The linearized Stationary Boltzmann Equation in a bounded Convex Domain

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1 An application of the Velocity averaging lemma

The Boltzmann equation reads

$$\partial_t F + v \cdot \nabla_x F = Q(F, F). \tag{1}$$

Here, $F = F(t, x, v) \geq 0$ is the distribution function for the particles located in the position x with velocity v at time t. The collision operator Q is defined as

$$Q(F,G) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} \Big(F(v')G(v'_*) - F(v)G(v_*) \Big) B(|v - v_*|, \theta) \, d\theta d\epsilon dv_*, \tag{2}$$

where v' and v'_* are the velocities after the elastic collision of two particles whose velocities are v and v_*, respectively, before the encounter. Here, the cross-section B is chosen according to the type of interaction between particles. We set

$$e_1 = \frac{v_* - v}{|v_* - v|}$$

and choose $e_2 \in \mathbb{S}^2$ and $e_3 \in \mathbb{S}^2$ such that $\{e_1, e_2, e_3\}$ forms an orthonormal basis for $\mathbb{R}^3$, and define

$$\alpha = \cos\theta e_1 + \sin\theta \cos\epsilon e_2 + \sin\theta \sin\epsilon e_3.$$

Then,

$$v' = v + \big((v_* - v)\cdot\alpha\big)\alpha, \tag{3}$$
$$v'_* = v_* - \big((v_* - v)\cdot\alpha\big)\alpha. \tag{4}$$

As have been adopted by many authors, regarding the cross-section, we consider Grad's angular cutoff potential [14] by assuming

$$0 \le B(|v - v_*|, \theta) \le C|v - v_*|^\gamma \cos\theta \sin\theta. \tag{5}$$

Our discussion includes hard sphere model ($\gamma = 1$), cutoff hard potential ($0 < \gamma < 1$), and cutoff Maxwellian molecular gases ($\gamma = 0$). Consider the stationary solution

$$F = M + M^{\frac{1}{2}} f, \tag{6}$$

where

$$M(v) = \pi^{-\frac{3}{2}} e^{-|v|^2}.$$

Plugging the expression (6) into (1) and discarding the nonlinear term, we arrive at the stationary linearized Boltzmann equation

$$v \cdot \nabla_x f(x, v) = L(f)(x, v), \tag{7}$$

with linearized collision operator L, which reads

$$L(f) = M^{-\frac{1}{2}}\big(Q(M, M^{\frac{1}{2}} f) + Q(M^{\frac{1}{2}} f, M)\big). \tag{8}$$

Under the assumption (5), L can be decomposed into a multiplicative operator and an integral operator

$$L(f) = -\nu(v) f + K(f). \tag{9}$$

Here, ν is a function of the velocity variable v behaving like $(1 + |v|)^\gamma$, i.e., there exist two positive constants ν_0 and ν_1, depending only on γ, such that

$$0 < \nu_0(1 + |v|)^\gamma < \nu(v) < \nu_1(1 + |v|)^\gamma, \tag{10}$$

for all $v \in \mathbb{R}^3$. The integral operator K reads

$$K(f)(x, v) = \int_{\mathbb{R}^3} f(x, v_*) k(v_*, v)\, dv_*,$$

where the collision kernel k is symmetric, that is, $k(v, v_*) = k(v_*, v)$. Notice that the assumption of the cross-section here is different from and more general than that in [4, 6]. The significant difference is that the operator K in the case we consider does not guarantee to have regularity in velocity variables.

Under the decomposition (9), then we consider the boundary value problem

$$\begin{cases} \nu(v)f(x,v) + v\cdot\nabla_x f(x,v) = K(f)(x,v), & \text{for } x\in\Omega,\ v\in\mathbb{R}^3, \\ \qquad\qquad\qquad f|_{\Gamma_-}(q,v) = g(q,v), & \text{for } (q,v)\in\Gamma_-. \end{cases} \tag{11}$$

The celebrated velocity averaging lemma reveals that the combination of transport and averaging in velocity yields regularity in space variable [12]. Recall the velocity averaging lemma in [10, 13]: Suppose u is an L^2 solution to the transport equation

$$v\cdot\nabla_x u = G(x,v), \qquad (x,v)\in\mathbb{R}^n\times\mathbb{R}^n,$$

where $G\in L^2$. Let

$$\bar{u}(x) := \int_{\mathbb{R}^n} u(x,v)\psi(v)\,dv,$$

where ψ is a bounded function with compact support. Then, we have

$$\bar{u}(x)\in\tilde{H}^{1/2}(\mathbb{R}^n).$$

Here, the Sobolev space is generalized to non-integer order via the Fourier transform as follows.

Definition 1.1. *We say $u:\mathbb{R}^3\to\mathbb{R}$ is in $\tilde{H}^s_x(\mathbb{R}^3)$ if*

$$\|u\|_{\tilde{H}^s_x(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{\frac{s}{2}}|\mathcal{F}(u)(\xi)|^2\,d\xi\right)^{\frac{1}{2}} < \infty, \tag{12}$$

where $\mathcal{F}(u)(\xi)$ is the Fourier transform of u, i.e.,

$$\mathcal{F}(\xi) = (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} u(x)e^{-i\xi\cdot x}\,dx$$

This velocity averaging lemma demonstrates that the regularity in the transport direction can be converted to the regularity in space variables after averaging with weight ψ. Hence, it is natural to adopt this technique to the study of regularity problem of linearized Boltzmann equation in the whole space [9].

In case the source term $\Psi(x,v)$ is imposed, the inhomogeneous stationary linearized Boltzmann equation in the whole space reads

$$\nu(v)f + v\cdot\nabla_x f = K(f) + \Psi(x,v). \tag{13}$$

We can rewrite it as an integral equation

$$\begin{aligned} f(x,v) &= \int_0^\infty e^{-\nu(v)t}[K(f)(x-vt,v)+\Psi(x-vt,v)]\,dt \\ &=: S(K(f)+\Psi) \\ &= SK(f)+S(\Psi), \end{aligned} \tag{14}$$

where

$$S(h)(x,v) := \int_0^\infty e^{-\nu(v)t} h(x-vt,t)\,dt. \tag{15}$$

Performing the Picard iteration, formally we can derive that

$$f = \sum_{k=0}^{\infty} S(KS)^k(\Psi). \tag{16}$$

By carefully adapting the idea of velocity averaging lemma, the following lemma was proved in [9].

Lemma 1.2. *The operator* $KSK : L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3)) \to L^2_v(\mathbb{R}^3; \tilde{H}^{s+\frac{1}{2}}_x(\mathbb{R}^3))$ *is bounded for any* $s \geq 0$.

Here, the mixed fractional Sobolev space is defined as follows.

Definition 1.3. *We say* $u : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *is in* $L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3))$ *if*

$$\|u\|_{L^2_v(\mathbb{R}^3;\tilde{H}^s_x(\mathbb{R}^3))} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (1+|\xi|^2)^{\frac{s}{2}} |\mathcal{F}(u)(\xi,v)|^2\,d\xi dv\right)^{\frac{1}{2}} < \infty, \tag{17}$$

where $\mathcal{F}(u)(\xi,v)$ *is the Fourier transform of* u *with respect to the space variable.*

However, when it comes to bounded space domains, the main tool of velocity averaging lemma, namely, the method of Fourier transform, does not translate well on a bounded domain. In [7], for the incoming boundary data, the authors overcome the obstacles arising from geometry and obtain a fractional regularity result in space variables for the linearized stationary Boltzmann equations in a bounded convex domain.

To this aim, for a bounded domain, we adopt the fractional Sobolev space through the Slobodeckij semi-norm.

Definition 1.4. *Let* $s \in (0,1)$*,* $\Omega \subset \mathbb{R}^3$ *open. We say* $f(x,v) \in L^2_v(\mathbb{R}^3; H^s_x(\Omega))$ *if* $f \in L^2_v(\mathbb{R}^3; L^2_x(\Omega))$ *and*

$$\int_{\mathbb{R}^3}\int_\Omega\int_\Omega \frac{|f(x,v)-f(y,v)|^2}{|x-y|^{3+2s}}\,dxdydv < \infty, \tag{18}$$

with

$$\|f\|_{L^2_v(\mathbb{R}^3;H^s_x(\Omega))} = \left(\|f\|^2_{L^2(\Omega\times\mathbb{R}^3)} + \int_{\mathbb{R}^3}\int_\Omega\int_\Omega \frac{|f(x,v)-f(y,v)|^2}{|x-y|^{3+2s}}\,dxdydv\right)^{\frac{1}{2}}. \tag{19}$$

Notice that Definition 1.3 and Definition 1.4 of fractional Sobolev spaces are equivalent on the whole space. In other words, for $0 < s < 1$, there exist two positive constants $C_1 = C_1(s)$ and $C_2 = C_2(s)$ such that

$$C_1\|u\|_{L^2_v(\mathbb{R}^3;H^s_x(\mathbb{R}^3))} \le \|u\|_{L^2_v(\mathbb{R}^3;\tilde{H}^s_x(\mathbb{R}^3))} \le C_2\|u\|_{L^2_v(\mathbb{R}^3;H^s_x(\mathbb{R}^3))} \tag{20}$$

for any $u \in L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3))$.

Here, we shall first introduce our main result and then explain the multiple obstacles we encountered and how we overcome them. We consider a bounded convex domain which satisfies the following assumption.

Definition 1.5. *We say a* C^2 *bounded convex domain* Ω *in* $\mathbb{R}^3$ *satisfies the positive curvature condition if* $\partial\Omega$ *is of positive Gaussian curvature.*

Remark 1.6. *Positive curvature condition implies uniform convexity, which would also imply strict convexity. If the domain is compact, then its being strict convexity is equivalent to being uniform convexity. On the contrary, a uniformly convex domain does not necessarily satisfy positive curvature condition.*

We consider the incoming boundary value problem for linearized Boltzmann equation in Ω

$$\begin{cases} v\cdot\nabla_x f(x,v) = L(f)(x,v), & \text{for } x \in \Omega,\ v \in \mathbb{R}^3, \\ \quad f|_{\Gamma_-}(q,v) = g(q,v), & \text{for } (q,v) \in \Gamma_-, \end{cases} \tag{21}$$

where

$$\Gamma_- := \{(q,v) \in \partial\Omega \times \mathbb{R}^3 :\ n(q)\cdot v < 0\},$$

and $n(q)$ is the unit outward normal of $\partial\Omega$ at q. In this context, L satisfies one of hard sphere, cutoff hard, and cutoff Maxwellian potentials.

We assume the following two conditions on the incoming data g.

Assumption 1.7. *There are positive constants a, C such that*

$$|g(q_1, v)| \leq C\, e^{-a|v|^2} \tag{22}$$

and

$$|g(q_1, v) - g(q_2, v)| \leq C\, |q_1 - q_2|, \tag{23}$$

for any $(q_1, v) \in \Gamma_-$ and $(q_2, v) \in \Gamma_-$.

The main regularity result obtained in [7] is the following theorem.

Theorem 1.8. *Suppose Ω satisfies the positive curvature condition in Definition 1.5, linearized collision operator L satisfies angular cutoff assumption (5), and incoming data g satisfies Assumption 1.7. Then the solution $f \in L^2(\Omega \times \mathbb{R}^3)$ for stationary linearized Boltzmann equation (21) belongs to $L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega))$ for any $0 < \epsilon < 1$.*

We shall sketch the proof and reveal the difficulties induced by geometry and the method we tackle the problem. For the technical details, see [7]. Let $x \in \Omega$ and $v \in \mathbb{R}^3$. We define

Definition 1.9.

$$\begin{aligned}
\tau_-(x, v) &= \inf_{t>0} \{t :\, x - vt \notin \Omega\}, \\
q_-(x, v) &= x - \tau_-(x, v)v, \\
\tau_+(x, v) &= \inf_{t>0} \{t :\, x + vt \notin \Omega\}, \\
q_+(x, v) &= x + \tau_+(x, v)v.
\end{aligned}$$

We can write down the integral equation

$$\begin{aligned}
f(x, v) =& e^{-\nu(v)\tau_-(x,v)} g(q_-(x, v), v) \\
&+ \int_0^{\tau_-(x,v)} e^{-\nu(v)s} K(f)(x - sv, v)\, ds.
\end{aligned} \tag{24}$$

Hereafter, we define

$$(Jg)(x, v) := e^{-\nu(v)\tau_-(x,v)} g(q_-(x, v), v), \tag{25}$$

$$(S_\Omega f)(x, v) := \int_0^{\tau_-(x,v)} e^{-\nu(v)s} f(x - sv, v) ds. \tag{26}$$

Notice that $S_\Omega : L^p(\Omega\times\mathbb{R}^3) \to L^p(\Omega\times\mathbb{R}^3)$ and $J : L^p(\Gamma_-; d\sigma) \to L^p(\Omega\times\mathbb{R}^3)$ are bounded for $1 \le p \le \infty$ with

$$d\sigma = |v \cdot n(q)|\, d\Sigma(q) dv,$$

where $\Sigma(q)$ is the surface element on $\partial\Omega$ at q. Performing Picard iteration, we have

$$\begin{aligned} f(x,v) =& J(g) + S_\Omega K(f) \\ =& J(g) + S_\Omega K J(g) + S_\Omega K S_\Omega K(f) \\ =& J(g) + S_\Omega K J(g) + S_\Omega K S_\Omega K J(g) + S_\Omega K S_\Omega K S_\Omega K(f) \\ =& J(g) + S_\Omega K J(g) + S_\Omega K S_\Omega K J(g) + S_\Omega K S_\Omega K S_\Omega K J(g) \\ & + S_\Omega K S_\Omega K S_\Omega K S_\Omega K(f) \\ =& g_0 + g_1 + g_2 + g_3 + f_4, \end{aligned} \tag{27}$$

where

$$g_i := (S_\Omega K)^i J(g), \tag{28}$$

$$f_i := (S_\Omega K)^i (f). \tag{29}$$

We observe that each g_i is directly under influence of boundary data and the geometry of the domain. Our strategy is to prove $g_i \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\mathbb{R}^3))$. And, concerning the remaining term f_4, we shall match up the regularity of boundary terms.

We point out the difference between the cases for the whole space and a bounded domain. Suppose $f \in L^2(\Omega \times \mathbb{R}^3)$ and consider its zero extension $\widetilde{f}$ in $\mathbb{R}^3 \times \mathbb{R}^3$. Notice that

$$SK(\widetilde{f})\Big|_\Omega = S_\Omega K(f). \tag{30}$$

Therefore, applying Lemma 1.2, we have

Corollary 1.10. *The operator* $KS_\Omega K : L^2(\Omega \times \mathbb{R}^3) \to L^2_v(\mathbb{R}^3; H^{1/2}_x(\Omega))$ *is bounded.*

Furthermore, from the geometric properties, we have

Lemma 1.11. *The operator* $S_\Omega K S_\Omega K : L^2(\Omega \times \mathbb{R}^3) \to L^2_v(\mathbb{R}^3; H^{1/2}_x(\Omega))$ *is bounded.*

Therefore, if we end iteration at f_2, we can already claim $f \in L^2_v(\mathbb{R}^3; H^{1/2}_x(\Omega))$.

Considering piling up the regularity, however, notice that

$$SKSK(\widetilde{f})\Big|_{\Omega} \neq S_\Omega K S_\Omega K(f). \tag{31}$$

It seemingly comes to the limit of this strategy. However, surprisingly, some subtle properties of functional space on a bounded convex domain satisfies positive curvature condition allow us to improve the regularity one step further. We use $\widetilde{F}$ to denote the zero extension of F from $\Omega \times \mathbb{R}^3$ to $\mathbb{R}^3 \times \mathbb{R}^3$. We have

Lemma 1.12. $\widetilde{S_\Omega K S_\Omega K} : L^2(\Omega \times \mathbb{R}^3) \to L^2_v(\mathbb{R}^3; H_x^{\frac{1}{2}-\epsilon}(\mathbb{R}^3))$ *is bounded for any* $\epsilon \in (0, \frac{1}{2})$*. Furthermore, there is a constant* C *independent of* ϵ *and* f *such that*

$$\|\widetilde{S_\Omega K S_\Omega K f}\|_{L^2_v(\mathbb{R}^3; H_x^{\frac{1}{2}-\epsilon}(\mathbb{R}^3))} \leq \frac{C}{\sqrt{\epsilon}} \|f\|_{L^2(\Omega \times \mathbb{R}^3)}. \tag{32}$$

That is, zero extension only reduces infinitesimal regularity. Therefore, after zero extension, we can repeat our strategy and obtain the desired result.

Regarding the existence result of boundary value problem (21), it has been studied by Guiraud [16] for convex domains and, for general domains, by Esposito, Guo, Kim, and Marra [11].

In the paper of Esposito, Guo, Kim, and Marra [11], they proved the solution is continuous away from the grazing set. With stronger assumption on cross-section B, namely,

$$\begin{aligned} &B(|v - v_*|, \theta) = |v - v_*|^\gamma \beta(\theta), \\ &0 \leq \beta(\theta) \leq C \sin\theta \cos\theta, \end{aligned} \tag{33}$$

the interior Hölder estimate was established in [5] and later improved to interior pointwise estimate for first derivatives [6]. Notice that, in [5, 6], the fact K improves regularity in velocity are key properties used. The idea is to move the regularity in velocity to space through transport and collision. This idea was inspired by the mixture lemma by Liu and Yu [20]. In contrast, in the present result, we do not need the smoothing effect of K in velocity; the integral operator K itself provides "velocity averaging" and therefore regularity. Regarding regularity issues for the time dependent Boltzmann equation, we refer the interested readers to [17, 18].

2 Small domain problem

Concerning the problem (21) in a small convex domain, in [8], the authors give the necessary and sufficient conditions for the existence theory in $H^1(\Omega \times \mathbb{R}^3)$. We assume that $L(f)$ satisfies the following assumption.

Assumption A. *The operator $L(f)$ can be decomposed into the multiplicative term $-\nu(v)f(x,v)$ and the integral operator term*

$$K(f)(x,v) := \int_{\mathbb{R}^3} k(v,v^*)f(x,v^*)dv^* \tag{34}$$

such that

$$k(v,v^*) = k(v^*,v), \tag{35}$$

$$\nu_0(1+|v|)^\gamma \leq \nu(v) \leq \nu_1(1+|v|)^\gamma, \tag{36}$$

$$|k(v,v^*)| \lesssim \frac{1}{|v-v^*|(1+|v|+|v^*|)^{1-\gamma}} e^{-\frac{1-\rho}{4}(|v-v^*|^2+(\frac{|v|^2-|v^*|^2}{|v-v^*|})^2)}, \tag{37}$$

$$|\nabla_v k(v,v^*)| \lesssim \frac{1+|v|}{|v-v^*|^2(1+|v|+|v^*|)^{1-\gamma}} e^{-\frac{1-\rho}{4}(|v-v^*|^2+(\frac{|v|^2-|v^*|^2}{|v-v^*|})^2)}, \tag{38}$$

$$|\nabla_v \nu(v)| \lesssim (1+|v|)^{\gamma-1}, \tag{39}$$

where $\rho \in (0,1)$ and $\gamma \in [0,1]$.

Remark 2.1. *1. If we adopt the idea of Grad [14] and consider the Grad angular cut-off potentials which include the hard sphere, hard potential, and Maxwellian molecular condition, then the condition of* (37) *and the upper bound of* (36) *hold. See Caflisch [1].*

2. It is worth mentioning that the commonly used cross section $B(|v-v^*|,\theta) = b|v-v^*|^\gamma \cos\theta$, where b is a positive constant, leads to all the estimates in Assumption A.

We assume that the domain Ω possesses the following property.

Assumption B. *$\Omega \subset \mathbb{R}^3$ is a C^2 bounded domain such that $\partial\Omega$ is of positive Gaussian curvature.*

Theorem 2.2. *Suppose L satisfies* **Assumption A**, *then there exists $\epsilon > 0$ such that: for any domain Ω satisfying* **Assumption B** *with* $\mathrm{diam}(\Omega) < \epsilon$, *the boundary value problem* (21) *has a unique solution $f \in H^1(\Omega \times \mathbb{R}^3)$ if and only if $Jg \in H^1(\Omega \times \mathbb{R}^3)$.*

We remark that the condition $Jg \in H^1$ in the statement of Theorem 2.2 is implicit. To demonstrate that there is a wide class of functions satisfying this condition, we have the following lemma.

Lemma 2.3. *Let $g : \Gamma^- \to \mathbb{R}$. Suppose g satisfies the following conditions.*

$$\int_{\mathbb{R}^3} \int_{\Gamma_v^-} |g(z,v)|^2 d\Sigma(z) dv < \infty, \tag{40}$$

$$\int_{\mathbb{R}^3} \frac{\nu(v)^2}{|v|^2} \int_{\Gamma_v^-} |g(z,v)|^2 d\Sigma(z) dv < \infty, \tag{41}$$

$$\int_{\mathbb{R}^3} \int_{\Gamma_v^-} |\nabla_x g(z,v)|^2 d\Sigma(z) dv < \infty, \tag{42}$$

$$\int_{\mathbb{R}^3} \int_{\Gamma_v^-} |\nabla_v g(z,v)|^2 N^(z,v)^2 d\Sigma(z) dv < \infty, \tag{43}$$

where

$$\Gamma_v^- := \{x \in \partial\Omega \mid n(x) \cdot v < 0\},$$
$$N(x,v) := -n(q(x,v)) \cdot \frac{v}{|v|},$$
$$|\nabla_x h(z,v)|^2 := g^{ij} \nabla_{\frac{\partial}{\partial x_i}} h \nabla_{\frac{\partial}{\partial x_j}} h,$$

$d\Sigma(z)$ is the surface measure of $\partial\Omega$ and g^{ij} is the (i,j) element of the inverse matrix of the metric tensor on $\partial\Omega$. Then we have $Jg \in H^1(\Omega \times \mathbb{R}^3)$.

The detailed definition of $|\nabla_x h(z,v)|^2$ and the proof is presented in the subsection. It can be verified by the following standard scaling analysis:

$$v \cdot \nabla_x f = \frac{1}{\kappa}(-\nu(v) f + K(f)), \quad (x,v) \in \Omega' \times \mathbb{R}^3. \tag{44}$$

Notice that by defining $f_\kappa(x,v) := f(\kappa x, v)$, we have

$$v \cdot \nabla_x f_\kappa = -\nu(v) f_\kappa + K(f_\kappa), (x,v) \in \Omega \times \mathbb{R}^3, \tag{45}$$

where $\Omega := \frac{1}{\kappa}\Omega'$.

We note that the smallness of domain or equivalently the diluteness of gas has been used to build up the existence theories for the Boltzmann equation, e.g., [2, 15]. In this article, we also take the advantage of the smallness of domain.

Lemma 2.4. *Let Ω be a bounded domain with C^2 boundary with positive Gaussian curvature. Then, there exists a positive constant $C(\Omega)$ depending on Ω such that for any $z \in \partial\Omega$ we have*

$$|z - q(z,v)| \leq C(\Omega) N(z,v). \tag{46}$$

2.1 Sketch of Proof

Here, we briefly sketch the idea of the proof of Theorem 1.1. A formal Picard iteration gives the following solution formula for (21)

$$f = \sum_{i=0}^{\infty} (S_\Omega K)^i Jg. \tag{47}$$

This is a valid solution of (21) if the right hand side of (47) converges in H^1. We first consider the L^2 convergence. We have

Lemma 2.5. *For any $h \in L^2(\Omega \times \mathbb{R}^3)$, we have*

$$\|S_\Omega K h\|_{L^2(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{2}} \|h\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{48}$$

By Lemma 2.5 above, we can see that $S_\Omega K$ is a contraction mapping in L^2 provided the diameter of Ω is small. This provides the L^2 convergence of (47) in case $\mathrm{diam}(\Omega)$ is small. However, concerning the H^1 convergence of (47), we do not have a direct analogy of Lemma 2.5. Instead, we establish the following key lemmas:

Lemma 2.6. *Given $h \in H^1(\Omega \times \mathbb{R}^3)$, we have*

$$\|S_\Omega K h\|_{H^1(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{2}} \|h\|_{H^1(\Omega\times\mathbb{R}^3)} + C(\Omega)\|h\|_{L^2(\partial\Omega\times\mathbb{R}^3)}, \tag{49}$$

where $\|h\|_{L^2(\partial\Omega\times\mathbb{R}^3)}$ is defined in trace sense.

Lemma 2.7. *Let $C(\Omega)$ be as defined in Lemma 2.4, then for any $h \in L^2(\Omega \times \mathbb{R}^3)$ we have*

$$\|S_\Omega K h\|_{L^2(\partial\Omega\times\mathbb{R}^3)} \lesssim C(\Omega)\|h\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{50}$$

Hence, we have

Corollary 2.8. *Let $C(\Omega)$ be as defined in Lemma 2.4, then for any $h \in L^2(\Omega \times \mathbb{R}^3)$ we have*

$$\|S_\Omega K S_\Omega K h\|_{H^1(\Omega\times\mathbb{R}^3)} \lesssim \mathrm{diam}(\Omega)^{\frac{1}{2}} \|S_\Omega K h\|_{H^1(\Omega\times\mathbb{R}^3)} + C(\Omega)^2\|h\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{51}$$

By Lemma 2.6 we have

$$\begin{aligned} &\|(S_\Omega K)^i Jg\|_{H^1(\Omega\times\mathbb{R}^3)} \\ \lesssim\, &\mathrm{diam}(\Omega)^{\frac{1}{2}} \|(S_\Omega K)^{i-1} Jg\|_{H^1(\Omega\times\mathbb{R}^3)} + C(\Omega)^2 \|(S_\Omega K)^{i-2} Jg\|_{L^2(\Omega\times\mathbb{R}^3)}. \end{aligned}$$

For small diam(Ω), we have

$$
\begin{aligned}
&\|(S_\Omega K)^i Jg\|_{H^1(\Omega\times\mathbb{R}^3)}\\
\le&\frac{1}{2}\|(S_\Omega K)^{i-1}Jg\|_{H^1(\Omega\times\mathbb{R}^3)}+CC(\Omega)^2\|(S_\Omega K)^{i-2}Jg\|_{L^2(\Omega\times\mathbb{R}^3)}.
\end{aligned}
\tag{52}
$$

Combining (52) with Lemma 2.5, we conclude the H^1 convergence of (47) for the case where diam(Ω) is small. Notice that the constant $C(\Omega)$ depends on Ω. More precisely, it depends on the maximum radius of curvature. Nevertheless, it is the diameter of Ω that affects the convergence, no mater what value $C(\Omega)$ is. Uniqueness of the H^1 solution follows from the contraction mapping argument on the L^2 space.

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