Helmholtz-Weyl decomposition on a time dependent domain with an application to time periodic Navier-Stokes flows with large flux

Takahiro Okabe
Graduate School of Engineering Science

Abstract

In this résumé, we study the Helmholtz-Weyl decomposition on a time dependent bounded domain $\Omega(t)$ in $\mathbb{R}^3$. Especially, we investigate the domain dependence of each component in the decomposition, namely, the harmonic vector fields (i.e., div and rot free vectors), vector potentials, and scalar potentials equipped with suitable boundary conditions, when $\Omega(t)$ moves along to $t \in \mathbb{R}$.

Under the domain perturbation, a new approach for analyzing the parameter-dependence of the solutions is achieved, even though the transformed elliptic systems have nontrivial null spaces. As an application, we consider a time periodic solution of the incompressible Navier-Stokes equations for some large boundary data with non-zero fluxes.

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the smooth boundary. We consider the Helmholtz-Weyl decomposition for a solenoidal vector field b on Ω, i.e., $\operatorname{div} b = 0$ in Ω, such as

$$b = h + \operatorname{rot} w \quad \text{in } \Omega, \tag{1.1}$$

where h is a harmonic vector field, $\operatorname{div} h = 0$, $\operatorname{rot} h = 0$ in Ω with the boundary condition $h \times \nu = 0$ on $\partial\Omega$, and w satisfies $\operatorname{div} w = 0$ in Ω with $w \cdot \nu = 0$ on $\partial\Omega$. Here, ν denotes the unit outward normal vector to $\partial\Omega$. Such a decomposition for smooth vectors was proved by Weyl [26], and Bendali, Dominguez and Gallic [2] expanded it in the Sobolev space $H^s(\Omega)$, $s \geq 0$. In recent years, Kozono and Yanagisawa [15] derived (1.1) for L^r-vector fields, $1 < r < \infty$. Furthermore, many relevant studies have been developed by Foiaş and Temam [7], Yoshida and Giga [27], Griesinger [11] and Bolik and von Wahl [4], and so on.

In this article, introducing a time dependent domain $\Omega(t)$ for $t \in \mathbb{R}$, we investigate the Helmholtz-Weyl decomposition (1.1) on each $\Omega(t)$, i.e.,

$$b(t) = h(t) + \operatorname{rot} w(t) \quad \text{in } \Omega(t). \tag{1.2}$$

For (1.2), our main interest is to clarify the domain dependence of the harmonic vector field $h(t)$ and the vector potential $w(t)$ when $\Omega(t)$ moves along to time $t \in \mathbb{R}$. Especially, we try to establish the (time-)parameter dependence of $h(t)$ and $w(t)$, i.e., the time continuity and time differentiability for $h(t)$ and $w(t)$ in suitable function spaces.

The Helmholtz-Weyl decomposition has been developed and applied in various fields of the mathematical and numerical analysis for the electromagnetism, magnetohydrodynamics and also fluid mechanics. In particular, it is suggested that (1.1) plays an effective role in the solvability of the boundary value problem of the incompressible Navier-Stokes equations. Indeed, using (1.1) Kozono and Yanagisawa [16] constructed the stationary solution for large boundary data. The second author applied it to the time periodic problem of the Navier-Stokes equations. From this viewpoint, to investigate the domain dependence of (1.2) gives us a way to apply it to such problems on a time dependent domain $\Omega(t)$.

As an application of the decomposition, we consider the time periodic problem of the incompressible Navier-Stokes equations on a time dependent domain $\Omega(t) \subset \mathbb{R}^3$ which is bounded with smooth boundary $\partial\Omega(t)$ for $t \in \mathbb{R}$:

$$\begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi = f \quad \text{in } \bigcup_{t\in\mathbb{R}} \Omega(t) \times \{t\}, \\ \operatorname{div} v = 0 \quad \text{in } \bigcup_{t\in\mathbb{R}} \Omega(t) \times \{t\}, \\ v = \beta \quad \text{on } \bigcup_{t\in\mathbb{R}} \partial\Omega(t) \times \{t\}. \end{cases} \tag{N-S}$$

Here, $v = v(x,t) = \big(v^1(x,t), v^2(x,t), v^3(x,t)\big)$ and $\pi(x,t)$ denote the unknown velocity and pressure of the fluid at $(x,t) \in \bigcup_{t\in\mathbb{R}} \Omega(t) \times \{t\}$, respectively. While $f = f(x,t) = \big(f^1(x,t), f^2(x,t), f^3(x,t)\big)$ is the given external force and $\beta = \beta(x,t) = \big(\beta^1(x,t), \beta^2(x,t), \beta^3(x,t)\big)$ is the given boundary data.

2 Preliminaries

2.1 Assumption on the domain

To begin with, we impose the following assumption on $\{\Omega(t)\}_{t\in\mathbb{R}}$.

Assumption. $\{\Omega(t)\}_{t\in\mathbb{R}}$ is a family of bounded domains in $\mathbb{R}^3$ with smooth boundary $\partial\Omega(t)$ and satisfies the followings.

(i) For the noncylindrical domain $Q_\infty = \bigcup_{t\in\mathbb{R}} \Omega(t) \times \{t\}$, there exist a cylindrical domain $\widetilde{Q}_\infty = \widetilde{\Omega} \times \mathbb{R}$ and a level-preserving C^∞ diffeomorphism $\Phi : \overline{Q}_\infty \to \overline{\widetilde{Q}}_\infty$;

$$(y,s) = \Phi(x,t) := \big(\phi(x,t), t\big) := \big(\phi^1(x,t), \phi^2(x,t), \phi^3(x,t), t\big)$$

such that

$$\det\left(\frac{\partial \phi^i}{\partial x^j}(x,t)\right) \equiv J(t)^{-1} > 0 \quad \text{for } (x,t) \in \overline{Q}_\infty. \tag{2.1}$$

(ii) There exists $K \in \mathbb{N}$ such that for all $t \in \mathbb{R}$, $\partial\Omega(t)$ consists of $K+1$ connected components $\Gamma_0(t), \dots, \Gamma_K(t)$ of C^∞ closed surfaces which satisfy that $\Gamma_1(t), \dots, \Gamma_K(t)$

lie inside of $\Gamma_0(t)$, $\Gamma_k(t) \cap \Gamma_{k'}(t) = \emptyset$ for $k \neq k'$, and that

$$\partial\Omega(t) = \bigcup_{k=0}^{K} \Gamma_k(t). \tag{2.2a}$$

Furthermore, there exists $L \in \mathbb{N}$ such that

$$\dot{\Omega}(t) := \Omega(t) \setminus \Sigma(t) \text{ is simply connected for } t \in \mathbb{R}, \text{ where } \Sigma(t) := \bigcup_{\ell=1}^{L} \Sigma_\ell(t), \tag{2.2b}$$

and $\Sigma_1(t), \dots, \Sigma_L(t)$ are C^∞ surfaces transversal to $\partial\Omega(t)$ with $\Sigma_\ell(t) \cap \Sigma_{\ell'}(t) = \emptyset$ for $\ell \neq \ell'$, for all $t \in \mathbb{R}$. Here, we may assume that $\phi\big(\Sigma(t), t\big) = \phi\big(\Sigma(t'), t'\big)$ for all $t, t' \in \mathbb{R}$, without loss of generality.

Remark 2.1. As is mentioned in Miyakawa and Teramoto [20, Theorem 4.3], the condition (2.1) is of no restriction.

In this article, we express the inverse of Φ as

$$\Phi^{-1}(y, s) = \big(\phi^{-1}(y, s), s\big) := \big(\phi_1^{-1}(y, s), \phi_2^{-1}(y, s), \phi_3^{-1}(y, s), s\big) \quad \text{for } (y, s) \in \widetilde{Q}_\infty. \tag{2.3}$$

Moreover, for $T > 0$ we put

$$Q_T := \bigcup_{0<t<T} \Omega(t) \times \{t\} \quad \text{and} \quad \widetilde{Q}_T := \widetilde{\Omega} \times (0, T). \tag{2.4}$$

Here, it should be noted that since $\phi(\cdot, t) : \Omega(t) \ni x \mapsto y \in \widetilde{\Omega}$ is a diffeomorphism, we introduce a vector field $\widetilde{u}$ on $\bigcup_{0<t<\infty} \Omega(t) \times \{t\}$ are defined by, for $i = 1, 2, 3$,

$$\widetilde{u}^i(y, s) := \sum_\ell \frac{\partial y^i}{\partial x^\ell} u^\ell(\phi^{-1}(y, s), s) := \sum_\ell \frac{\partial \phi^i}{\partial x^\ell}\big(\phi^{-1}(y, s), s\big) u^\ell\big(\phi^{-1}(y, s), s\big), \tag{2.5}$$

for $(y, s) \in \widetilde{\Omega} \times (0, \infty)$, and a scalar function $\widetilde{\pi}$ on $\widetilde{\Omega} \times (0, \infty)$ defined by

$$\widetilde{\pi}(y, s) := \pi\big(\phi^{-1}(y, s), s\big), \tag{2.6}$$

for $(y, s) \in \widetilde{\Omega} \times (0, \infty)$, when the vector field u and the function π on $\bigcup_{0<t<\infty} \Omega(t) \times \{t\}$ are given.

3 Main results

In order to state our main results, let us introduce the Helmholtz-Weyl decomposition on a bounded domain Ω, according to Kozono and Yanagisawa [15]. For this purpose, we introduce the following function spaces,

$$\begin{aligned}
V_{\mathrm{har}}(\Omega) &:= \big\{h \in C^\infty(\overline{\Omega})\,;\, \operatorname{div} h = \operatorname{rot} h = 0 \text{ in } \Omega,\ \ h \times \nu = 0 \text{ on } \partial\Omega\big\},\\
X_{\mathrm{har}}(\Omega) &:= \big\{u \in C^\infty(\overline{\Omega})\,;\, \operatorname{div} u = \operatorname{rot} u = 0 \text{ in } \Omega,\ \ u \cdot \nu = 0 \text{ on } \partial\Omega\big\},\\
X^2_\sigma(\Omega) &:= \big\{u \in L^2(\Omega)\,;\, \operatorname{div} u = 0 \text{ in } \Omega,\ \ \operatorname{rot} u \in L^2(\Omega),\ \ u \cdot \nu = 0 \text{ on } \partial\Omega\big\},\\
Z^2_\sigma(\Omega) &:= \big\{u \in X^2_\sigma(\Omega)\,;\, u \perp X_{\mathrm{har}}(\Omega)\big\},
\end{aligned}$$

for $\Omega \subset \mathbb{R}^3$ which is a bounded domain with the smooth boundary, where ν denotes the outward unit normal vector to $\partial\Omega$ and we consider $u \cdot \nu = 0$ in weak sense, i.e., $\langle u \cdot \nu, \psi \rangle_{\partial\Omega} = 0$ for all $\psi \in C^\infty(\overline{\Omega})$, where

$$\langle u \cdot \nu, \psi \rangle_{\partial\Omega} := (u, \nabla\psi) + (\operatorname{div} u, \psi) \quad \text{for all } \psi \in C^\infty(\overline{\Omega}).$$

Kozono and Yanagisawa [15] established the Helmholtz-Weyl decomposition for L^r-vector fields, for $1 < r < \infty$. In our situation, since we deal with only H^1-framework, we prepare the following proposition, excerpting the H^1-part from [15, Theorem 2.1 and 2.4].

Proposition 3.1 ([15, Theorem 2.1 and 2.4]). *(i) For every $b \in H^1(\Omega)$ which satisfies $\operatorname{div} b = 0$ in Ω, then there uniquely exist a harmonic vector field $h \in V_{\mathrm{har}}(\Omega)$ and a vector potential $w \in Z^2_\sigma(\Omega) \cap H^2(\Omega)$ such that*

$$h + \operatorname{rot} w = b \quad \text{in } \Omega, \tag{3.1}$$

with the estimate

$$\|h\|_{H^1(\Omega)} + \|w\|_{H^2(\Omega)} \le C\|b\|_{H^1(\Omega)},$$

where the constant $C > 0$ depends on Ω, but is independent of b.

(ii) For every $f \in H^1(\Omega)$, there uniquely exist a harmonic vector field $h \in V_{\mathrm{har}}(\Omega)$, a vector potential $w \in Z^2_\sigma(\Omega) \cap H^2(\Omega)$ and a scalar potential $p \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$h + \operatorname{rot} w + \nabla p = f \quad \text{in } \Omega,$$

with the estimate

$$\|h\|_{H^1(\Omega)} + \|w\|_{H^2(\Omega)} + \|p\|_{H^2(\Omega)} \le C\|f\|_{H^1(\Omega)},$$

where the constant $C > 0$ depends on Ω, but is independent of f.

Remark 3.1. In the decomposition, the uniqueness and regularity of the vector potential w within $Z^2_\sigma(\Omega)$ immediately follows from [15, Theorem 2.4], since $w \perp X_{\mathrm{har}}(\Omega)$.

Here, the following theorem is on the domain dependence, i.e., time dependence of the Helmholtz-Weyl decomposition.

Theorem 3.1. *Let $\{\Omega(t)\}_{t\in\mathbb{R}}$ be as in Assumption. Let $T > 0$ and let $b(t) \in H^1\big(\Omega(t)\big)$ satisfy $\operatorname{div} b(t) = 0$ in $\Omega(t)$ for $t \in [0, T]$. Suppose the Helmholtz-Weyl decomposition of $b(t)$ in $\Omega(t)$, i.e.,*

$$h(t) + \operatorname{rot} w(t) = b(t) \quad \text{in } \Omega(t),$$

where $h(t) \in V_{\mathrm{har}}\big(\Omega(t)\big)$ and $w(t) \in Z^2_\sigma\big(\Omega(t)\big) \cap H^2\big(\Omega(t)\big)$ are uniquely determined, for all $t \in [0, T]$.

(i) If the transformed vector field $\widetilde{b}(t)$ on $\widetilde{\Omega}$ from $b(t)$ on $\Omega(t)$, defined by (2.5) satisfies

$$\widetilde{b} \in C\big([0, T]; H^1(\widetilde{\Omega})\big),$$

then the transformed vector fields $\widetilde{h}(t)$ and $\widetilde{w}(t)$ on $\widetilde{\Omega}$ from $h(t)$ and $w(t)$ on $\Omega(t)$, respectively, satisfy

$$\widetilde{h} \in C\big([0, T]; H^1(\widetilde{\Omega})\big) \quad \text{and} \quad \widetilde{w} \in C\big([0, T]; H^2(\widetilde{\Omega})\big).$$

(ii) Furthermore, if $\widetilde{b} \in C^1\big([0,T]; H^1(\widetilde{\Omega})\big)$ *then*

$$\widetilde{h} \in C^1\big([0,T]; H^1(\widetilde{\Omega})\big) \quad \textit{and} \quad \widetilde{w} \in C^1\big([0,T]; H^2(\widetilde{\Omega})\big).$$

Remark 3.2. (a) The time dependence of $\widetilde{h}(t)$ and $\widetilde{w}(t)$ comes from the variation of the domain and the time regularity of data $\widetilde{b}(t)$. In particular, for the harmonic vector fields, since the basis $\nabla q_1(t), \dots, \nabla q_K(t)$ of $V_{\mathrm{har}}(\Omega)$ are determined only by the domain and since the contribution from $b(t)$ appears only in the coefficients, we can separately discuss these two aspects for $\widetilde{h}(t)$.

(b) We can extend Theorem 3.1 to $b(t) \in W^{1,r}\big(\Omega(t)\big)$ with $\operatorname{div} b(t) = 0$ in $\Omega(t)$, $t \in [0,T]$, for all $1 < r < \infty$, just by replacing the norm by one on the L^r-Sobolev spaces. Namely, $\widetilde{h} \in C^m\big([0,T]; W^{1,r}(\widetilde{\Omega})\big)$ and $\widetilde{w} \in C^m\big([0,T]; W^{2,r}(\widetilde{\Omega})\big)$, provided $\widetilde{b} \in C^m\big([0,T]; W^{1,r}(\widetilde{\Omega})\big)$ with some $m \in \{0,1\}$.

Since we can decompose $f(t) = b(t) + \nabla p(t)$ with $\operatorname{div} b(t) = 0$ with $\Omega(t)$ for every $f \in H^1\big(\Omega(t)\big)$, investigating the time dependence of $p(t)$, we immediately obtain the following generalization.

Corollary 3.2. *Let* $\big\{\Omega(t)\big\}_{t\geq 0}$ *be as in Assumption. Let* $T > 0$ *and let* $f(t) \in H^1\big(\Omega(t)\big)$ *for* $t \in [0,T]$. *Suppose the Helmholtz-Weyl decomposition of* $f(t)$ *in* $\Omega(t)$, *i.e.,*

$$h(t) + \operatorname{rot} w(t) + \nabla p(t) = f(t) \quad \textit{in } \Omega(t),$$

where $h(t) \in V_{\mathrm{har}}\big(\Omega(t)\big)$, $w(t) \in Z^2_\sigma\big(\Omega(t)\big) \cap H^2\big(\Omega(t)\big)$ *and* $p(t) \in H^2\big(\Omega(t)\big) \cap H^1_0\big(\Omega(t)\big)$ *are uniquely determined, for all* $t \in [0,T]$.

(i) If the transformed vector filed $\widetilde{f}(t)$ *in* $\widetilde{\Omega}$ *from* $f(t)$, *by* (2.5) *satisfies*

$$\widetilde{f} \in C\big([0,T]; H^1(\widetilde{\Omega})\big),$$

then the transformed vector fields $\widetilde{h}(t)$, $\widetilde{w}(t)$ *on* $\widetilde{\Omega}$ *by* (2.5) *and the transformed function* $\widetilde{p}(t)$ *on* $\widetilde{\Omega}$ *by* (2.6) *satisfy*

$$\widetilde{h} \in C\big([0,T]; H^1(\widetilde{\Omega})\big), \quad \widetilde{w} \in C\big([0,T]; H^2(\widetilde{\Omega})\big) \quad \textit{and} \quad \widetilde{p} \in C\big([0,T]; H^2(\widetilde{\Omega}) \cap H^1_0(\widetilde{\Omega})\big).$$

(ii) Furthermore, if $\widetilde{f} \in C^1\big([0,T]; H^1(\widetilde{\Omega})\big)$ *then*

$$\widetilde{h} \in C^1\big([0,T]; H^1(\widetilde{\Omega})\big), \quad \widetilde{w} \in C^1\big([0,T]; H^2(\widetilde{\Omega})\big) \quad \textit{and} \quad \widetilde{p} \in C^1\big([0,T]; H^2(\widetilde{\Omega}) \cap H^1_0(\widetilde{\Omega})\big).$$

Under Assumption, it was shown that $\dim V_{\mathrm{har}}\big(\Omega(t)\big) = K$ for each $t \in \mathbb{R}$, by [15]. Moreover, $V_{\mathrm{har}}\big(\Omega(t)\big)$ is spanned by the gradient flows of the solutions of the following Laplace equations with the Dirichlet boundary condition, i.e., for $k = 1, \dots, K$,

$$\begin{cases} \Delta q_k(t) = 0 & \text{in } \Omega(t), \\ q_k(t)|_{\Gamma_j(t)} = \delta_{k,j} & \text{for } j = 0, \dots, K. \end{cases} \tag{3.2}$$

By the Schmidt orthonormalization, the orthonormal basis $\eta_1(t), \dots, \eta_K(t)$ of $V_{\mathrm{har}}\big(\Omega(t)\big)$ in L^2-sense is obtained with the relation $\eta_j(t) = \sum_{k=1}^{K} \alpha_{jk}(t)\nabla q_k(t)$, $j = 1, \dots, K$ with some coefficients $\alpha_{jk}(t) \in \mathbb{R}$ for $j, k = 1, \dots, K$.

To consider the boundary value problem of the incompressible Navier-Stokes equations, as a compatibility condition, let us consider $\beta(t) \in H^{\frac{1}{2}}\big(\partial\Omega(t)\big)$ satisfies the general flux condition

$$\int_{\partial\Omega(t)} \beta(t)\cdot\nu(t)\, dS = 0 \quad \text{for } t \in \mathbb{R}, \tag{G.F.C.}$$

where $\nu(t)$ denotes the outward unit normal vector to $\partial\Omega(t)$.

Then, we state our existence theorem of the time periodic solutions of (N-S).

Theorem 3.3. *Let $T > 0$ and let $\big\{\Omega(t)\big\}_{t\in\mathbb{R}}$ be as in Assumption. Let f be external force on Q_∞ with $f \in L^2(Q_T)$ and let $\beta(t) \in H^{\frac{1}{2}}\big(\partial\Omega(t)\big)$ with (G.F.C.) for all $t \in \mathbb{R}$ satisfy $\widetilde{\beta} \in C^1\big(\mathbb{R}; H^{\frac{1}{2}}(\partial\widetilde{\Omega})\big)$. Furthermore, we assume $\phi^{-1}(\cdot, t+T) = \phi^{-1}(\cdot, t)$, $f(t+T) = f(t)$ and $\beta(t+T) = \beta(t)$ for all $t \in \mathbb{R}$. If*

$$\sup_{0\le t\le T} \left\| \sum_{\ell,k=1}^{K} \alpha_{k\ell}(t) \left(\int_{\Gamma_\ell(t)} \beta(t)\cdot\nu(t)\, dS \right) \eta_k(t) \right\|_{L^3(\Omega(t))} < \frac{1}{C_s}, \tag{3.3}$$

then there exist a solenoidal extension $b(t) \in H^1\big(\Omega(t)\big)$ of $\beta(t)$ with $\widetilde{b} \in C^1\big([0,T]; H^1(\widetilde{\Omega})\big)$, an initial data $a \in L^2\big(\Omega(0)\big)$ and a weak solution u of (N-S) such that $u(T) = a$ in $L^2_\sigma\big(\Omega(0)\big)$. Here, $C_s = 3^{-\frac{1}{2}}2^{\frac{2}{3}}\pi^{-\frac{2}{3}}$ is the best constant of the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$.

Remark 3.3. (a) We note that $\alpha_{jk}(t)$ and $\eta_k(t)$ are determined only by $\Omega(t)$ and C_s is an absolute constant. Hence, the condition (3.3) are formulated only by the given domain $\Omega(t)$ and data $\beta(t)$ on $\partial\Omega(t)$, independent of the diffeomorphism or the fixed domain $\widetilde{\Omega}$.

(b) Compared to the previous results of the second author [22], since $\alpha_{k\ell}$ and η_k depend on t, the assumption (3.3) gives us the relation of the balances of the domain variations and the flux. Indeed, we shall give the following corollaries.

Corollary 3.4. *Let $T > 0$ and let $\{\Omega(t)\}_{t\in\mathbb{R}}$ be as in Assumption with $\phi^{-1}(\cdot, t+T) = \phi^{-1}(\cdot, t)$ for all $t \in \mathbb{R}$. Assume that $\Omega(t) = \lambda(t)\Omega(0) = \{\lambda(t)x \in \mathbb{R}^3\,;\, x \in \Omega(0)\}$ with some $\lambda \in C^\infty(\mathbb{R})$ satisfying $\lambda(t) > 0$ and $\lambda(t+T) = \lambda(t)$, for all $t \in \mathbb{R}$. Let $\beta(t) \in H^{\frac{1}{2}}\big(\partial\Omega(t)\big)$ with (G.F.C.) for all $t \in \mathbb{R}$ satisfy $\widetilde{\beta} \in C^1\big(\mathbb{R}; H^{\frac{1}{2}}(\widetilde{\Omega})\big)$ and $\beta(t+T) = \beta(t)$ for all $t \in \mathbb{R}$. If*

$$\frac{1}{\lambda(t)} \sum_{k=1}^{K} \left| \int_{\Gamma_k(t)} \beta(t)\cdot\nu(t)\, dS \right| < C_0 \quad \textit{for all } t \in [0,T], \tag{3.4}$$

then β satisfies the assumption (3.3) in Theorem 3.3. Here, the constant $C_0 > 0$ depends only on $\Omega(0)$.

Remark 3.4. We put $q_k(y,t) := q_k\big(y/\lambda(t)\big)$ for $y \in \Omega(t)$, where q_k is the solution of (3.2) on $\Omega(0)$. Then we easily see that $q_k(t)$ is the solution of (3.2) on $\Omega(t)$ and (3.4) follows from $\alpha_{jk}(t) = \frac{1}{\sqrt{\lambda(t)}}\alpha_{jk}(0)$ and $\eta_k(t) = \frac{1}{\lambda(t)\sqrt{\lambda(t)}}\eta_k(0)$.

Corollary 3.5. *Let $T > 0$ and let $\Omega(t)$ be an annulus $A_{R_0,R_1}(t) := \{x \in \mathbb{R}^3 \,;\, 0 < R_1(t) < |x| < R_0(t)\}$ for $t \in \mathbb{R}$ with some $R_0, R_1 \in C^\infty(\mathbb{R})$ satisfying $R_0(t+T) = R_0(t)$ and $R_1(t+T) = R_1(t)$, for all $t \in \mathbb{R}$. Let $\beta(t) \in H^{\frac{1}{2}}\big(\partial\Omega(t)\big)$ with (G.F.C.) for all $t \in \mathbb{R}$ satisfy $\widetilde{\beta} \in C^1\big(\mathbb{R}; H^{\frac{1}{2}}(\widetilde{\Omega})\big)$ and $\beta(t+T) = \beta(t)$ for all $t \in \mathbb{R}$. If*

$$2^{-\frac{2}{3}}3^{-\frac{1}{3}}\pi^{-\frac{2}{3}}\left(\frac{1}{R_1(t)^3} - \frac{1}{R_0(t)^3}\right)^{\frac{1}{3}}\left|\int_{|x|=R_1(t)} \beta(t)\cdot\nu(t)\,dS\right| < \frac{1}{C_s} \quad \textit{for all } t \in [0,T],$$

then β satisfies the assumption (3.3) *in Theorem 3.3.*

Remark 3.5. (a) For the stationary problem of (N-S), the same situation was considered by Kozono and Yanagisawa [16].

(b) Solving (3.2), we explicitly see that $\nabla q_1(x,t) = -\left(\frac{1}{R_1(t)} - \frac{1}{R_0(t)}\right)^{-1}\frac{x}{|x|^2}$ is a basis of $V_{\mathrm{har}}\big(\Omega(t)\big)$. So, we can transform (3.3) into one in Corollary 3.5 by a direct calculation.

4 Approach to our problem

The essential difficulty of our problem comes from the analysis of the domain dependence of the vector potential in the Helmholtz-Weyl decomposition.

To overcome this difficulty, we focus on the constant as in the estimate of the Helmholtz-Weyl decomposition.

Introducing a Banach space $H^1_{\mathrm{div}}(\Omega) := \{u \in H^1(\Omega)\,;\, \operatorname{div} u = 0 \text{ in } \Omega\}$, due to Proposition 3.1, we define a bounded linear operator

$$f_\Omega : H^1_{\mathrm{div}}(\Omega) \ni b \mapsto f_\Omega[b] = w \in H^2(\Omega) \cap Z^2_\sigma(\Omega) \quad \text{with } w \text{ is in (3.1)}.$$

Here, we put

$$C(\Omega) := \sup_{b \in H^1_{\mathrm{div}}(\Omega)\setminus\{0\}} \frac{\|f_\Omega[b]\|_{H^2(\Omega)}}{\|b\|_{H^1(\Omega)}} < \infty.$$

Then, the following theorem plays an essential role to control the parameter dependence of the vector potentials.

Theorem 4.1. *Let $T > 0$. Then it holds that*

$$\sup_{0 \le t \le T} C\big(\Omega(t)\big) < \infty. \tag{4.1}$$

Due to this uniform estimate of the constant, we achieve the continuity of the vector potential in the decomposition.

Theorem 4.2. *Let $T > 0$ and let $b(t) \in H^1\big(\Omega(t)\big)$ with $\operatorname{div} b(t) = 0$ in $\Omega(t)$ for $t \in \mathbb{R}$ and let $w(t) \in Z^2_\sigma\big(\Omega(t)\big) \cap H^2\big(\Omega(t)\big)$ and $h(t) \in V_{\rm har}\big(\Omega(t)\big)$ satisfy $h(t) + \operatorname{rot} w(t) = b(t)$ in $\Omega(t)$. if $\widetilde{b} \in C\big([0,T]; H^1(\widetilde{\Omega})\big)$ then*

$$\widetilde{w} \in C\big([0,T]; H^2(\widetilde{\Omega})\big). \tag{4.2}$$

Theorem 4.2 was derived by using the uniform estimate of the constant, Theorem 4.1, and the a priori estimate associated with the elliptic system for the vector potential of the decomposition.

However, to derive time differentiability, Theorem 4.1 and the approach based on the a priori estimate were not enough to control the difference quotient $(\widetilde{w}(t+h) - \widetilde{w}(t))/h$. In order to avoid this difficulty, we go back to the way to choose the vector potential $w(t) \in Z^2_\sigma\big(\Omega(t)\big) = X^2_\sigma\big(\Omega(t)\big) \cap X_{\rm har}\big(\Omega(t)\big)^\perp$. Then, we find out the fact that the orthogonality to $X_{\rm har}\big(\Omega(t)\big)$ is preserved under our diffeomorphism, even though we never expect that the diffeomorphism preserves specific qualitative properties in the function spaces. Indeed, for $t \in \mathbb{R}$ we introduce a C^∞ diffeomorphism $\varphi(\cdot,t)$ from $\overline{\Omega(t)}$ to $\overline{\Omega(t_0)}$, using the relation $\Omega(t) \xrightarrow{\phi(\cdot,t)} \widetilde{\Omega} \xrightarrow{\phi^{-1}(\cdot,t_0)} \Omega(t_0)$, defined by

$$\Omega(t_0) \ni \tilde{x} = \varphi(x,t) := \phi^{-1}\big(\phi(x,t), t_0\big) \quad \text{for } x \in \overline{\Omega(t)}. \tag{4.3}$$

We often use the notation $\tilde{x}$ for an coordinate of $\Omega(t_0)$ for some fixed $t_0 \in \mathbb{R}$. Moreover, via a coordinate transformation with $\varphi(\cdot,t)$, there is a one-to-one relation between vector fields u on $\Omega(t)$ and $\tilde{u}$ on $\Omega(t_0)$ such that

$$\tilde{u}^i = \sum_\ell \frac{\partial \tilde{x}^i}{\partial x^\ell} u^\ell, \quad \textit{i.e.}, \quad \tilde{u}^i(\tilde{x}) = \sum_\ell \frac{\partial \varphi^i}{\partial x^\ell}(\varphi^{-1}(\tilde{x},t),t) u^\ell\big(\varphi^{-1}(\tilde{x},t)\big), \quad i = 1,2,3. \tag{4.4}$$

We have the following lemma.

Lemma 4.1. *Let $w(t) \in Z^2_\sigma\big(\Omega(t)\big) \cap H^2\big(\Omega(t)\big)$ for $t \in \mathbb{R}$. Then, $\tilde{w}(t) \in Z^2_\sigma\big(\Omega(t_0)\big) \cap H^2\big(\Omega(t_0)\big)$.*

By Lemma 4.1, we note that $\big(\tilde{w}(t_0+\varepsilon) - \tilde{w}(t_0)\big)/\varepsilon \in Z^2_\sigma\big(\Omega(t_0)\big) \cap H^2\big(\Omega(t_0)\big)$. Moreover, we have

$$\begin{aligned} &\operatorname{rot} \frac{\tilde{w}(t_0+\varepsilon) - w(t_0)}{\varepsilon} \\ &\qquad = \frac{\tilde{b}(t_0+\varepsilon) - b(t_0)}{\varepsilon} - \frac{\tilde{h}(t_0+\varepsilon) - h(t_0)}{\varepsilon} - \frac{\operatorname{Rot}(t_0+\varepsilon) - \operatorname{rot}}{\varepsilon} \tilde{w}(t_0+\varepsilon). \end{aligned} \tag{4.5}$$

Here, $\operatorname{Rot}(t)$ denotes the transformed rotation by the coordinate transform. Hence, we find that the L.H.S. of (4.5) is the Helmholtz-Weyl decomposition of the R.H.S. and is differentiable at t_0. By using the estimate of the Helmholtz-Weyl decomposition on $\Omega(t_0)$ we obtain the time differentiability of the vector potential $\widetilde{w}(t_0)$. Here, it should be noted that the uniform estimate, Theorem 4.1, still plays an important role to control $\tilde{w}(t_0+\varepsilon)$ in R.H.S. of (4.5).

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math., **17** (1964), 35–92.

[2] A. Bendali, J. M. Dominguez, S. Gallic, *A variational approach for the vector potential formulation of the Stokes and Navier-Stokes problems in three dimensional domains*, J. Math. Anal. Appl., **107** (1985), 537–560.

[3] M.E. Bogovskiĭ, *Solution of the first boundary value problem for an equation of continuity of an incompressible medium.*, Dokl. Akad. Nauk SSSR, **248** (1979), 1037–1040.

[4] J. Bolik and W. von Wahl, *Estimating* $\nabla \mathbf{u}$ *in terms of* $\operatorname{div} \mathbf{u}$, $\operatorname{curl} \mathbf{u}$, *either* $(\nu, \mathbf{u})$ *or* $\nu \times \mathbf{u}$ *and the topology*, Math. Methods Appl. Sci., **20** (1997), 737–744.

[5] W. Borchers and H. Sohr, *On the equations* $\operatorname{rot} \mathbf{v} = \mathbf{g}$ *and* $\operatorname{div} \mathbf{u} = f$ *with zero boundary conditions*, Hokkaido Math. J., **19** (1990), 67–87.

[6] R. Courant and D. Hilbert, Methoden der mathematischen Physik. II., Springer-Verlag, Berlin-New York, 1968, xv+469.

[7] C. Foiaş and R. Temam, *Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **5** (1978), 28–63.

[8] H. Fujita and N. Sauer, *On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries*, J. Fac. Sci. Univ. Tokyo Sect. I, **17** (1970), 403–420.

[9] D. Fujiwara and S. Ozawa, *The Hadamard variational formula for the Green functions of some normal elliptic boundary value problems*, Proc. Japan Acad. Ser. A Math. Sci., **54** (1978), 215–220.

[10] P. R., Garabedian, M. Schiffer, *Convexity of domain functionals*, J.Analyse Math., **2** (1953), 281–368.

[11] R. Griesinger, *Decompositions of* L^q *and* $H_0^{1,q}$ *with respect to the operator* rot, Math. Ann., **288** (1990), 245–262.

[12] J. Hadamard, *Mémoire sur le probleme d'analyse relatif à l'equilibre des plaques élastiques encastrées*, Memoires des Savants Etrangers, vol.33 (1908) (cf. Oeuvres, De Jacqes Hadamard Tom II, Centre National de la Recherche Scientifiques **2**, Anatole France, 1968, 515–631).

[13] A. Inoue, M. Wakimoto, *On existence of solutions of the Navier-Stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **24** (1977), 303–319.

[14] H. Kozono and E. Ushikoshi, *Hadamard variational formula for the Green's function of the boundary value problem on the Stokes equations*, Arch. Ration. Mech. Anal., **208** (2013), 1005–1055.

[15] H. Kozono and T. Yanagisawa, *L^r-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains*, Indiana Univ. Math. J., **58** (2009), 1853–1920.

[16] H. Kozono and T. Yanagisawa, *Leray's problem on the stationary Navier-Stokes equations with inhomogeneous boundary data*, Math. Z. **262** (2009), 27–39.

[17] O.A. Ladyzhenskaya, *Initial-boundary problem for Navier-Stokes equations in domains with time-varying boundaries*, 1968, Semin. Math., V. A. Steklov Math. Inst., Leningrad 11, 35-46 (1968); translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklov 11, 97-128 (1968).

[18] J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications. Vol. I, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York-Heidelberg, 1972.

[19] K. Masuda, *Weak solutions of Navier-Stokes equations*, Tohoku Math. J. (2), **36** (1984), 623–646.

[20] T. Miyakawa and Y. Teramoto, *Existence and periodicity of weak solutions of the Navier-Stokes equations in a time dependent domain*, Hiroshima Math. J., **12** (1982), 513–528.

[21] H. Morimoto, *On existence of periodic weak solutions of the Navier-Stokes equations in regions with periodically moving boundaries*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **18** (1971/72), 499–524.

[22] T. Okabe, *Periodic solutions of the Navier-Stokes equations with the inhomogeneous time-dependent boundary data under the general flux condition*, J. Evol. Equ., **11** (2011), 265–286.

[23] J. Peetre, *On Hadamard's variational formula*. J. Differential Equations., **36** no. 3 (1980), 335–346.

[24] R. Salvi, *On the existence of periodic weak solutions of Navier-Stokes equations in regions with periodically moving boundaries*, Acta Appl. Math., **37** (1994), 169–179.

[25] R. Temam, Navier-Stokes equations, Theory and numerical analysis, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI, 2001.

[26] H. Weyl, *The method of orthogonal projection in potential theory*, Duke Math. J., **7** (1940), 411–444.

[27] Z. Yoshida and Y. Giga, *Remarks on spectra of operator rot*, Math. Z., **204** (1990), 235–245.

Graduate School of Engineering Science
Osaka University
Toyonaka 560-8531
JAPAN
E-mail address: okabe@sigmath.es.osaka-u.ac.jp