$, we need to use a good property of positive Gaussian curvature.

Lemma 3.7 ([3]). *Let Ω be a C^2 bounded convex domain of positive Gaussian curvature. Then, there exists a positive constant $C_3(\Omega)$ depending only on Ω such that for any $(z,v) \in \Gamma^-$ we have*

$$|z - q(z,-v)| \leq C_3(\Omega) N(z,v).$$

By Lemma 3.7, we have

$$\tau(z,-v) \leq C_3(\Omega) \frac{N(z,v)}{|v|},$$

with which we may improve the estimate (6) as

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} |(\nabla_x \tau(x,v)) e^{-\nu(v)\tau(x,v)} Kh(q(x,v),v)|^p e^{p\alpha|v|^2}\, dxdv \\
\leq &\int_{\mathbb{R}^3} \int_\Omega \frac{1}{N(x,v)^p |v|^p} e^{-p\nu_0 \tau(x,v)} |Kh(q(x,v),v)|^p e^{p\alpha|v|^2}\, dxdv \\
= &\int_{\mathbb{R}^3} \int_{\Gamma^-_v} \int_0^{\tau(z,-v)} \frac{1}{N(z,v)^p |v|^p} e^{-p\nu_0 s} |Kh(z,v)|^p N(z,v)|v| e^{p\alpha|v|^2}\, ds d\Sigma(z) dv \\
\lesssim &C_3(\Omega) \int_{\partial\Omega} \int_{\Gamma^-_z} \frac{1}{|v|^p} \frac{1}{N(z,v)^{p-2}} |Kh(z,v)|^p\, e^{p\alpha|v|^2} dv d\Sigma(z) \\
\lesssim &C_3(\Omega) \|h\|^p_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}
\end{aligned}$$

for $2 \leq p < 3$. We may apply this effect to the v derivative to obtain the following estimate.

Lemma 3.8. *Let Ω be a C^2 bounded convex domain of positive Gaussian curvature, and let $C_3(\Omega)$ be a constant defined in Lemma 3.7. Then, given $h \in W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ with $2 \leq p < 3$ and $0 \leq \alpha < (1-\rho)/2$, where ρ is the constant in* **Assumption A**, *we have*

$$\|S_\Omega K h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} \lesssim \operatorname{diam}(\Omega)^{\frac{1}{p}} \|h\|_{W^{1,p}_\alpha(\Omega\times\mathbb{R}^3)} + \|h\|_{L^p_\alpha(\Omega\times\mathbb{R}^3)} + C_3(\Omega)^{\frac{1}{p}} \|h\|_{L^p_\alpha(\partial\Omega\times\mathbb{R}^3)}.$$

In the same way as in the proof of the statement (i), by Lemma 3.8 and Lemma 3.5, we may conclude that the series (5) converges in $W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ for $2 \leq p < 3$ and

$0 \le \alpha < (1-\rho)/2$ if $\mathrm{diam}(\Omega)$ is sufficiently small, which is the second statement of Theorem 1.1.

We show the third statement of Theorem 1.1 in the hard sphere model $\gamma = 1$ by just providing counterexamples. For the detailed argument, see [5].

For the first statement, we have the following lemma.

Lemma 3.9. *For fixed $1 \le p < 2$ and $0 \le \alpha < (1-\rho)/2$, there exist a bounded convex domain Ω and a boundary data g such that the boundary value problem* (2) *has a solution in $L^2(\Omega \times \mathbb{R}^3) \cap W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ but this solution does not belong to $W^{1,2}_\alpha(\Omega \times \mathbb{R}^3)$.*

An example of the domain Ω and the boundary data g is given as follows. We choose Ω as a small bounded convex domain such that

$$D_{r_1} := \{x = (0, x_2, x_3) \in \mathbb{R}^3 \mid |x| < r_1\} \subset \partial\Omega$$

with a small radius r_1 and

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x| < r_1, x_1 < 0\} \subset \Omega.$$

Also, let φ_1 be a smooth cut-off function on $\partial\Omega$ such that $0 \le \varphi_1 \le 1$, $\varphi_1(x) = 1$ for $x \in D_{r_1/4}$, and $\varphi_1(x) = 0$ for $x \in \partial\Omega \setminus D_{r_1/2}$. We pose the boundary data g of the form:

$$g(x, v) = \varphi_1(x) e^{-\frac{1}{2}|v|^2}, \quad (x, v) \in \Gamma^-.$$

A zoomed picture of the function φ_1 near $x = 0$ is given as Figure 1.

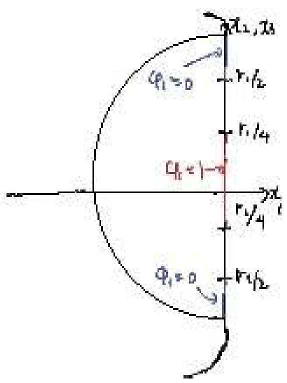

Figure 1: An example of Lemma 3.9

For the statement (ii), we have the following lemma.

Lemma 3.10. *For fixed $2 \le p < 3$ and $0 \le \alpha < (1-\rho)/2$, there exist a bounded convex domain Ω with its boundary of positive Gaussian curvature and a boundary data g such that the boundary value problem* (2) *has a solution in $L^3(\Omega \times \mathbb{R}^3) \cap W^{1,p}_\alpha(\Omega \times \mathbb{R}^3)$ but this solution does not belong to $W^{1,3}_\alpha(\Omega \times \mathbb{R}^3)$.*

An example of the domain Ω and the boundary data g is given as follows. Let Ω be a small ball centered at the origin with radius r. We introduce the spherical coordinates on the boundary: $x = (r\cos\theta, r\sin\theta\cos\phi, r\sin\theta\sin\phi)$ for $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$. With these coordinates, for $\theta_0 \in (0,\pi)$, let $\partial\Omega_{\theta_0} := \{x \in \partial\Omega \mid 0 \leq \theta < \theta_0\}$. Take $0 < \theta_1 < \theta_2 < \pi$ and a smooth cut-off function φ_2 on $\partial\Omega$ such that $\varphi_2(x) = 1$ for $x \in \partial\Omega_{\theta_1}$, $\varphi_2(x) = 0$ for $x \in \partial\Omega \setminus \partial\Omega_{\theta_2}$, and $0 \leq \varphi_2(x) \leq 1$ for $x \in \partial\Omega_{\theta_2} \setminus \partial\Omega_{\theta_1}$. We pose the boundary data g of the form:

$$g(x,v) = \varphi_2(x)e^{-\frac{1}{2}|v|^2}, \quad (x,v) \in \Gamma^-.$$

A picture of the function φ_2 is given as Figure 2.

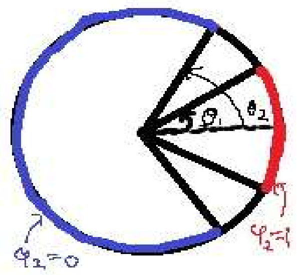

Figure 2: An example of Lemma 3.10

4 $W^{1,p}$ regularity for the nonlinear problem

In this section, we give a proof of Theorem 1.6 based on [6].

The integral form of (4) reads:

$$f = S_\Omega Kf + Jg + S_\Omega\phi, \quad \phi = \Gamma(f,f). \tag{7}$$

The formal solution for given ϕ is

$$f = \sum_{i=0}^{\infty}(S_\Omega K)^i(Jg + S_\Omega\phi).$$

Based on the relationship, we introduce the following iteration scheme:

$$\begin{cases} v\cdot\nabla_x f_1 + \nu(v)f_1 = Kf_1 & \text{in } \Omega\times\mathbb{R}^3, \\ f_1 = g & \text{on } \Gamma^-, \end{cases}$$

and

$$\begin{cases} v\cdot\nabla_x f_{j+1} + \nu(v)f_{j+1} = Kf_{j+1} + \Gamma(f_j,f_j) & \text{in } \Omega\times\mathbb{R}^3, \\ f_{j+1} = g & \text{on } \Gamma^-. \end{cases}$$

We make use of the smallness assumption in order to apply the contraction mapping theorem in $\hat{L}^\infty_\alpha$.

For the linear theory, in the same way as in the proof of Lemma 3.1, we have the following estimate.

Lemma 4.1. *Suppose* **Assumption A** *and let* $0 \le \alpha < (1-\rho)/2$. *Also, let* Ω *be a bounded convex domain with* C^2 *boundary of positive Gaussian curvature with uniform circumscribed and interior radii* R *and* r *respectively. Then, given* $h \in \hat{L}^\infty_\alpha$, *we have*

$$\|S_\Omega K h\|_{\infty,\alpha} \lesssim (1 + \operatorname{diam}(\Omega))|h|_{\infty,\alpha} + (Rr)^{\frac{1}{2}} \left(1 + \frac{R}{r}\right) \|h\|_{\infty,\alpha}.$$

By the contraction mapping argument, we obtain the following existence result.

Corollary 4.2. *Let* ϕ *be a function such that* $S_\Omega \phi \in \hat{L}^\infty_\alpha$. *Suppose* **Assumption A**. *Then, given* $0 < \alpha < (1-\rho)/2$, *where* ρ *is the constant in* **Assumption A**, *there exists a positive constant* δ *such that: For any bounded convex domain* Ω *with* C^2 *boundary of positive Gaussian curvature with uniform circumscribed and interior sphere radii* R *and* r *respectively, if*

$$\max\left\{\operatorname{diam}(\Omega), (Rr)^{\frac{1}{2}} \left(1 + \frac{R}{r}\right)\right\} < \delta,$$

there exists a solution $f \in \hat{L}^\infty_\alpha$ *to the integral equation* (7). *Moreover, we have*

$$\|f\|_{\infty,\alpha} \lesssim \|S_\Omega \phi\|_{\infty,\alpha} + \|Jg\|_{\infty,\alpha}.$$

For the nonlinear theory, we need to verify that $S_\Omega \Gamma(f_i, f_i)$ belongs $\hat{L}^\infty_\alpha$ and that the generated function $\{f_i\}$ converges to a function f in $\hat{L}^\infty_\alpha$. It suffices to show the following lemma. For a proof, see [6].

Lemma 4.3. *Let* $0 \le \alpha < (1-\rho)/2$. *where* ρ *is the constant in* **Assumption A**. *Also, let* Ω *be a bounded convex domain with* C^2 *boundary of positive Gaussian curvature with uniform circumscribed and interior radii* R *and* r *respectively. Then, for* $h_1, h_2 \in \hat{L}^\infty_\alpha$, *we have*

$$\|S_\Omega \Gamma(h_1, h_2)\|_{\infty,\alpha} \lesssim \left(1 + \operatorname{diam}(\Omega) + (Rr)^{\frac{1}{2}} \left(1 + \frac{R}{r}\right)\right) \|h_1\|_{\infty,\alpha} \|h_2\|_{\infty,\alpha}.$$

By Corollary 4.2 and Lemma 4.3, we have

$$\begin{aligned}
\|f_{i+1}\|_{\infty,\alpha} \lesssim& \|S_\Omega \Gamma(f_i, f_i)\|_{\infty,\alpha} + \|Jg\|_{\infty,\alpha} \\
&\lesssim \left(1 + \operatorname{diam}(\Omega) + (Rr)^{\frac{1}{2}} \left(1 + \frac{R}{r}\right)\right) \|f_i\|^2_{\infty,\alpha} + \|Jg\|_{\infty,\alpha}.
\end{aligned}$$

Hence, by the assumption that $\operatorname{diam}(\Omega)$ and $(Rr)^{\frac{1}{2}} \left(1 + \frac{R}{r}\right)$ is small enough, we have

$$\|f_{i+1}\|_{\infty,\alpha} \le C\|f_i\|^2_{\infty,\alpha} + C\|Jg\|_{\infty,\alpha}$$

for some constant $C > 1$. We further take $\delta > 0$ so small that $\delta < 1/4C^2$ to achieve

$$\|f_1\|_{\infty,\alpha} \le \frac{1}{4C} \le \frac{1}{2C}.$$

Also, if $\|f_i\|_{\infty,\alpha} \le 1/2C$ for some i, we have

$$\|f_{i+1}\|_{\infty,\alpha} \le \frac{1}{2}\|f_i\|_{\infty,\alpha} + \frac{1}{4C} \le \frac{1}{2C}.$$

Hence, by induction, the $\hat{L}^\infty_\alpha$ norm of the sequence f_i is uniformly bounded by $1/2C$. Furthermore, by the subtraction, we have

$$\begin{cases} v \cdot \nabla_x(f_{i+1} - f_i) + \nu(v)(f_{i+1} - f_i) \\ \quad = K(f_{i+1} - f_i) + \Gamma(f_i, f_i) - \Gamma(f_{i-1}, f_{i-1}), \quad (x, v) \in \Omega \times \mathbb{R}^3, \\ f_{i+1}(x, v) - f_i(x, v) = 0, \quad (x, v) \in \Gamma^-. \end{cases}$$

Notice that $\Gamma(f_i, f_i) - \Gamma(f_{i-1}, f_{i-1}) = \Gamma(f_i, f_i - f_{i-1}) + \Gamma(f_i - f_{i-1}, f_{i-1})$. Hence, we have

$$\begin{aligned} \|f_{i+1} - f_i\|_{\infty,\alpha} &\lesssim \left(1 + \mathrm{diam}(\Omega) + (Rr)^{\frac{1}{2}}\left(1 + \frac{R}{r}\right)\right) \\ &\quad \times \left(\|f_i\|_{\infty,\alpha}\|f_i - f_{i-1}\|_{\infty,\alpha} + \|f_i - f_{i-1}\|_{\infty,\alpha}\|f_{i-1}\|_{\infty,\alpha}\right) \\ &\lesssim \left(1 + \mathrm{diam}(\Omega) + (Rr)^{\frac{1}{2}}\left(1 + \frac{R}{r}\right)\right)\frac{1}{C}\|f_i - f_{i-1}\|_{\infty,\alpha}. \end{aligned}$$

In the last line, we use the uniform bound $1/2C$ of $\|f_{i+1}\|_{\infty,\alpha}$. With small enough $\mathrm{diam}(\Omega)$ and $(Rr)^{\frac{1}{2}}\left(1 + \frac{R}{r}\right)$ we have

$$\|f_{i+1} - f_i\|_{\infty,\alpha} \lesssim \frac{1}{C}\|f_i - f_{i-1}\|_{\infty,\alpha}.$$

By taking C sufficiently large, we finally deduce that

$$\|f_{i+1} - f_i\|_{\infty,\alpha} \le \frac{1}{2}\|f_i - f_{i-1}\|_{\infty,\alpha},$$

which implies that $\{f_i\}$ is a Cauchy sequence in $\hat{L}^\infty_\alpha$. Hence we achieve the convergence in $\hat{L}^\infty_\alpha$ of the iteration scheme when $\mathrm{diam}(\Omega)$, $(Rr)^{\frac{1}{2}}\left(1 + \frac{R}{r}\right)$ and $\|Jg\|_{\infty,\alpha}$ is small enough. This completes the proof of Theorem 1.6.

5 Conclusion

In this article, we discussed $W^{1,p}$ regularity of solutions to the incoming boundary value problem of nonlinear and linearized Boltzmann equations in small convex domains. At least in the linearized level, we observe that the geometry of the domain plays a crucial role for the range of the exponent p. Concerning the nonlinear problem, we recently removed the smallness assumption on the domain to obtain the same regularity result [4].

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