Stability of stationary compressible Navier—Stokes flows on the 3D whole space

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1 Introduction

This article gives a survay of [5]. We consider the initial value problem for the compressible Navier-Stokes equation:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) = \mu \Delta v + (\mu + \mu')\nabla \mathrm{div}\, v - \nabla P(\rho) + \rho F(x), \\ (\rho, v)|_{t=0} = (\rho_0, v_0), \quad (\rho, v)(t,x) \to (\rho_\infty, 0). \end{cases} \tag{1}$$

Here $t \geq 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $v = (v_1, v_2, v_3)$ is the fluid velocity, ρ is the fluid density, ρ_∞ is a given positive constant, P is a given pressure, μ and μ' are given viscosity coefficients and $F = (F_1, F_2, F_3)$ is a given stationary external force. In this article, we assume that the viscous coefficients μ and μ' are constants that satisfy $\mu > 0$ and $2\mu/3 + \mu' \geq 0$, and the pressure P is a smooth function of the density ρ in a neighborhood of ρ_∞ with $P'(\rho_\infty) > 0$. The corresponding stationary problem of the initial value problem (1) is the following.

$$\begin{cases} \mathrm{div}(\rho^* v^*) = 0, \\ \mathrm{div}(\rho^* v^* \otimes v^*) = \mu \Delta v^* + (\mu + \mu')\nabla \mathrm{div}\, v^* - \nabla P(\rho^*) + \rho^* F(x), \\ (\rho^*, v^*)(x) \to (\rho_\infty, 0). \end{cases} \tag{2}$$

The existence results of the stationary problem (2) and the initial value problem (1) ware obtained by Shibata and Tanaka [10] when a stationery force F is small and an initial perturbation $(\rho_0 - \rho^*, v_0 - v^*)$ is small in the Sobolev space H^3. In [11], Shibata and Tanaka derived the decay rates of the perturbations

$$\|(\rho - \rho^*, v - v^*)(t)\|_{\dot{H}^1} \lesssim_\epsilon (1+t)^{-\frac{1}{2}+\epsilon} \|(\rho_0 - \rho^*, v_0 - v^*)\|_{L^{\frac{6}{5}} \cap H^3}, \tag{3}$$

where $\epsilon > 0$ is an arbitrary constant. Here $\dot{H}^s$ denotes the homogeneous Sobolev space whose definition is given in Section 2 below. On the other hand, the decay estimate of the perturbation around the motionless state $(\rho_\infty, 0)$ is given by

$$\|(\rho - \rho_\infty, v)(t)\|_{\dot{H}^s} \lesssim (1+t)^{-\frac{s}{2} - \frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right)} \|(\rho_0 - \rho_\infty, v_0)\|_{L^p \cap H^3} \tag{4}$$

for $0 \leq s \leq 2$ and $1 \leq p \leq 2$. (Cf. [6], [7], [8], [9].)

This article aims to give a summary of resent result on stability analysis of a solution of the stationary problem (2). We shall consider the stationary solution obtained in the following theorem.

Theorem 1.1 ([5]). *There exists a constant $\delta_0 > 0$ such that if $F \in \dot{B}^{-\frac{3}{2}}_{2,\infty} \cap \dot{H}^3$ and*

$$\|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}\cap \dot{H}^3} \leq \delta_0,$$

then there exists a unique stationary solution $(\rho^*, v^*) = (\sigma^* + \rho_\infty, v^*)$ of (2) such that

$$\|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}\cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^5} \lesssim \delta_0, \tag{5}$$

Theorem 1.1 show the existence of the stationary solution in the framework of the homogeneous Besov space $\dot{B}^s_{p,r}$ whose definition is given in Section 2 below. The estimate (5) for the stationary solution will use for deriving the decay estimate of the perturbation. The following theorem show the existence of the global solution of the perturbation around the stationary solution which obtained in Theorem 1.1.

Theorem 1.2 ([5]). *Let (ρ^*, v^*) be a stationary solution of (2) satisfying (5) with $\|F\|_{\dot{B}^{-3/2}_{2,\infty}\cap \dot{H}^3}$ sufficiently small. Then, there exists a constant $\delta_1 > 0$ such that if an initial perturbation $(\sigma_0, w_0) = (\rho_0 - \rho^*, v_0 - v^*)$ satisfy*

$$\|(\sigma_0, w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} \leq \delta_1,$$

then there exists a global solution $(\rho, v) = (\sigma + \rho^*, w + v^*)$ of (1) satisfying $(\sigma, w) \in C^0([0,\infty); \dot{B}^{1/2}_{2,\infty} \cap \dot{H}^3)$ and

$$\sup_{t>0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} \lesssim \delta_1.$$

We now state our main theorem which derive the decay rate of the perturbation around the stationary solution obtained in Theorem 1.1.

Theorem 1.3 ([5]). *Let (ρ^*, v^*), (ρ, v) be as in Theorem 1.1 and Theorem 1.2, with*

$$\|(\rho_0 - \rho^*, v_0 - v^*)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} + \|F\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}\cap \dot{H}^3}$$

sufficiently small. If the initial perturbation $(\rho_0 - \rho^*, v_0 - v^*) \in L^p$ for some $1 \leq p \leq 2$, then the decay estimate

$$\|(\rho - \rho^*, v - v^*)(t)\|_{\dot{H}^s\cap \dot{H}^3} \lesssim_s (1+t)^{-\frac{s}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \|(\rho_0 - \rho^*, v_0 - v^*)\|_{L^p\cap H^3} \tag{6}$$

holds for $-3/2 < s < 3/2$ with $s/2 + 3/2(1/p - 1/2) > 0$.

In this article, we will give outline of the proof of Theorem 1.3. We present the notation used throughout this paper and the basic facts of the homogeneous Besov spaces in Section 2, and give the proof of Theorem 1.3 in Section 3.

2 Preliminaries

In this section, we shall introduce the following notations and some function spaces. The notation $A \lesssim_\alpha B$ means that there exists a constant C depending on α such that $A \le CB$. The notation $A \sim_\alpha B$ means that $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$. We denote a commutator by $[X,Y] \equiv XY - YX$. We write $\mathcal{S}$ for the set of all Schwartz functions on $\mathbb{R}^3$, and we write $\mathcal{S}'$ for the set of all tempered distributions on $\mathbb{R}^3$. The notations $\hat{\cdot}$, $\mathcal{F}$ stand for the Fourier transform

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) \equiv \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)dx,$$

and the notation $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. The symbol $\mathbb{P}$ denotes the Helmholtz projection: $\mathbb{P}u \equiv u - \Delta^{-1}\nabla \mathrm{div}\, u$, $u \in \mathcal{S}'$. We denote the $L^2(\mathbb{R}^3)$ inner product by $\langle u, v\rangle \equiv \int_{\mathbb{R}^3} uv dx$.

The rest of this section introduces the homogeneous Sobolev and Besov spaces and presents some basic facts. For any $s \in \mathbb{R}$, we define the homogeneous Sobolev space $\dot{H}^s = \dot{H}^s(\mathbb{R}^3)$ by

$$\dot{H}^s = \left\{ u \in \mathcal{S}' \;\middle|\; \hat{u} \in L^1_{loc}(\mathbb{R}^3),\ \|u\|_{\dot{H}^s} = \||\cdot|^s \hat{u}\|_{L^2} < \infty \right\}.$$

Next, we give the definition of the homogeneous Besov space. We choose $\phi \in C^\infty(\mathbb{R}^3)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^3 \mid 3/4 \le |\xi| \le 8/3\}$ such that

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \ne 0.$$

Define the dyadic blocks $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ by the Fourier multiplier

$$\dot{\Delta}_j u \equiv \mathcal{F}^{-1}[\phi(2^{-j}\cdot)\hat{u}].$$

The homogeneous low frequency cutoff operator is denoted by

$$\dot{S}_j u \equiv \sum_{j'<j} \dot{\Delta}_{j'} u, \quad j \in \mathbb{Z}. \tag{7}$$

Fix $\phi_0 \in C_0^\infty(\mathbb{R}^3)$ satisfying $\phi_0(0) \ne 0$. Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$. Then, the homogeneous Besov space $\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^3)$ is given by

$$\dot{B}^s_{p,r} \equiv \left\{ u \in \mathcal{S}' \;\mid\; \lim_{j\to-\infty} \|\mathcal{F}^{-1}[\phi_0(2^{-j}\cdot)\hat{u}]\|_{L^\infty} = 0,\ \ \|u\|_{\dot{B}^s_{p,r}} < \infty \right\},$$
$$\|u\|_{\dot{B}^s_{p,r}} \equiv \left\| (2^{js}\|\dot{\Delta}_j u\|_{L^p})_{j\in\mathbb{Z}} \right\|_{\ell^r}.$$

We present some lemmas for the homogeneous Besov space.

Lemma 2.1. *Let $s \in \mathbb{R}$, $1 \le p, r \le \infty$, $u \in \mathcal{S}'$ and $v \in \mathcal{S}$. Then, we have the following duality estimates:*

$$\langle u, v\rangle \lesssim \|u\|_{\dot{B}^s_{p,r}} \|u\|_{\dot{B}^{-s}_{p',r'}} \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\psi} \langle u, \psi\rangle,$$

where the supremum is taken over the Schwartz functions ψ with $\|\psi\|_{\dot{B}^{-s}_{p',r'}} \le 1$ and $0 \notin \mathrm{supp}\, \mathcal{F}\psi$.

Lemma 2.2. *Let $s_1, s_2 \in \mathbb{R}$ satisfy $s_1, s_2 < 3/2$ and $s_1 + s_2 > 0$. Let $1 \leq r_1, r_2 \leq \infty$ satisfy $1/r_1 + 1/r_2 = 1/r$. Then, for any $u \in \dot{B}^{s_1}_{2,r_1}$ and $v \in \dot{B}^{s_2}_{2,r_2}$, we have $uv \in \dot{B}^{s_1+s_2-3/2}_{2,r}$ and*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{3}{2}}_{2,r}} \lesssim_{s_1,s_2} \|u\|_{\dot{B}^{s_1}_{2,r_1}} \|v\|_{\dot{B}^{s_2}_{2,r_2}}.$$

In the cases $s_1 \leq 3/2$, $s_2 < 3/2$ with $s_1 + s_2 \geq 0$, we have

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{3}{2}}_{2,\infty}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,\infty}}$$

for any $u \in \dot{B}^{s_1}_{2,1}$ and $v \in \dot{B}^{s_2}_{2,\infty}$.

Lemma 2.3. *Let $\Phi \in C^\infty(\mathbb{R}^3)$, $1 \leq r \leq \infty$ and $u, v \in \dot{B}^s_{2,r} \cap \dot{B}^{3/2}_{2,1}$ with $-3/2 \leq s < 3/2$ or $s = 3/2$, $r = 1$. Then, we have*

$$\|\Phi(u) - \Phi(v)\|_{\dot{B}^s_{2,r}} \lesssim_\Phi (1 + \|(u,v)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})\|u - v\|_{\dot{B}^s_{2,r}}.$$

Lemma 2.4. *Let $-3/2 < s < 5/2$, $1 \leq r \leq \infty$. Then, we have*

$$\left\| \left(2^{js} \|[\dot{\Delta}_j, h\partial_k]u\|_{L^2} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} \lesssim_{s,\phi_0} \|\nabla h\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|u\|_{\dot{B}^s_{2,r}},$$

where $1 \leq k \leq 3$ and u, h are scalar functions.

As for the proofs of Lemmas 2.1, 2.3 and 2.4, see [2, Lemma 2.100, Proposition 2.29, Theorem 2.47, 2.52], [4, Lemma 1.6 ii)] for example.

3 The proof of Theorem 1.3

Throughout this section, we fix the stationary solution $(\rho^*, v^*) = (\sigma^* + \rho_\infty, v^*)$ of (2) and the global solution $(\rho, v) = (\sigma + \rho^*, w + v^*)$ of (1) satisfying

$$\delta \equiv \|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty} \cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^5} + \sup_{t>0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3} < \infty.$$

We shall prove the following theorem.

Theorem 3.1. *Let $-3/2 \leq s_0 \leq 1/2$. If*

$$\delta = \|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty} \cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^5} + \sup_{t>0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^3}$$

is small enough, then, for any $-3/2 < s < 3/2$ with $s_0 \leq s$, we have

$$\|(\sigma, w)(t)\|_{\dot{B}^s_{2,\infty} \cap \dot{H}^3} \lesssim_s (1+t)^{-\frac{s-s_0}{2}} \|(\sigma_0, w_0)\|_{\dot{B}^{s_0}_{2,\infty} \cap \dot{H}^3}, \tag{8}$$

where $(\sigma, w) = (\rho - \rho^*, v - v^*)$ and $(\sigma_0, w_0) = (\rho_0 - \rho^*, v_0 - v^*)$.

The proof of Theorem 3.1 is carried out by decomposing the perturbation into low- and high-frequency parts with respect to the Fourier space. For fixed $j_0 \in \mathbb{Z}$, we decompose the perturbation (σ, w) as follows:

$$(\sigma, w) = (\sigma_L, w_L) + (\sigma_H, w_H), \tag{9}$$

where $(\sigma_L, w_L) \equiv (\dot{S}_{j_0}\sigma, \dot{S}_{j_0}w)$. For any $T > 0$, $\epsilon > 0$ and $-3/2 \leq s_0 \leq 1/2$, we define the quantity $\mathcal{D}_{\epsilon,s_0}(T)$ by

$$\mathcal{D}_{\epsilon,s_0}(T) \equiv \sup_{\substack{-3/2+\epsilon\leq\eta\leq 3/2-\epsilon,\\ s_0\leq\eta}} \sup_{0\leq t\leq T} (1+t)^{\frac{\eta-s_0}{2}} \|(\sigma, w)(t)\|_{\dot{B}^{\eta}_{2,\infty}\cap \dot{H}^3}. \tag{10}$$

3.1 Estimate for the low frequency part

To estimate for the low frequency part, We shall rewrite the equation of the perturbation in the momentum formulation. (Cf. [12], [1].) Let $n = (m - m^*)/P'(\rho_\infty)^{1/2}$ with $m = \rho v$, $m^* = \rho^* v^*$. Then, (σ, n) satisfies the equation Then, the pair of functions $V = (\sigma, n) = (\rho - \rho^*, n)$ satisfies the system of equations:

$$\begin{cases} \partial_t \sigma + \gamma_1 \mathrm{div}\, n = 0, \\ \partial_t n - \mathcal{A}_1 n + \gamma_1 \nabla\sigma = h + \rho_\infty^{-1}\sigma F(x), \end{cases} \tag{11}$$

where $\gamma_1 = P'(\rho_\infty)^{1/2}$, $\mathcal{A}_1 = \mu/\rho_\infty \Delta + (\mu+\mu')/\rho_\infty \nabla\mathrm{div}$; h is defined by

$$h = \sum_{i=1}^{4} h_i \tag{12}$$

with

$$\begin{aligned} h_1 &= -\mathrm{div}\left(\frac{n\otimes m}{\rho} + \frac{m^*\otimes n}{\rho} + \rho_\infty^{-1}\left(\Psi(\sigma^*+\sigma) - \Psi(\sigma^*)\right) m^*\otimes m\right), \\ h_2 &= -\nabla\left(\Pi(\sigma^*, \sigma)\sigma\right), \quad h_3 = \mathcal{A}_1\left(\left(\Psi(\sigma^*+\sigma) - \Psi(0)\right) n\right), \\ h_4 &= \mathcal{A}_1\left((\Psi(\sigma^*+\sigma) - \Psi(\sigma^*)) m^*\right), \\ \Pi(\zeta_1, \zeta_2) &= \int_0^1 \left(P'(\zeta_1 + \theta\zeta_2 + \rho_\infty) - P'(\rho_\infty)\right) d\theta, \quad \Psi(\zeta) = \frac{1}{\zeta + \rho_\infty}. \end{aligned}$$

Let e^{tA} be the semigroup associated with the linear equation (11):

$$e^{tA}U_0 = \mathcal{F}^{-1}\left[e^{t\hat{A}(\xi)}\widehat{U_0}\right], \quad U_0 = (U_{0,1}, \ldots, U_{0,4})^{\mathsf{T}} \in \mathcal{S}'(\mathbb{R}^3)^4, \tag{13}$$

where $\hat{A}(\xi)$ is the matrix of the form:

$$\hat{A}(\xi) = \begin{bmatrix} 0 & -i\gamma_1\xi^{\mathsf{T}} \\ -i\gamma_1\xi & -\nu|\xi|^2 \mathrm{I}_3 - (\nu+\nu')\xi\otimes\xi \end{bmatrix}. \tag{14}$$

Here $\nu = \mu/\rho_\infty$, $\nu' = \mu'/\rho_\infty$, $\xi = (\xi_1, \xi_2, \xi_3)^\mathsf{T} \in \mathbb{R}^3$, $\xi \otimes \xi = \xi\xi^\mathsf{T}$ and I_3 is the 3×3 identity matrix. By direct calculation, the eigenvalues of $\hat{A}(\xi)$ are

$$\lambda_\pm(\xi) = -\frac{2\nu + \nu'}{2}|\xi|^2 \pm \frac{\sqrt{(2\nu+\nu')^2|\xi|^4 - 4\gamma_1^2|\xi|^2}}{2}, \quad \lambda_0(\xi) = -\nu|\xi|^2. \tag{15}$$

We set $P_\pm(\xi)$:

$$P_\pm(\xi) = \frac{V_\pm \otimes V_\pm}{V_\pm \cdot V_\pm} \quad \text{with} \quad V_\pm = \begin{bmatrix} -i\lambda_\pm^{-1}\gamma|\xi|^2 \\ \xi \end{bmatrix}$$

for $|\xi| \neq 0, \eta_0$, where $\eta_0 = \gamma/(\nu + \nu'/2)$, $V_\pm \cdot V_\pm \equiv V_\pm^\mathsf{T} V_\pm$, and set the eigenprojection $P_0(\xi)$:

$$P_0(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I}_3 - \frac{\xi \otimes \xi}{|\xi|^2} \end{bmatrix}.$$

Since $P_+(\xi) + P_-(\xi) + P_0(\xi) = \mathrm{I}_4$, we have the spectral resolution

$$e^{t\hat{A}(\xi)} = e^{\lambda_+ t}P_+(\xi) + e^{\lambda_- t}P_-(\xi) + e^{\lambda_0 t}P_0(\xi) \quad \text{for} \quad |\xi| \neq 0, \eta_0. \tag{16}$$

If $|\xi| = \eta_0$, then we have

$$e^{t\hat{A}(\xi)} = e^{-\nu_0|\xi|^2 t} \begin{bmatrix} 1 - \nu_0|\xi|^2 t & -i\gamma\xi^\mathsf{T} t \\ -i\gamma\xi t & (1 - \nu_0|\xi|^2 t)\frac{\xi\otimes\xi}{|\xi|^2} \end{bmatrix} + e^{-\nu|\xi|^2 t}P_0(\xi), \tag{17}$$

where $\nu_0 = \nu + \nu'/2$.

The following lemma shows some smoothing estimate for the low frequency part of the semigroup e^{tA} and its adjoint e^{tA^*}. This lemma has been proved in [2, Proposition 10.22]. (Cf. [3], [4], [13].)

Lemma 3.2. *Let $j_0 \in \mathbb{Z}$, $s \in \mathbb{R}$. Set $e_L^{tA} \equiv \dot{S}_{j_0} e^{tA}$, $e_L^{tA^*} \equiv \dot{S}_{j_0} e^{tA^*}$, where $\dot{S}_{j_0}$ is the low frequency cut-off operator defined in* (7).

(i) *For any $U_0 \in \dot{B}^s_{2,r}$ and $\alpha \geq 0$, we have*

$$\|e_L^{tA}U_0\|_{\dot{B}^{s+\alpha}_{2,r}}, \ \|e_L^{tA^*}U_0\|_{\dot{B}^{s+\alpha}_{2,r}} \lesssim_{\alpha, j_0} (1+t)^{-\frac{\alpha}{2}}\|U_0\|_{\dot{B}^s_{2,r}} \tag{18}$$

for any $1 \leq r \leq \infty$.

(ii) *The following time-space integral estimate holds:*

$$\int_0^\infty \|e_L^{tA}U_0\|_{\dot{B}^{s+2}_{2,1}} dt, \ \int_0^\infty \|e_L^{tA^*}U_0\|_{\dot{B}^{s+2}_{2,1}} dt \lesssim_{j_0} \|U_0\|_{\dot{B}^s_{2,1}} \tag{19}$$

for any $U_0 \in \dot{B}^s_{2,1}$.

We now show the following proposition which derives the time decay estimate for the low frequency part of the perturbation (σ_L, w_L).

Proposition 3.3. *Let $-3/2 \le s_0 \le 1/2$, and let $\epsilon > 0$ be a small number. If*

$$\delta = \|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}\cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^5} + \sup_{t>0} \|(\sigma, w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3}$$

is sufficiently small, then, for any $T > 0$, we have

$$\sup_{0\le t\le T} (1+t)^{\frac{s-s_0}{2}} \|(\sigma_L, w_L)(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{\epsilon,j_0} \|(\sigma_0, w_0)\|_{\dot{B}^{s_0}_{2,\infty}} + \delta\mathcal{D}_{\epsilon,s_0}(T), \tag{20}$$

where $-3/2+\epsilon \le s \le 3/2-\epsilon$ with $s_0 \le s$. Here, $\mathcal{D}_{\epsilon,s_0}(T)$ is the quantity defined in (10).

Proof. Let $n = (m - m^*)/\gamma_1$ with $m = \rho v$, $m^* = \rho^* v^*$ and $\gamma_1 = P'(\rho_\infty)^{1/2}$. Then, (σ, n) satisfies the equation (11). Let e^{tA} be the semigroup defined in (13). Then, the Duhamel principle gives

$$V_L(t) = e^{tA}_L V_0 + \int_0^t e^{(t-\tau)A}_L \begin{bmatrix} 0 \\ h + \rho_\infty^{-1}\sigma F(x) \end{bmatrix} (\tau)\, d\tau, \tag{21}$$

where $V_L \equiv \dot{S}_{j_0} V$, $e^{tA}_L \equiv \dot{S}_{j_0} e^{tA}$ and $V_0 = V(0)$. By the definition of n, we have $\gamma_1 n = \rho_\infty w + (\sigma v - \sigma^* w)$. Let $n_L = \dot{S}_{j_0} n$. Then, for any $-3/2 < s < 3/2$, Lemma 2.2 shows

$$\|w_L\|_{\dot{B}^s_{2,\infty}} \lesssim_s \|n_L\|_{\dot{B}^s_{2,\infty}} + \delta\|(\sigma, w)\|_{\dot{B}^s_{2,\infty}}.$$

Thus, to prove Proposition 3.3, it is sufficient to show the inequality

$$\sup_{0\le t\le T} (1+t)^{\frac{s-s_0}{2}} \|(\sigma_L, n_L)(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{\epsilon,p,j_0} \|(\sigma_0, w_0)\|_{\dot{B}^{s_0}_{2,\infty}} + \delta\mathcal{D}_{\epsilon,s_0}(T),$$

where $-3/2+\epsilon \le s \le 3/2-\epsilon$ with $s_0 \le s$. The Duhamel principle gives

$$\begin{bmatrix} \sigma_L \\ n_L \end{bmatrix}(t) = e^{tA}_L \begin{bmatrix} \sigma_0 \\ n_0 \end{bmatrix} + \int_0^t e^{(t-\tau)A}_L \begin{bmatrix} 0 \\ h + \rho_\infty^{-1}\sigma F(x) \end{bmatrix}(\tau)\, d\tau, \tag{22}$$

where $e^{tA}_L \equiv \dot{S}_{j_0} e^{tA}$ and the function h is defined in (12). Let us denote $V_0 = (\sigma_0, n_0)^\mathsf{T}$, $V = (\sigma, n)^\mathsf{T}$. Then, by Lemma 2.2 and Lemma 3.2 (i), we have

$$\left\|e^{tA}_L V_0\right\|_{\dot{B}^s_{2,\infty}} \lesssim_{j_0} (1+t)^{-\frac{s-s_0}{2}} \|V_0\|_{\dot{B}^{s_0}_{2,\infty}} \lesssim (1+t)^{-\frac{s-s_0}{2}} \|(\sigma_0, w_0)\|_{\dot{B}^{s_0}_{2,\infty}}. \tag{23}$$

We shall prove the following estimate.

$$\|h\|_{\dot{B}^\beta_{2,\infty}} \lesssim_\beta \delta\|V\|_{\dot{B}^{\beta+2}_{2,\infty}} \quad \text{for any } -\frac{5}{2} < \beta < -\frac{1}{2}. \tag{24}$$

By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} \|h_1\|_{\dot{B}^\beta_{2,\infty}} &\lesssim \left\| \frac{n\otimes m}{\rho} + \frac{m^*\otimes n}{\rho} + \rho_\infty^{-1}\left(\Psi(\sigma^*+\sigma) - \Psi(\sigma^*)\right) m^*\otimes m \right\|_{\dot{B}^{\beta+1}_{2,\infty}} \\ &\lesssim_\beta \|(\sigma^*, v^*, \sigma, n)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \|U\|_{\dot{B}^{\beta+2}_{2,\infty}} \lesssim \delta\|U\|_{\dot{B}^{\beta+2}_{2,\infty}}, \\ \|h_2\|_{\dot{B}^\beta_{2,\infty}} &\lesssim \|\Pi(\sigma^*, \sigma)\sigma\|_{\dot{B}^{\beta+1}_{2,\infty}} \lesssim \|(\sigma^*, \sigma)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \|\sigma\|_{\dot{B}^{\beta+2}_{2,\infty}} \end{aligned}$$

for any $-5/2 < \beta < -1/2$. By Lemma 2.2 and Lemma 2.3, we obtain bounds for h^3, h^4 as

$$\begin{aligned}
\|h_3\|_{\dot{B}^{\beta}_{2,\infty}} &\lesssim \|(\Psi(\sigma^*+\sigma)-\Psi(0))\,\nabla n\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\qquad\qquad + \|\nabla\left(\Psi(\sigma^*+\sigma)-\Psi(0)\right) n\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\lesssim_\beta \|\sigma^*\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|n\|_{\dot{B}^{\beta+2}_{2,\infty}} + \|U\|_{\dot{B}^{\frac{3}{2}}_{2,\infty}}\|U\|_{\dot{B}^{\beta+2}_{2,\infty}} \lesssim \delta\|U\|_{\dot{B}^{\beta+2}_{2,\infty}}, \\
\|h_4\|_{\dot{B}^{\beta}_{2,\infty}} &\lesssim_\beta \|(\Psi(\sigma^*+\sigma)-\Psi(\sigma^*))\,\nabla m^*\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\qquad\qquad + \|\nabla\left(\Psi(\sigma^*+\sigma)-\Psi(\sigma^*)\right) m^*\|_{\dot{B}^{\beta+1}_{2,\infty}} \\
&\lesssim_\beta \|m^*\|_{\dot{B}^{\frac{3}{2}}_{2,1}}\|\sigma\|_{\dot{B}^{\beta+2}_{2,\infty}} \lesssim \delta\|\sigma\|_{\dot{B}^{\beta+2}_{2,\infty}}
\end{aligned}$$

for any $-5/2 < \beta < -1/2$. Thus, we have the desired estimate (24).

Let us denote the second term in (22) by

$$\begin{aligned}
N_L(t) &= \int_0^t e_L^{(t-\tau)A}\begin{bmatrix}0\\ h\end{bmatrix}(\tau)\,d\tau + \int_0^t e_L^{(t-\tau)A}\begin{bmatrix}0\\ \rho_\infty^{-1}\sigma F(x)\end{bmatrix}(\tau)\,d\tau \\
&\equiv N_L^1(t) + N_L^2(t).
\end{aligned} \tag{25}$$

We estimate the term $N_L^1(t)$. We first treat the case $-1/2 < s \le 3/2-\epsilon$ with $s_0 \le s$. In this case, we estimate $N_L^1(t)$ by using the duality argument. Let $\psi = (\psi_1,\dots,\psi_4)^{\mathsf{T}} \in \mathcal{S}^4$. Fix a real number α_0 satisfying $s-s_0 < \alpha_0 < s-s_0+2$ and $s+1/2+\epsilon \le \alpha_0 \le s+5/2-\epsilon$. This α_0 can be taken if $\epsilon < 1$. Lemma 2.1 then yields

$$\begin{aligned}
\left\langle N_L^1(t),\psi\right\rangle &\lesssim \int_{\frac{t}{2}}^{t} \|h(\tau)\|_{\dot{B}^{s-2}_{2,\infty}}\|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\,d\tau \\
&\qquad\qquad + \int_0^{\frac{t}{2}} \|h(\tau)\|_{\dot{B}^{s-\alpha_0}_{2,\infty}}\|e_L^{(t-\tau)A^*}\psi\|_{\dot{B}^{\alpha_0-s}_{2,1}}\,d\tau.
\end{aligned}$$

By Lemma 2.2,

$$\|V(\tau)\|_{\dot{B}^{s}_{2,\infty}} \lesssim (1+\|U(\tau)\|_{\dot{B}^{\frac{3}{2}}_{2,1}})\|U(\tau)\|_{\dot{B}^{s}_{2,\infty}} \lesssim \|U(\tau)\|_{\dot{B}^{s}_{2,\infty}}, \tag{26}$$

where $U = (\sigma, w)^{\mathsf{T}}$. Then, the estimate (24) with $\beta = s-2$ shows

$$\|h(\tau)\|_{\dot{B}^{s-2}_{2,\infty}} \lesssim (\delta_1 + \|U(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{B}^{\frac{3}{2}}_{2,1}})\|U(\tau)\|_{\dot{B}^{s}_{2,\infty}} \lesssim \delta\|U(\tau)\|_{\dot{B}^{s}_{2,\infty}}.$$

By using the time-space integral estimate

$$\int_0^\infty \|e_L^{\tau A^*}\psi\|_{\dot{B}^{2-s}_{2,1}}\,d\tau \lesssim \|\psi\|_{\dot{B}^{-s}_{2,1}}$$

which follows from Lemma 3.2 (ii), we obtain

$$\int_{\frac{t}{2}}^{t} \|h(\tau)\|_{\dot B^{s-2}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot B^{2-s}_{2,1}}\, d\tau \lesssim_s \delta \int_{\frac{t}{2}}^{t} \|U(\tau)\|_{\dot B^{s}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot B^{2-s}_{2,1}}\, d\tau$$
$$\lesssim \delta \mathcal{D}_{\epsilon,s_0}(T)(1+t)^{-\frac{s-s_0}{2}} \int_0^\infty \|e_L^{\tau A^*}\psi\|_{\dot B^{2-s}_{2,1}}\, d\tau$$
$$\lesssim_{j_0} \delta \mathcal{D}_{\epsilon,s_0}(T)(1+t)^{-\frac{s-s_0}{2}} \|\psi\|_{\dot B^{-s}_{2,1}}.$$

The estimate (24) with $\beta = s-\alpha_0$ shows

$$\|h(\tau)\|_{\dot B^{s-\alpha_0}_{2,\infty}} \lesssim (\delta_1 + \|U(\tau)\|_{\dot B^{\frac{1}{2}}_{2,\infty}\cap \dot B^{\frac{3}{2}}_{2,1}})\|U(\tau)\|_{\dot B^{s-\alpha_0+2}_{2,\infty}} \lesssim \delta \|U(\tau)\|_{\dot B^{s-\alpha_0+2}_{2,\infty}}.$$

By using the time decay estimate

$$\|e_L^{(t-\tau)A}\psi\|_{\dot B^{\alpha_0-s}_{2,\infty}} \lesssim (1+t-\tau)^{-\frac{\alpha_0}{2}} \|\psi\|_{\dot B^{-s}_{2,\infty}},$$

which follows from Lemma 3.2 (i), we obtain

$$\int_0^{\frac{t}{2}} \|h(\tau)\|_{\dot B^{s-\alpha_0}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot B^{\alpha_0-s}_{2,1}}\, d\tau \lesssim_s \delta \int_0^{\frac{t}{2}} \|U(\tau)\|_{\dot B^{s-\alpha_0+2}_{2,\infty}} \|e_L^{(t-\tau)A^*}\psi\|_{\dot B^{\alpha_0-s}_{2,1}}\, d\tau$$
$$\lesssim \delta \mathcal{D}_{\epsilon,s_0}(T)\|\psi\|_{\dot B^{-s}_{2,1}} \int_0^{\frac{t}{2}} (1+\tau)^{-\frac{s-s_0+2-\alpha_0}{2}} (1+t-\tau)^{-\frac{\alpha_0}{2}}\, d\tau$$
$$\lesssim_{\epsilon,j_0} \delta \mathcal{D}_{\epsilon,s_0}(T)(1+t)^{-\frac{s-s_0}{2}} \|\psi\|_{\dot B^{-s}_{2,1}}.$$

As ψ is arbitrary, applying Lemma 2.1, we obtain the inequality

$$\left\|N_L^1(t)\right\|_{\dot B^{s}_{2,\infty}} \lesssim_{\epsilon,j_0} \delta \mathcal{D}_{\epsilon,s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}} \tag{27}$$

for $-1/2 < s \le 3/2-\epsilon$, $-3/2 \le s_0 \le 1/2$ with $s_0 \le s$.

Next, we show the inequalities (27) for $-3/2+\epsilon \le s \le -1/2$, $-3/2 \le s_0 \le 3/2$ with $s_0 \le s$. We take $\epsilon_1 > 0$ which satisfies $\epsilon_1 < 1/2$. Then, using Lemma 3.2 (i) with $\alpha = 2+s-\epsilon_1$ and the estimate 24 with $\beta = -2+\epsilon_1$, we have

$$\|N_L^1(t)\|_{\dot B^{s}_{2,\infty}} \lesssim_{j_0} \int_0^t (1+t-\tau)^{-\frac{1}{2}(2+s-\epsilon_1)} \|h(\tau)\|_{\dot B^{-2+\epsilon_1}_{2,\infty}}\, d\tau$$
$$\lesssim \delta \int_0^t (1+t-\tau)^{-\frac{1}{2}(2+s-\epsilon_1)} \|U(\tau)\|_{\dot B^{\epsilon_1}_{2,\infty}}\, d\tau$$
$$\lesssim_{\epsilon_1,p} \delta \mathcal{D}_{\epsilon,s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}}.$$

By Lemma 2.2 and Lemma 3.2 (i) with $\alpha = s+3/2$, we have

$$\|N_L^2(t)\|_{\dot B^{s}_{2,\infty}} \lesssim_{j_0} \int_0^t (1+t-\tau)^{-\frac{s}{2}-\frac{3}{4}} \|\sigma(\tau)F\|_{\dot B^{-\frac{3}{2}}_{2,\infty}}\, d\tau$$
$$\lesssim_\epsilon \|F\|_{\dot B^{-\frac{3}{2}+\epsilon}_{2,1}} \int_0^t (1+t-\tau)^{-\frac{s}{2}-\frac{3}{4}} \|\sigma(\tau)\|_{\dot B^{\frac{3}{2}-\epsilon}_{2,\infty}} d\tau$$
$$\lesssim_\epsilon \delta \mathcal{D}_{\epsilon,s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}},$$

where $-3/2+\epsilon \le s \le 3/2-\epsilon$, $-3/2 \le s_0 \le 3/2$ with $s_0 \le s$. Hence, we obtain

$$\|N_L(t)\|_{\dot{B}^s_{2,\infty}} \lesssim_{\epsilon,j_0} \delta \mathcal{D}_{\epsilon,s_0}(T)\,(1+t)^{-\frac{s-s_0}{2}} \tag{28}$$

for $-3/2+\epsilon \le s \le -1/2$, $-3/2 \le s_0 \le 3/2$ with $s_0 \le s$. □

3.2 Estimate for the high frequency part

We show the following estimate for the high frequency part of the perturbation (σ_H, w_H).

Proposition 3.4. *If*

$$\delta = \|\sigma^*\|_{\dot{B}^{-\frac{1}{2}}_{2,\infty}\cap \dot{H}^4} + \|v^*\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^5} + \sup_{t>0} \|(\sigma,w)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3}$$

is small enough, then we have the following estimate for all $T>0$ and small $\epsilon>0$,

$$\sup_{0\le t\le T} (1+t)^{\frac{s-s_0}{2}} \|(\sigma_H, w_H)(t)\|_{\dot{H}^3} \lesssim_{j_0} \|(\sigma_0,w_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}\cap \dot{H}^3} + \delta \mathcal{D}_{\epsilon,s_0}(T), \tag{29}$$

where $-3/2+\epsilon \le s \le 3/2-\epsilon$, $-3/2 \le s_0 \le 1/2$ with $s_0 \le s$. Here, $\mathcal{D}_{\epsilon,s_0}(T)$ is the quantity defined in (10).

Proof. The perturbation (σ, w) satisfies the following equation

$$\begin{cases} \partial_t \sigma + \rho_\infty \mathrm{div}\, w = f(\sigma, w), \\ \partial_t w - \mathcal{A}_0 w + \rho_\infty \gamma_2 \nabla \sigma = g(\sigma, w), \\ (\sigma, w)|_{t=0} = (\sigma_0, w_0), \end{cases} \tag{30}$$

where $\gamma_2 = P'(\rho_\infty)/\rho_\infty^2$, $\nu_0 = \mu/\rho_\infty$, $\nu_0' = \mu'/\rho_\infty$, $\mathcal{A}_0 \equiv \nu_0 \Delta + (\nu_0+\nu_0')\nabla \mathrm{div}$, $(\sigma_0, w_0) \equiv (\rho_0 - \rho^*, v_0 - v^*)$; f and g are defined by the following:

$$f(\sigma,w) = -\mathrm{div}\,\{\sigma v + \sigma^* w\}\,,\ g(\sigma, w) = \sum_{i=1}^{4} g^i$$

with

$$\begin{aligned} g^1 &= -v^* \cdot \nabla w - w \cdot \nabla v^* - w \cdot \nabla w, \\ g^2 &= -(\Phi(\sigma^*+\sigma) - \Phi(\sigma^*))\nabla \sigma^* - (\Phi(\sigma^*+\sigma) - \Phi(0))\nabla \sigma, \\ g^3 &= (\Psi(\sigma^*+\sigma) - \Psi(\sigma^*))\mathcal{A}_0(v^*+w), \quad g^4 = (\Psi(\sigma^*) - \Psi(0))\mathcal{A}_0 w, \\ \Phi(\zeta) &= \frac{P'(\zeta+\rho_\infty)}{\zeta+\rho_\infty}, \quad \Psi(\zeta) = \frac{1}{\zeta+\rho_\infty}. \end{aligned}$$

Let $f_H = f - \dot{S}_j f$ and $g_H = g - \dot{S}_j g$. for any multi-index $\alpha_1, \alpha_2 \in \mathbb{Z}^3$ with $|\alpha_1| = 3$, $|\alpha_2| = 2$, we have the following identities

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_x^{\alpha_1}(\gamma_2\sigma_H, w_H)\|_{L^2}^2 + \nu\|\nabla \partial_x^{\alpha_1} w_H\|_{L^2}^2 + (\nu+\nu')\|\mathrm{div}\,\partial_x^{\alpha_1} w_H\|_{L^2}^2 \\ &\qquad\qquad = \gamma_2\langle \partial_x^{\alpha_1} f_H, \partial_x^{\alpha_1}\sigma_H\rangle + \langle \partial_x^{\alpha_1} g_H, \partial_x^{\alpha_1} w_H\rangle, \\ &\frac{d}{dt}\langle \nabla \partial_x^{\alpha_2}\sigma_H, \partial_x^{\alpha_2} w_H\rangle + \rho_\infty\gamma_2\|\nabla\partial_x^{\alpha_2}\sigma_H\|_{L^2}^2 = \rho_\infty\|\mathrm{div}\,\partial_x^{\alpha_2} w_H\|_{L^2}^2 + \langle \mathcal{A}\partial_x^{\alpha_2} w_H, \nabla\partial_x^{\alpha_2}\sigma_H\rangle \\ &\qquad\qquad + \langle \nabla\partial_x^{\alpha_2} f_H, \partial_x^{\alpha_2} w_H\rangle + \langle \partial_x^{\alpha_2} g_H, \nabla \partial_x^{\alpha_2}\sigma_H\rangle. \end{aligned}$$

By using Lemma 2.2 and the identity

$$\langle v\cdot\nabla\partial_x^{\alpha_1}\sigma_H, \partial_x^{\alpha_1}\sigma_H\rangle = -\frac{1}{2}\langle \mathrm{div} v\,\partial_x^{\alpha_1}\sigma_H, \partial_x^{\alpha_1}\sigma_H\rangle,$$

we have

$$\begin{aligned}\langle\partial_x^{\alpha_1}f_H, \partial_x^{\alpha_1}\sigma_H\rangle &= \langle(v\cdot\nabla\partial_x^{\alpha_1}\sigma)_H, \partial_x^{\alpha_1}\sigma_H\rangle + \langle((\mathrm{div}\,v)\partial_x^{\alpha_1}\sigma)_H, \partial_x^{\alpha_1}\sigma_H\rangle \\ &\quad + \sum_{0<\beta\le\alpha_1}\langle \mathrm{div}(\partial_x^{\beta}v\,\partial_x^{\alpha_1-\beta}\sigma)_H, \partial_x^{\alpha_1}\sigma_H\rangle + \langle\partial_x^{\alpha_1}\mathrm{div}(\sigma^* w)_H, \partial_x^{\alpha_1}\sigma_H\rangle \\ &\lesssim \|\mathrm{div} v\|_{L^\infty}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2}^2 + \|v\|_{\dot B^{\frac{3}{2}}_{2,1}\cap\dot B^{\frac{5}{2}}_{2,1}}\|\partial_x^{\alpha_1}\sigma\|_{L^2}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} \\ &\quad + \|\nabla v\|_{H^3}\|\nabla\sigma\|_{H^2}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} + \|\nabla\sigma^*\|_{H^3}\|\nabla w\|_{H^3}\|\partial_x^{\alpha_1}\sigma_H\|_{L^2} \\ &\lesssim \delta\big(\|(\sigma,w)\|_{\dot B^{s}_{2,\infty}\cap\dot H^3} + \|w_H\|_{\dot H^4}\big)\|\partial_x^{\alpha_1}\sigma_H\|_{L^2}.\end{aligned}$$

By Lemma 2.2 and Lemma 2.3, we have

$$\langle\partial_x^{\alpha_1}g_H, \partial_x^{\alpha_1}w_H\rangle \lesssim \delta\|(\sigma,v)\|^2_{\dot B^{s}_{2,\infty}\cap\dot H^3} + \delta(\|\partial_x^{\alpha_1}w_H\|^2_{L^2} + \|\partial_x^{\alpha_2}\nabla\sigma_H\|^2_{L^2}).$$

and

$$\begin{aligned}&\sum_{|\alpha_2|=2}\langle\partial_x^{\alpha_2}g_H, \partial_x^{\alpha_2}\nabla\sigma_H\rangle \\ &\qquad\lesssim \delta\|(\sigma,v)\|^2_{\dot B^{s}_{2,\infty}\cap\dot H^3} + \delta(\|\partial_x^{\alpha_1}w_H\|^2_{L^2} + \|\partial_x^{\alpha_2}\nabla\sigma_H\|^2_{L^2}).\end{aligned}$$

Since $\|\partial_x^{\alpha_1}w_H\|_{L^2} \lesssim_{j_0} \|\nabla\partial_x^{\alpha_1}w_H\|_{L^2}$, if $\kappa>0$ is small enough, then

$$\frac{d}{dt}\mathcal{E}_\kappa(t) + c_0\mathcal{E}_\kappa(t) \lesssim_{j_0} \delta\|(\sigma,v)\|^2_{\dot B^{s}_{2,\infty}\cap\dot H^3},$$

where $0<t<T$, $c_0>0$ is a constant and

$$\mathcal{E}_\kappa(t) = \sum_{|\alpha_1|=4}\|\partial_x^{\alpha_1}(\gamma_2\sigma_H, w_H)(t)\|^2_{L^2} + \sum_{|\alpha_2|=3}\kappa\langle\partial_x^{\alpha_2}\nabla\sigma_H(t), \partial_x^{\alpha_2}w_H(t)\rangle.$$

Since $\mathcal{E}_\kappa \sim \|(\sigma_H, w_H)\|_{\dot H^3}$ if $\kappa>0$ is small, by Grönwall's inequality, we have

$$\begin{aligned}\|(\sigma_H, w_H)(t)\|^2_{\dot H^4} &\lesssim e^{-c_0 t}\|(\sigma_H, w_H)(0)\|^2_{\dot H^3} \\ &\qquad + c_1\int_0^t e^{-c_0(t-\tau)}\|(\sigma,w)(\tau)\|^2_{\dot B^{s}_{2,\infty}\cap\dot H^3}d\tau \\ &\lesssim (1+t)^{-s+s_0}(\|(\sigma_0,w_0)\|^2_{\dot H^3} + c_1\mathcal{D}_{s_0}(T)^2),\end{aligned}$$

where $0<t<T$ and $\mathcal{D}_{\epsilon,s_0}(T)$ is the quantity defined in (10). □

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. By Proposition 3.3 and Proposition 3.4, if δ is small enough, then we have

$$\|(\sigma_H, w_H)(t)\|_{\dot B^{s}_{2,\infty}\cap\dot H^3} \lesssim_{j_0} (1+t)^{-\frac{s-s_0}{2}}\|(\sigma_0,w_0)\|_{\dot B^{s_0}_{2,\infty}\cap\dot H^3}, \tag{31}$$

where $t \geq 0$, $-3/2 + \epsilon \leq s \leq 3/2 - \epsilon$ and $-3/2 \leq s_0 \leq 1/2$ with $s_0 \leq s$. By the interpolation inequality (see [2][Proposition 2.22] for example)

$$\|u\|_{\dot{H}^s} \lesssim_{s_1,s_2,\theta} \|u\|_{\dot{B}^{s_1}_{2,\infty}}^{1-\theta} \|u\|_{\dot{B}^{s_2}_{2,\infty}}^{\theta} \quad \text{with} \quad s = (1-\theta)s_1 + \theta s_2,\ \theta \in (0,1), \tag{32}$$

we obtain

$$\|(\rho - \rho^*, v - v^*)(t)\|_{\dot{H}^s \cap \dot{H}^3} \lesssim_s (1+t)^{-\frac{s}{2} - \frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right)} \|(\rho_0 - \rho^*, v_0 - v^*)\|_{L^p \cap H^3} \tag{33}$$

holds for $-3/2 < s < 3/2$, $1 \leq p \leq 2$ with $s/2 + 3/2(1/p - 1/2) > 0$. □

Acknowledgement

The author is supported by JSPS KAKENHI Grant Number JP23KJ0942.

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