L_1 approach to the compressible viscous fluid flows in the half-space

Jou-Chun Kuo   Yoshihiro Shibata

1 Introduction

Let $1 < q < \infty$ and $-1 + N/q \leq s < 1/q$, where $N \geq 2$ is the space dimension. In this paper, we use the L_1–$B^{s+1}_{q,1} \times B^s_{q,1}$ maximal regularity framework to show the local well-posedness of the Navier-Stokes equations describing the isotropic motion of the compressible viscous fluid flows in the half-space. Let

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}, \quad \partial\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}.$$

The equations considered in this paper read as

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{v}) = 0 & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \rho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) - \alpha \Delta \mathbf{v} - \beta \nabla \operatorname{div} \mathbf{v} + \nabla P(\rho) = 0 & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \mathbf{v}|_{\partial\mathbb{R}^N_+} = 0, \quad (\rho, \mathbf{v}) = (\rho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N_+. \end{cases} \tag{1.1}$$

Here, ρ and $\mathbf{v} = (v_1, \cdots, v_N)$ are respective unkown density and velocity functions, while the initial datum $(\rho_0, \mathbf{v}_0)$ is assumed to be given. Moreover, α and β denote respective the viscosity coefficients and the second viscosity coefficients satisfying the conditions

$$\alpha > 0, \quad \alpha + \beta > 0,$$

and $P(\rho)$ is a smooth function defined on $(0, \infty)$ satisfying $P'(\rho) > 0$, that is, the barotropic fluid is considered.

Since the 1950s, lots of mathematicians have contributed to the research on the local well-posedness and global well-posedness of compressible viscous fluids. Regarding the local well-posedness, Solonnikov [17] proved it in $W^{2,1}_q$ with $N < q < \infty$, while Tani [20] proved it in the Hölder spaces. Ströhmer [18] applied the analytic semigroup approach and Enomoto and Shibata [8] proved it in the L_p-L_q maximal regularity class, where $\mathcal{R}$ boundedness of solution operators have been used. When the fluid domain is the whole space, the local well-posedness was proved by Charve and Danchin [4] in the L_1 in time framework.

As for the global well-posedness, Matsumura and Nishida [11, 12] proved it by energy estimates in the three-dimensional whole space, the half space and exterior domains. After that, semigroup approach has been establised. Ströhmer [19] proved the global well-posedness by the semigroup theory. He formulated the system in Lagrange coordinates. To this end, the convection term $\mathbf{v} \cdot \nabla \rho$ can be eliminated and the transformed system can be regarded as a pure parabolic type system. Therefore, the derivative loss from the mass conservation equation vanishes. This idea has also been applied to the maximal regularity approach. As for the maximal L^p regularity $(1 < p < \infty)$, Shibata [14] proved the global well-posedness in exterior domain, which is an improvement of Matsumra and Nishida ’ s theory since he minimized the requirement of the regularity of initial data. On the other hand, for the endpoint case $p = 1$, which is the maximal L_1 regularity. R. Danchin and R. Tolksdorf [7] proved the local and global well-posedness of equations (1.1) in the L_1 in time and $B^{N/q}_{q,1} \times B^{N/q-1}_{q,1}$ in space maximal regularity framework for some $q \in (2, min(4, 2N/(N-2))$, and the main assumption is that the fluid domain is bounded. Especially, they consider only the case where $s = -1 + N/q$ in our notation for thier local well-posedness theory. To obtain the L_1 in time maximal regularity of solutions to the linearized compressible Navier-Stokes equations, in [7] they used their extended version of Da Prato and Grisvard theory [5], which was a first result concerning L_1 maximal regularity for continuous analytic semigroups. In [7] , they assumed that the fluid domain is bounded, which seems to be necessary to obtain the linear theory for Lamé equations cf. [7, Sect. 3] in their argument. However, our strategy of obtaining the L_1 in time maximal regularity is completely different from [5] , and our analysis enable us to treat the unbounded domain. Our result here is in the half space, which is the model problem.

1.1 Notation

Let us summerize the symbols and functional spaces in this paper. Let $L_q(\Omega)$, and $W^m_q(\Omega)$ denote the standard Lebesgue space, and Sobolev space on a domain Ω in N dimensional Euclidean space $\mathbb{R}^N$, while $\|\cdot\|_{L_q(\Omega)}$, and $\|\cdot\|_{W^m_q(\Omega)}$ denote their norms. For time interval I, $L_q(I, X)$ and $W^1_q(I, X)$ denote respective X-valued Lebesgue space and Sobolev space of order 1. $W^\alpha_q(I, X) = (L_q(I, X), W^1_q(I, X))_{[\alpha]}$ for $\alpha \in (0, 1)$. Here, the complex interpolation functiors are denoted by $(\cdot, \cdot)_{[\theta]}$ for $\theta \in (0, 1)$ and $1 \le r \le \infty$. For $1 \le q < \infty$, we write

$$\|f\|_{L_q(I,X)} = \Big(\int_I \|f(t)\|_X^q \, \mathrm{d}t\Big)^{1/q}, \quad \|e^{-\gamma t} f\|_{L_q(I,X)} = \Big(\int_I (e^{-\gamma t}\|f(t)\|_X)^q \, \mathrm{d}t\Big)^{1/q}.$$

For differentiation with respect to space variables $x = (x_1, \dots, x_N)$, $D^\delta f := \partial^\delta_x f = \partial^{|\delta|} f / \partial x_1^{\delta_1} \cdots \partial x_N^{\delta_N}$ for multi-index $\delta = (\delta_1, \dots, \delta_N)$ with $|\delta| = \delta_1 + \cdots + \delta_N$. For the notational simplicity, we write $\nabla f = \{\partial^\delta_x f \mid |\delta| = 1\}$, $\nabla^2 f = \{\partial^\delta_x f \mid |\delta| = 2\}$, $\bar\nabla f = (f, \nabla f)$, and $\bar\nabla^2 f = (f, \nabla f, \nabla^2 f)$. For a Banach space X, $\mathcal{L}(X)$ denotes the set of all bounded linear operators from X into itself and $\|\cdot\|_{\mathcal{L}(X)}$ denotes its norm. Let $\mathbf{I}$ denote the identity operator and $\mathbb{I}$ the $N \times N$ identity matrix. For any Banach space X with norm $\|\cdot\|_X$, $X^N = \{\mathbf{f} = (f_1, \dots, f_N) \mid f_i \in X \ \ (i = 1, \dots, N)\}$ and $\|\mathbf{f}\|_X = \sum_{i=1}^N \|f_i\|_X$. For a vector $\mathbf{v}$ and a matrix $\mathbb{A}$, $\mathbf{v}^\top$ and $\mathbb{A}^\top$ denote respective the transpose of $\mathbf{v}$ and the transpose of $\mathbb{A}$. The letter C denotes a generic constant and $C_{a,b,\cdots} = C(a, b, \cdots)$ denotes the constant depending on quantities a, b, $\cdots$. C, $C_{a,b,\cdots}$, and $C(a, b, \cdots)$ may change from line to line.

Finally, we shall give the definition of inhomogeneous Besov space $B^s_{q,r}$. To this end, we need to introduce Littlewood-Paley decomposition. Let $\phi \in \mathcal{S}(\mathbb{R}^N)$ with $\operatorname{supp}\phi = \{\xi \in \mathbb{R}^N \mid 2^{-1} \le |\xi| \le 2\}$ such that $\sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$. Then, define

$$\phi_k := \mathcal{F}^{-1}_\xi[\phi(2^{-k}\xi)], \quad k \in \mathbb{Z}, \qquad \psi = \mathcal{F}^{-1}_\xi[1 - \sum_{k\in\mathbb{N}} \phi(2^{-k}\xi)].$$

For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ we denote

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} := \begin{cases} \|\psi * f\|_{L_p(\mathbb{R}^N)} + \Big(\sum_{k\in\mathbb{N}} \Big(2^{sk}\|\phi_k * f\|_{L_p(\mathbb{R}^N)}\Big)^q\Big)^{1/q} & \text{if } 1 \le q < \infty, \\ \|\psi * f\|_{L_p(\mathbb{R}^N)} + \sup_{k\in\mathbb{N}} \Big(2^{sk}\|\phi_k * f\|_{L_p(\mathbb{R}^N)}\Big) & \text{if } q = \infty. \end{cases}$$

Here, $f * g$ means the convolution between f and g. Then inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ are defined as the sets of all $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|f\|_{B^s_{p,q}(\mathbb{R}^N)} < \infty$.

Let $s \in \mathbb{R}$, $p \in (1, \infty)$, and $q \in [1, \infty]$. Then the space $B^s_{q,r}(\mathbb{R}^N_+)$ is the collection of all $f \in \mathcal{D}'(\mathbb{R}^N_+)$ such that there exists a function $g \in B^s_{q,r}(\mathbb{R}^N)$ with $g|_{\mathbb{R}^N_+} = f$, where $\mathcal{D}'(\mathbb{R}^N_+)$ be the collection of all complex-valued distributions on $\mathbb{R}^N_+$. Moreover, the norm of $f \in B^s_{q,r}(\mathbb{R}^N_+)$ is given by

$$\|f\|_{B^s_{q,r}(\mathbb{R}^N_+)} = \inf \|g\|_{B^s_{q,r}(\mathbb{R}^N)},$$

where the infimum is taken over all $g \in B^s_{q,r}(\mathbb{R}^N)$ such that its restriction $g|_{\mathbb{R}^N_+}$ coincides with f.

Let $\Omega \in \{\mathbb{R}^N, \mathbb{R}^N_+\}$. It is well-known that $B^s_{p,q}(\Omega)$ may be *characterized* by means of real interpolation. In fact, for $-\infty < s_0 < s_1 < \infty$, $1 < p < \infty$, $1 \le q \le \infty$, and $0 < \theta < 1$, it follows that

$$B^{\theta s_0 + (1-\theta)s_1}_{p,q}(\Omega) = \big(H^{s_0}_p(\Omega), H^{s_1}_p(\Omega)\big)_{\theta,q},$$

cf. [13, Theorem 8], [21, Theorem 2.4.2]. Here, the real interpolation functors are denoted by $(\cdot,\cdot)_{\theta,q}$.

1.2 Main theorem

Our main result of this paper reads as follows.

Theorem 1.1. *Let $N - 1 < q < 2N$ and $-1 + N/q \le s < 1/q$. Let ρ_* be a positive constant describing the mass density of the reference body, and let $\tilde{\eta}_0 \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$. Set $\eta_0(x) = \rho_* + \tilde{\eta}_0(x)$. Assume that there exist two positive constants $\rho_1 < \rho_2$ such that*

$$\rho_1 < \rho_* < \rho_2, \quad \rho_1 < P'(\rho_*) < \rho_2, \quad \rho_1 < \eta_0(x) < \rho_2, \quad \rho_1 < P'(\eta_0(x)) < \rho_2 \quad (x \in \overline{\mathbb{R}^N_+}). \tag{1.2}$$

Then, there exist small numbers $T > 0$ and $\sigma_0 > 0$ such that for any initial data $\rho_0 = \rho_* + \tilde{\rho}_0$ with $\tilde{\rho}_0 \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$ and $\mathbf{v}_0 \in B^s_{q,1}(\mathbb{R}^N_+)$, problem (1.1) admits unique solutions ρ and $\mathbf{v}$ satisfying the regularity conditions:

$$\begin{aligned} \rho - \rho_0 &\in L_1((0,T), B^{s+1}_{q,1}(\mathbb{R}^N_+)) \cap W^1_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)), \\ \mathbf{v} &\in L_1((0,T), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)^N) \end{aligned} \tag{1.3}$$

provided that $\|\tilde{\rho}_0 - \tilde{\eta}_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} \le \sigma_0$.

Remark 1.1. The condition $-1 + N/q \le s < 1/q$ requires that $-1 + N/q < 1/q$. Thus, the condition $N - 1 < q$ is necessary for our argument. On the other hand, the requirement of $s < 1/q$ comes from our linear theory. To use the Abidi-Paicu-Haspot theory for the Besov space estimate of the products of functions (cf. Lemma 2.4 in Sect. 2 below), we have to assume that $-N/q < s < N/q$ when $q \ge 2$ and $-N/q' < s < N/q$ when $1 < q < 2$. Since $-1 + N/q \le s$ when $q \ge 2$, we need to assume that $-N/q' < -1 + N/q$, which is fulfilled by $2N > q$. When $1 < q < 2$, we need to assume that $-N/q' < -1 + N/q$, which is fulfilled by $N > 1$. Thus, $N - 1 < q < 2N$ is necessary for our argument.

1.3 Problem Reformulation

To prove Theorem 1.1, it is advantageous to transfer equations (1.1) to equations in Lagrange coordinates. In fact, the convection term $\mathbf{v} \cdot \nabla \rho$ can be eliminated, the derivative loss from the mass conservation equation vanishes.

Let $\mathbf{u}(x,t)$ be the velocity field in Lagrange coordinates: $x = (x_1, \ldots, x_N)$ and we consider Lagrange transformation:

$$y = X_{\mathbf{u}}(x,t) := x + \int_0^t \mathbf{u}(x,\tau)\, d\tau,$$

where equations (1.1) are written in Euler coordinates: $y = (y_1, \ldots, y_N)$. If

$$\Big\|\int_0^T \nabla \mathbf{u}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_{L_\infty(\mathbb{R}^N_+)} \le c_0 \tag{1.4}$$

with some small constant $c_0 > 0$, and then for each $t \in (0,T)$, the map: $X_{\mathbf{u}}(x,t) = y$ is a C^1 diffeomorphism from $\mathbb{R}^N_+$ onto $\Phi(\mathbb{R}^N_+)$ under the assumption that $\mathbf{u} \in L_1((0,T), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N)$ with $-1+N/q \le s < 1/q$ (cf. Danchin et al [6]). Moreover, using an argument due to Ströhmer [18], we have $\Phi(\mathbb{R}^N_+) = \mathbb{R}^N_+$, and so as a conclusion, $\Phi(\mathbb{R}^N_+)$ is a C^1 diffeomorphism from $\mathbb{R}^N_+$ onto $\mathbb{R}^N_+$.

We shall drive equations in Lagrange coordinates. Let $\mathbb{A}_{\mathbf{u}}$ is the Jacobi matrix of transformation: $y = X_{\mathbf{u}}(x,t)$ for each $t \ge 0$, that is

$$\mathbb{A}_{\mathbf{u}} = \frac{\partial x}{\partial y} = (\frac{\partial y}{\partial x})^{-1} = \Big(\mathbb{I} + \int_0^t \nabla\mathbf{u}(x,\tau)\,\mathrm{d}\tau\Big)^{-1} = \sum_{j=0}^\infty \Big(\int_0^t \nabla\mathbf{u}(x,\tau)\,\mathrm{d}\tau\Big)^j,$$

which is well-defined under the smallness assumption (1.4), where $\mathbb{I}$ denotes the $N\times N$ identity matrix. We have the following well-known formulas:

$$\begin{aligned}
\nabla_y = \mathbb{A}_{\mathbf{u}}^\top\nabla_x, \quad \mathrm{div}_y(\,\cdot\,) = \mathbb{A}_{\mathbf{u}}^\top\colon \nabla_x(\,\cdot\,) = \mathrm{div}_x(\mathbb{A}_{\mathbf{u}}(\,\cdot\,)),\\
\nabla_y\,\mathrm{div}_y(\,\cdot\,) = \mathbb{A}_{\mathbf{u}}^\top\nabla_x((\mathbb{A}_{\mathbf{u}}^\top - \mathbb{I})\colon \nabla_x(\,\cdot\,)) + \mathbb{A}_{\mathbf{u}}^\top\nabla_x\,\mathrm{div}_x(\,\cdot\,),\\
\Delta_y(\,\cdot\,) = \mathrm{div}_y\,\nabla_y(\,\cdot\,) = \mathrm{div}_x(\mathbb{A}_{\mathbf{u}}\mathbb{A}_{\mathbf{u}}^\top\nabla_x(\,\cdot\,)) = \mathrm{div}_x((\mathbb{A}_{\mathbf{u}}\mathbb{A}_{\mathbf{u}}^\top - \mathbb{I})\nabla_x(\cdot)) + \Delta_x(\,\cdot\,).
\end{aligned}$$

Then the system of equations (1.1) in Lagrange coordinates reads

$$\begin{cases}
\partial_t\rho + \rho\,\mathrm{div}\,\mathbf{u} = F(\rho,\mathbf{u}) & \text{in } \mathbb{R}^N_+\times(0,T),\\
\rho\partial_t\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\,\mathrm{div}\,\mathbf{u} + \nabla P(\rho) = \mathbf{G}(\rho,\mathbf{u}) & \text{in } \mathbb{R}^N_+\times(0,T),\\
\mathbf{u}|_{\partial\mathbb{R}^N_+} = 0, \quad (\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \mathbb{R}^N_+.
\end{cases} \tag{1.5}$$

Here, we have set

$$\begin{aligned}
F(\rho,\mathbf{u}) &= \rho((\mathbb{I} - \mathbb{A}_{\mathbf{u}}) : \nabla\mathbf{u})\\
\mathbf{G}(\rho,\mathbf{u}) &= (\mathbb{I} - (\mathbb{A}_{\mathbf{u}}^\top)^{-1})(\rho\partial_t\mathbf{u} - \alpha\Delta\mathbf{u}) + \alpha(\mathbb{A}_{\mathbf{u}}^\top)^{-1}\,\mathrm{div}((\mathbb{A}_{\mathbf{u}}\mathbb{A}_{\mathbf{u}}^\top - \mathbb{I}) : \nabla\mathbf{u})\\
&\quad + \beta\nabla((\mathbb{A}_{\mathbf{u}}^\top - \mathbb{I}) : \nabla\mathbf{u}).
\end{aligned}$$

For equations (1.5), we shall prove the following theorem, which is (1.1) with the equations in Lagrange coordinates.

Theorem 1.2. *Let $N-1 < q < 2N$ and $-1+N/q \le s < 1/q$. Let ρ_*, $\tilde\eta_0(x)$, and $\eta_0(x)$ be the same as in Theorem 1.1. Then, there exist constants $\sigma_0 > 0$ and $T > 0$ such that for any initial data $\rho_0 \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$ and $\mathbf{u}_0 \in B^s_{q,1}(\mathbb{R}^N_+)^N$, problem (1.5) admits unique solutions ρ and $\mathbf{u}$ satisfying the regularity conditions:*

$$\rho - \rho_0 \in W^1_1((0,T), B^{s+1}_{q,1}(\mathbb{R}^N_+)), \quad \mathbf{u} \in L_1((0,T), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)^N)$$

provided that $\|\rho_0 - \eta_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} \le \sigma_0$.

2 Spectral Analysis and L_1 Semigroup

To prove Theorem 1.2, the key issue is the L_1 maximal regularity theorem for the linearized equations of (1.5) at initial mass density $\eta_0(x) = \rho_* + \tilde\eta_0(x)$ with $\tilde\eta_0(x) \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$, which read as

$$\begin{cases} \partial_t\Pi + \eta_0(x)\operatorname{div}\mathbf{V} = F & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \eta_0(x)\partial_t\mathbf{V} - \alpha\Delta\mathbf{V} - \beta\nabla\operatorname{div}\mathbf{V} + \nabla(P'(\eta_0(x))\Pi) = \mathbf{G} & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \mathbf{V}|_{\partial\mathbb{R}^N_+} = 0, \quad (\Pi, \mathbf{V})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in } \mathbb{R}^N_+. \end{cases} \tag{2.1}$$

We shall prove the following theorem, which will be used to prove Theorem 1.2.

Theorem 2.1. *Let $N-1 < q < 2N$, $-1+N/q \le s < 1/q$, and $T > 0$. Let ρ_*, $\tilde\eta_0(x)$, and $\eta_0(x)$ be the same as in Theorem 1.1. Then, there exist positive constants $\gamma > 0$ and $C > 0$ such that for any initial data $(\rho_0, \mathbf{v}_0)$ and right membes $(F, \mathbf{G})$ such that $(\rho_0, \mathbf{v}_0) \in B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^s_{q,1}(\mathbb{R}^N_+)^N$,*

$$e^{-\gamma t}F \in L_1(\mathbb{R}_+, B^{s+1}_{q,1}(\mathbb{R}^N_+)), \quad e^{-\gamma t}\mathbf{G} \in L_1(\mathbb{R}_+, B^s_{q,1}(\mathbb{R}^N_+)^N),$$

then the initial boundary problem (2.1) admits unique solutions $(\Pi, \mathbf{V})$ with

$$e^{-\gamma t}\Pi \in W^1_1(\mathbb{R}_+, B^{s+1}_{q,1}(\mathbb{R}^N_+)), \quad e^{-\gamma t}\mathbf{V} \in L_1(\mathbb{R}_+, B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1(\mathbb{R}_+, B^s_{q,1}(\mathbb{R}^N_+)^N)$$

possessing the estimate:

$$\begin{aligned} &\|e^{-\gamma t}(\Pi, \partial_t\Pi)\|_{L_1(\mathbb{R}_+, B^{s+1}_{q,1}(\mathbb{R}^N_+))} + \|e^{-\gamma t}\mathbf{V}\|_{L_1(\mathbb{R}_+, B^{s+2}_{q,1}(\mathbb{R}^N_+))} + \|e^{-\gamma t}\partial_t\mathbf{V}\|_{L_1(\mathbb{R}_+, B^s_{q,1}(\mathbb{R}^N_+))} \\ &\quad \le C(\|(\rho_0, \mathbf{v}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^s_{q,1}(\mathbb{R}^N_+)} + \|e^{-\gamma t}(F, \mathbf{G})\|_{L_1(\mathbb{R}_+, B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^s_{q,1}(\mathbb{R}^N_+))}). \end{aligned}$$

Here and in the sequel, we set $\mathbb{R}_+ = (0, \infty)$, and

$$\|e^{-\gamma t}f\|_{L_1(\mathbb{R}_+, X)} = \int_0^\infty e^{-\gamma t}\|f(\cdot, t)\|_X \, dt.$$

In order to prove Theorem 2.1, we use the properties of solutions to the corresponding generalized resolvent problem:

$$\begin{cases} \lambda\rho + \eta_0 \operatorname{div}\mathbf{v} = f & \text{in } \mathbb{R}^N_+, \\ \eta_0(x)\lambda\mathbf{v} - \alpha\Delta\mathbf{v} - \beta\nabla\operatorname{div}\mathbf{v} + \nabla(P'(\eta_0)\rho) = \mathbf{g} & \text{in } \mathbb{R}^N_+, \\ \mathbf{v}|_{\partial\mathbb{R}^N_+} = 0. \end{cases} \tag{2.2}$$

To state our main result for equations (2.2), we introduce a parabolic sector Σ_μ defined by setting

$$\Sigma_\mu = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \mu\}, \tag{2.3}$$

where $\mu \in (0, \pi/2)$ and $\gamma > 0$. The set $\Sigma_\mu + \gamma$ is defined by

$$\Sigma_\mu + \gamma = \{\lambda + \gamma \mid \lambda \in \Sigma_\mu\}.$$

Moreover, functional spaces $\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$ and $\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)$ and their norms are defined by setting

$$\begin{aligned} \mathcal{H}^s_{q,1}(\mathbb{R}^N_+) &= \{(f, \mathbf{g}) \in B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^s_{q,1}(\mathbb{R}^N_+)^N\}, \\ \mathcal{D}^s_{q,1}(\mathbb{R}^N_+) &= \{(\rho, \mathbf{v}) \in B^{s+1}_{q,1}(\mathbb{R}^N_+) \times B^{s+2}_{q,1}(\mathbb{R}^N_+)^N \mid \mathbf{v}|_{\partial\mathbb{R}^N_+} = 0\}, \\ \|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} &= \|f\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{B^s_{q,1}(\mathbb{R}^N_+)}, \\ \|(f, \mathbf{g})\|_{\mathcal{D}^s_{q,1}(\mathbb{R}^N_+)} &= \|f\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)}. \end{aligned} \tag{2.4}$$

Then, we shall show the following theorem.

Theorem 2.2. *Let $1 < q < 2N$ and $-1+N/q \le s < 1/q$. Let ρ_*, $\tilde\eta_0(x)$, and $\eta_0(x)$ be the same as in Theorem 1.1. Then, the following three assertions hold.*

(1) *There exist constants $\gamma > 0$ and C such that for any $\lambda \in \Sigma_\mu + \gamma$ and $(f, \mathbf{g}) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$, problem (2.2) admits a unique solution $(\rho, \mathbf{v}) \in \mathcal{D}^s_{q,1}(\mathbb{R}^N_+)$ posssessing the estimate:*

$$\|\lambda(\rho, \mathbf{v})\|_{\mathcal{H}^s_{q,1}} + \|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathbf{v}\|_{B^s_{q,1}} \le C\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}$$

for every $\lambda \in \Sigma_\mu + \gamma$.

(2) *Let $\sigma > 0$ be a small number such that $-1 + 1/q < s - \sigma < s + \sigma < 1/q$. Then, there exist $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_i \in B^{s+2}_{q,1}(\mathbb{R}^N_+)$ $(i = 1, 2)$, $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, and there hold*

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathbf{v}_1\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathbf{v}_1\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}$$

for every $\lambda \in \Sigma_\mu + \gamma$ and $\mathbf{g} \in C^\infty_0(\mathbb{R}^N_+)$, as well as

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathbf{v}_2\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathbf{v}_2\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-2}\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}$$

for any $\lambda \in \Sigma_\mu + \gamma$ and $(f, \mathbf{g}) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$.

(3) *There exist constants γ and C such that for every $\lambda \in \Sigma_\mu + \gamma$ and $(f, \mathbf{g}) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$, and there hold*

$$\|\rho\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)},$$
$$\|\partial_\lambda\rho\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-2}\|(f, \mathbf{g})\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}.$$

In the statement of (1), (2) and (3), the constants γ and C depend on ρ_* and $\|\tilde\eta_0\|_{B^{s+1}_{q,1}}$.

2.1 Spectral Analysis

In this subsection, we shall prove Theorem 2.2 as a perturbation from Lamé equations, which read as

$$\eta_0(x)\lambda\mathbf{v} - \alpha\Delta\mathbf{v} - \beta\nabla\operatorname{div}\mathbf{v} = \mathbf{g} \quad \text{in } \mathbb{R}^N_+, \quad \mathbf{v}|_{\partial\mathbb{R}^N_+} = 0 \tag{2.5}$$

for spectral parameter $\lambda \in \Sigma_\mu + \gamma$ with large enough $\gamma > 0$. Thus, we start with the existence theorem for equations (2.5).

Theorem 2.3. *Let $1 < q < \infty$ and $-1 + 1/q < s < 1/q$. Let σ be a small positive number such that $-1 + 1/q < s - \sigma < s < s + \sigma < 1/q$. Let $\nu = s$ or $s \pm \sigma$. Assume that $\tilde\eta_0 \in B^{N/q}_{q,1}(\mathbb{R}^N_+)$. Moreover, $\eta_0(x)$ and ρ_* satisfy the assumptions (1.2). Then, there exist constants $\gamma_1 > 0$ and $C > 0$ depending on s, σ, and $\|\tilde\eta_0\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)}$ such that for any $\lambda \in \Sigma_\mu + \gamma_1$, problem (2.5) admits a unique solution $\mathbf{v} \in B^\nu_{q,1}(\mathbb{R}^N_+)^N$ satisfying the estimate:*

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathbf{v}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \le C\|\mathbf{g}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)},$$
$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathbf{v}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|\mathbf{g}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}.$$

Moreover, for any $\lambda \in \Sigma_\mu + \gamma_1$ and $\mathbf{g} \in C^\infty_0(\mathbb{R}^N_+)^N$, there holds

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathbf{v}\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C(1 + \|\tilde\eta_0\|_{B^s_{q,1}(\mathbb{R}^N_+)})|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{g}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}$$

as well as

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathbf{v}\|_{B^s_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{v}\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C(1 + \|\tilde\eta_0\|_{B^s_{q,1}(\mathbb{R}^N_+)})|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{g}\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.$$

Remark 2.1. $C_0^\infty(\mathbb{R}^N_+)$ is dense in $B^\nu_{q,1}(\mathbb{R}^N_+)$ provided that $-1+1/q<\nu<1/q$ and $1<q<\infty$.

We need the following lemma in this paper.

Lemma 2.4. *Let $1<q<\infty$ and $\nu\in\mathbb{R}$. If the condition $|\nu|<N/q$ for $q\geq 2$ holds and the condition $-N/q'<\nu<N/q$ for $1<q<2$ holds, then for any $u\in B^\nu_{q,1}(\mathbb{R}^N_+)$ and $v\in B^{N/q}_{q,\infty}(\mathbb{R}^N_+)\cap L_\infty(\mathbb{R}^N_+)$, there holds*

$$\|uv\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}\leq C_\nu\|u\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}\|v\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)}$$

for some constant $C>0$ independent of u and v.

Proof. For a proof, refer to [1, Cor. 2.5] and [9, Cor. 1]. □

By Theorem 2.3, we consider problem (2.2) of the Stokes system and prove Theorem 2.2. We insert the relation: $\rho=\lambda^{-1}(f-\eta_0\operatorname{div}\mathbf{v})$ obtained from the first equation in (2.2) into the second equations. Then, we have

$$\eta_0(x)\lambda\mathbf{v}-\alpha\Delta\mathbf{v}-\beta\nabla\operatorname{div}\mathbf{v}-\lambda^{-1}\nabla(P'(\eta_0)\eta_0\operatorname{div}\mathbf{v})=\mathbf{h}\quad\text{in }\mathbb{R}^N_+,\quad \mathbf{u}|_{\partial\mathbb{R}^N_+}=0,\tag{2.6}$$

where we have set $\mathbf{h}=\mathbf{g}-\lambda^{-1}\nabla(P'(\eta_0)f)$. In what follows, restore the notation of $\mathbb{R}^N_+$ like $B^s_{q,1}(\mathbb{R}^N_+)$, $\|\cdot\|_{B^s_{q,1}(\mathbb{R}^N_+)}$ etc.

As a first step to analyze equations (2.6), we shall prove the following theorem.

Theorem 2.5. *Let $N-1<q<2N$ and $-1+N/q\leq s<1/q$. Let $\sigma>0$ be a small number such that $-1+1/q<s-\sigma<s<s+\sigma<1/q$, and let $\nu=s$ or $s\pm\sigma$. Let $\eta_0(x)=\rho_*+\tilde\eta_0(x)$ with $\tilde\eta_0\in B^{s+1}_{q,1}(\mathbb{R}^N_+)$. Let $\gamma_1>0$ be the constant given in Theorem 2.3. Then, there exist $\gamma_2\geq\gamma_1$ and an operator family $\mathcal{S}(\lambda)$ such that $\mathcal{S}(\lambda)\in\operatorname{Hol}(\Sigma_\mu+\gamma_2,\mathcal{L}(B^s_{q,1}(\mathbb{R}^N_+),B^{s+2}_{q,1}(\mathbb{R}^N_+)))$, for any $\lambda\in\Sigma_\mu+\gamma_2$ and $\mathbf{h}\in B^\nu_{q,1}(\mathbb{R}^N_+)$ $\mathbf{v}=\mathcal{S}(\lambda)\mathbf{h}$ is a unique solution of equations (2.6), and there hold*

$$\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{S}(\lambda)\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\leq C\|\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)},$$
$$\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\partial_\lambda\mathcal{S}(\lambda)\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\leq C|\lambda|^{-1}\|\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}.$$

Moreover, there are two operator families $\mathcal{S}^i(\lambda)\in\operatorname{Hol}(\Sigma_\mu+\gamma_2,\mathcal{L}(B^\nu_{q,1}(\mathbb{R}^N_+),B^{\nu+2}_{q,1}(\mathbb{R}^N_+)))$ $(i=1,2)$ such that $\mathcal{S}(\lambda)=\mathcal{S}^1(\lambda)+\mathcal{S}^2(\lambda)$,

$$\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{S}^1(\lambda)\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\leq C|\lambda|^{-\frac{\sigma}{2}}\|\mathbf{h}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)},$$
$$\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\partial_\lambda\mathcal{S}^1(\lambda)\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|\mathbf{h}\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}$$

for any $\lambda\in\Sigma_\mu+\gamma_2$ and $\mathbf{h}\in C_0^\infty(\mathbb{R}^N_+)$, and

$$\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{S}^2(\lambda)\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\leq C|\lambda|^{-1}\|\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)},$$
$$\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\partial_\lambda\mathcal{S}^2(\lambda)\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\leq C|\lambda|^{-2}\|\mathbf{h}\|_{B^s_{q,1}(\mathbb{R}^N_+)}$$

for any $\lambda\in\Sigma_\mu+\gamma_2$ and $\mathbf{h}\in B^s_{q,1}(\mathbb{R}^N_+)$.

Here, the constants γ_2 and C depend on ρ_* and $\|\tilde\eta_0\|_{B^{s+1}_{q,1}}$.

We need the following lemma for the proof in this paper, which is the lemma for the Besov norm estimate of composite functions cf. [9, Proposition 2.4] and [3, Theorem 2.87].

Lemma 2.6. *Let $1 < q < \infty$. Let I be an open interval of $\mathbb{R}$. Let $\omega > 0$ and let $\tilde\omega$ be the smallest integer such that $\tilde\omega \geq \omega$. Let $F : I \to \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in BC^{\tilde\omega}(I, \mathbb{R})$. Assume that $v \in B^\omega_{q,r}$ has valued in $J \subset\subset I$. Then, $F(v) \in B^\omega_{q,1}$ and there exists a constant C depending only on ν, I, J, and N, such that*

$$\|F(v)\|_{B^\omega_{q,1}} \leq C(1 + \|v\|_{L_\infty})^{\tilde\omega}\|F'\|_{BC^{\tilde\omega}(I,\mathbb{R})}\|v\|_{B^\omega_{q,1}}.$$

Proof of Theorem 2.2. Recall the symbols defined in (2.4), which will be used below. Let $\mathbf{v} = \mathcal{S}(\lambda)(\mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f))$, and then $\mathbf{v}$ is a unique solution of equations (2.6) with $\mathbf{h} = \mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f)$. Using the formula $\mathcal{S}(\lambda) = \mathcal{S}^1(\lambda) + \mathcal{S}^2(\lambda)$, we divide $\mathbf{v}$ as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where

$$\mathbf{v}_1 = \mathcal{S}^1(\lambda)\mathbf{g}, \quad \mathbf{v}_2 = \mathcal{S}^2(\lambda)\mathbf{g} - \lambda^{-1}\mathcal{S}(\lambda)\nabla(P'(\eta_0)f).$$

Moreover, define ρ by $\rho = \lambda^{-1}(f - \eta_0 \operatorname{div}\mathbf{v})$. By using Lemmas 2.4 , 2.6 with the assumption: $N/q \leq s + 1$ and the above theorems, we complete the proof of Theorem 2.2. □

2.2 L_1 semigroup

Let $\mathcal{A}$ be an operator defined by

$$\mathcal{A}(\rho, \mathbf{v}) = (\eta_0 \operatorname{div}\mathbf{v},\ \eta_0(x)^{-1}(-\alpha\Delta\mathbf{v} - \beta\nabla \operatorname{div}\mathbf{v} + \nabla(P'(\eta_0(x)\rho)))$$

for $(\rho, \mathbf{u}) \in \mathcal{D}^s_{q,1}(\mathbb{R}^N_+)$. Then, problem (2.2) reads as

$$(\lambda\mathbf{I} + \mathcal{A})(\rho, \mathbf{v}) = (f, \eta_0(x)^{-1}\mathbf{g}). \tag{2.7}$$

Noticing that $\eta_0(x)^{-1} = \rho_*^{-1} - \tilde\eta_0(x)(\rho_*(\rho_* + \tilde\eta_0(x))^{-1}$, we see that there exists a constant $c_0 > 0$ depending on ρ_* and $\|\tilde\eta_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$ such that

$$c_0^{-1}\|\mathbf{g}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \leq \|\eta_0^{-1}\mathbf{g}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \leq c_0\|\mathbf{g}\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}.$$

for $\nu = s$ or $\nu = s\pm\sigma$. Thus, Theorem 2.2 holds for the equations (2.7), which replaces equations (2.2). Therefore, $\mathcal{A}$ generates a continuous analytic semigroup $\{T(t)\}_{t\geq 0}$ and solutions Π and $\mathbf{U}$ of equations (2.1) are given by

$$(\Pi, \mathbf{V}) = T(t)(\rho_0, \mathbf{v}_0) + \int_0^t T(t - s)(F(\cdot, s), \rho_0(\cdot)^{-1}\mathbf{G}(\cdot, s))\,\mathrm{d}s.$$

We now prove the L_1 in time maximal regularity of $\{T(t)\}_{t\geq 0}$. The idea of our proof here is due to Shibata [15], cf also Kuo [10] and Shibata and Watanabe [16]. Let $T_1(t)$ and $T_2(t)$ denote the mass density part of $T(t)$ and the velocity part of $T(t)$. Namely, $T(t)(\rho_0, \mathbf{v}_0) = (T_1(t)(\rho_0, \mathbf{v}_0), T_2(t)(\rho_0, \mathbf{v}_0))$ and $\rho = T_1(t)(\rho_0, \mathbf{v}_0)$ and $\mathbf{v} = T_2(t)(\rho_0, \mathbf{v}_0)$ satisfy equations (2.1) with $F = \mathbf{G} = 0$.

Theorem 2.7. *Let $N - 1 < q < 2N$ and $-1 + N/q \leq s < 1/q$. Let $\tilde\eta_0(x) \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$ and $\eta_0(x) = \rho_* + \tilde\eta_0(x)$ satisfies the assumption* (1.2)*. Let $\gamma > 0$ be a constant given in Theorem 2.2, which depends on ρ_* and $\|\tilde\eta_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$. Then, there exists a constant $C > 0$ depending on ρ_* and $\|\eta_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$ such that for any $(f, \mathbf{g}) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$, there holds*

$$\int_0^\infty e^{-\gamma t}(\|(\partial_t, \bar\nabla^2)T_2(t)(\rho_0, \mathbf{v}_0)\|_{B^s_{q,1}(\mathbb{R}^N_+)} + \|(1, \partial_t)T_1(t)(\rho_0, \mathbf{v}_0)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)})\,\mathrm{d}t$$
$$\leq C\|(\rho_0, \mathbf{v}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}.$$

Proof of Theorem 2.1. Let F_0 and $\mathbf{G}_0$ be zero extension of F and $\mathbf{G}$ outside of $(0,T)$ interval. Using $\{T(t)\}_{t\ge 0}$, we can write

$$(\Pi, \mathbf{V})(t) = T(t)(\rho_0, \mathbf{v}_0) + \int_0^t T(t-s)(F_0(\cdot,s), \rho_0^{-1}(\cdot)\mathbf{G}_0(\cdot,s))\,ds.$$

Let γ and C be the constant given in Theorem 2.7. By Fubini's theorem, we have

$$\begin{aligned}
&\int_0^\infty e^{-\gamma t}\|\bar\nabla^2 \int_0^t T_2(t-\ell)(F_0, \eta_0^{-1}\mathbf{G}_0)(\ell)\ \mathrm{d}\ell\|_{B^s_{q,1}(\mathbb{R}^N_+)}\ \mathrm{d}t\\
&\quad\le \int_0^\infty \Big\{\int_\ell^\infty e^{-\gamma t}\|\bar\nabla^2 T_2(t-\ell)(F_0, \eta_0^{-1}\mathbf{G}_0)(\ell)\|_{B^s_{q,1}(\mathbb{R}^N_+)}\ \mathrm{d}t\Big\}\ \mathrm{d}\ell\\
&\quad= \int_0^\infty e^{-\gamma\ell}\Big\{\int_0^\infty e^{-\gamma t}\|\bar\nabla^2 T_2(t)(F_0, \eta_0^{-1}\mathbf{G}_0)(\ell)\|_{B^s_{q,1}(\mathbb{R}^N_+)}\ \mathrm{d}t\Big\}\ \mathrm{d}\ell\\
&\quad\le C\int_0^\infty e^{-\gamma\ell}\|(F_0(\cdot,\ell), \eta_0^{-1}\mathbf{G}_0(\cdot,\ell))\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)}\ \mathrm{d}\ell\\
&\quad\le C\|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}.
\end{aligned}$$

Employing completely the same argument, we have

$$\int_0^\infty e^{-\gamma t}\|\int_0^t T_1(t-\ell)(F_0, \rho_0^{-1}\mathbf{G}_0)(\ell)\ \mathrm{d}\ell\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}\ \mathrm{d}t \le C\|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}.$$

Therefore, we have

$$\begin{aligned}
&\int_0^\infty e^{-\gamma t}(\|\rho(\cdot,t)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{v}(\cdot,t)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)})\,dt\\
&\qquad\le C(\|(\rho_0,\mathbf{v}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}),
\end{aligned}$$

which implies that

$$\begin{aligned}
&e^{-\gamma T}\int_0^T(\|\rho(\cdot,t)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{v}(\cdot,t)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)})\,dt\\
&\qquad\le C(\|(\rho_0,\mathbf{v}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_0^T(\|\rho(\cdot,t)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{v}(\cdot,t)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)})\,dt\\
&\qquad\le Ce^{\gamma T}(\|(\rho_0,\mathbf{v}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}).
\end{aligned}$$

Here, γ and C are constants depending on ρ_*, $\|\tilde\eta_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$.

To show the estimate of time derivatives, we use the relations:

$$\begin{aligned}
&\partial_t\Pi = -\eta_0\operatorname{div}\mathbf{v} + F,\\
&\partial_t\mathbf{V} = (\eta_0)^{-1}(\alpha\Delta\mathbf{V} + \beta\nabla\operatorname{div}\mathbf{V} - \nabla(P(\eta_0)\Pi) + \mathbf{G}),
\end{aligned}$$

and then,

$$\begin{aligned}
&\int_0^T(\|\partial_t\Pi(\cdot,t)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\partial_t\mathbf{V}(\cdot,t)\|_{B^s_{q,1}(\mathbb{R}^N_+)})\,dt\\
&\le C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}})(\int_0^T(\|\Pi(\cdot,t)\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} + \|\mathbf{V}(\cdot,t)\|_{B^{s+2}_{q,1}(\mathbb{R}^N_+)})\,dt + \|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))})\\
&\le C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}})e^{\gamma T}(\|(\rho_0,\mathbf{v}_0)\|_{\mathcal{H}^s_{q,1}(\mathbb{R}^N_+)} + \|(F,\mathbf{G})\|_{L_1((0,T),\mathcal{H}^s_{q,1}(\mathbb{R}^N_+))}).
\end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

3 Proof of Main Result

In this section, first we shall prove Theorem 1.2, which is the local well-posedness for the Navier-Stokes equations in Lagrange coordinates. Then, we convert the obtained solution back to Euler coordinates and prove the local well-posedness for the original compressible Navier-Stokes equations, which is Theorem 1.1.

Let $\eta_0(x) = \rho_* + \tilde\eta_0(x)$ with $\tilde\eta_0(x) \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$ and assume that $\eta_0(x)$ satisfy the assumption (1.2). Let $\rho_0(x) = \rho_* + \tilde\rho_0(x)$ with $\tilde\rho_0(x) \in B^{s+1}_{q,1}(\mathbb{R}^N_+)$. Let $\omega > 0$ be a small number determined late and assume that

$$\|\tilde\rho_0 - \tilde\eta_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} < \omega. \tag{3.1}$$

Let $\mathbf{u}_0 \in B^s_{q,1}(\mathbb{R}^N_+)^N$. We consider equations (1.5). By setting $\rho = \rho_0 + \theta$ we write equations (1.5) as follows:

$$\begin{cases} \partial_t\theta + \eta_0 \operatorname{div}\mathbf{u} = (\eta_0 - \rho_0 - \theta)\operatorname{div}\mathbf{u} + F(\theta+\rho_0, \mathbf{u}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \eta_0\partial_t\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\operatorname{div}\mathbf{u} + \nabla(P'(\eta_0)\theta) = -\nabla P(\rho_0) + \mathbf{G}(\theta+\rho_0,\mathbf{u}) & \\ \qquad + \tilde{\mathbf{G}}(\theta,\mathbf{u}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \mathbf{u}|_{\partial\mathbb{R}^N_+} = 0, \quad (\theta,\mathbf{u})|_{t=0} = (0,\mathbf{u}_0) & \text{in } \mathbb{R}^N_+, \end{cases}$$

where we have set $\tilde{\mathbf{G}}(\theta,\mathbf{u}) = (\eta_0 - \rho_0 - \theta)\partial_t\mathbf{u} - \nabla(P(\rho_0+\theta) - P(\rho_0) - P'(\eta_0)\theta)$. To prove Theorem 1.2, we use the Banach contraction mapping principle. To this end, we introduce an energy functional E_T and the underlying space $S_{T,\omega}$ defined by

$$E_T(\eta,\mathbf{w}) = \|(\eta,\partial_t\eta)\|_{L_1((0,T),B^{s+1}_{q,1}(\mathbb{R}^N_+))} + \|\mathbf{w}\|_{L_1((0,T),B^{s+2}_{q,1}(\mathbb{R}^N_+))} + \|\partial_t\mathbf{w}\|_{L_1((0,T),B^{s}_{q,1}(\mathbb{R}^N_+))},$$

$$S_{T,\omega} = \left\{ (\eta,\mathbf{w}) \;\middle|\; \begin{array}{l} \eta \in W^1_1((0,T),B^{s+1}_{q,1}(\mathbb{R}^N_+)), \\ \mathbf{w} \in L_1((0,T),B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,T),B^s_{q,1}(\mathbb{R}^N_+)^N), \\ (\eta,\mathbf{w})|_{t=0} = (0,\mathbf{u}_0), \quad E_T(\eta,\mathbf{w}) \le \omega, \quad \displaystyle\int_0^T \|\nabla\mathbf{w}(\cdot,\tau)\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)}\,\mathrm{d}\tau \le c_1 \end{array} \right\}.$$

Here, $T > 0$, $\omega > 0$ and $c_1 > 0$ are small constants chosen later. In particular, c_1 is chosen in such a way that

$$\Big\|\int_0^T \nabla\mathbf{w}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_{L_\infty(\mathbb{R}^N_+)} \le C\int_0^T \|\nabla\mathbf{w}(\cdot,\tau)\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)} \le Cc_1 \le c_0,$$

where c_0 is a constant appearing in (1.4). Thus, the constant c_1 guarantees that the Lagrange map $y = X_{\mathbf{w}}(x,t)$ is C^1 diffeomorphism from Ω onto Ω.

Given $(\theta,\mathbf{u}) \in S_{T,\omega}$, let η and $\mathbf{w}$ be solutions to the system of linear equations:

$$\begin{cases} \partial_t\eta + \eta_0\operatorname{div}\mathbf{w} = (\eta_0 - \rho_0 - \theta)\operatorname{div}\mathbf{u} + F(\rho_0+\theta,\mathbf{u}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \eta_0\partial_t\mathbf{w} - \alpha\Delta\mathbf{w} - \beta\nabla\operatorname{div}\mathbf{w} + \nabla(P'(\eta_0)\eta) = -\nabla P(\rho_0) & \\ \qquad + \mathbf{G}(\rho_0+\theta,\mathbf{u}) + \tilde{\mathbf{G}}(\theta,\mathbf{u}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \mathbf{w}|_{\partial\mathbb{R}^N_+} = 0, \quad (\eta,\mathbf{w})|_{t=0} = (0,\mathbf{u}_0) & \text{in } \mathbb{R}^N_+. \end{cases} \tag{3.2}$$

Let $\eta_{\mathbf{a}}$ and $\mathbf{w}_{\mathbf{a}}$ be solutions of the system of linear equations:

$$\begin{cases} \partial_t\eta_{\mathbf{a}} + \eta_0\operatorname{div}\mathbf{w}_{\mathbf{a}} = 0 & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \eta_0\partial_t\mathbf{w}_{\mathbf{a}} - \alpha\Delta\mathbf{w}_{\mathbf{a}} - \beta\nabla\operatorname{div}\mathbf{w}_{\mathbf{a}} + \nabla(P'(\eta_0)\eta_{\mathbf{a}}) = -\nabla P(\rho_0) & \text{in } \mathbb{R}^N_+ \times (0,\infty), \\ \mathbf{w}_{\mathbf{a}}|_{\partial\mathbb{R}^N_+} = 0, \quad (\eta_{\mathbf{a}},\mathbf{w}_{\mathbf{a}})|_{t=0} = (0,\mathbf{u}_0) & \text{in } \mathbb{R}^N_+. \end{cases}$$

We will choose $T > 0$ small enough later, and so for a while we assume that $0 < T < 1$. By Theorem 2.1, we know the unique existence of solutions $\eta_{\mathbf{a}}$ and $\mathbf{w}_{\mathbf{a}}$ satisfying the regularity conditions:

$$\eta_{\mathbf{a}} \in W^1_1((0,1), B^{s+1}_{q,1}(\mathbb{R}^N_+)), \quad \mathbf{w}_{\mathbf{a}} \in L_1((0,1), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,1), B^s_{q,1}(\mathbb{R}^N_+)^N)$$

as well as the estimates:

$$\begin{aligned} &\|(\eta_{\mathbf{a}}, \partial_t \eta_{\mathbf{a}})\|_{L_1((0,1), B^{s+1}_{q,1}(\mathbb{R}^N_+))} + \|(\partial_t, \bar{\nabla}^2)\mathbf{w}_{\mathbf{a}}\|_{L_1((0,1), B^s_{q,1}(\mathbb{R}^N_+))} \\ &\quad \le C e^{\gamma}(\|\mathbf{u}_0\|_{B^s_{q,1}(\mathbb{R}^N_+)} + \|\nabla P(\rho_0)\|_{B^s_{q,1}(\mathbb{R}^N_+)}). \end{aligned} \tag{3.3}$$

Here, γ is a constant depending on ρ_*, $\|\tilde{\eta}_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$ given in Theorem 2.1. Here and in the following, C denotes a general constant depending at most on ρ_* and $\|\tilde{\eta}_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$, which is changed from line to line, but independent of ω and T.

In view of (3.3), $\eta_{\mathbf{a}}$ and $\mathbf{w}_{\mathbf{a}}$ satisfy $E_1(\eta_{\mathbf{a}}, \mathbf{w}_{\mathbf{a}}) < \infty$, and so we choose $T \in (0,1)$ small enough in such a way that

$$E_T(\eta_{\mathbf{a}}, \mathbf{w}_{\mathbf{a}}) \le \omega/2. \tag{3.4}$$

Let ρ and $\mathbf{v}$ be solutions to the system of linear equations:

$$\begin{cases} \partial_t \rho + \eta_0 \operatorname{div} \mathbf{v} = (\eta_0 - \rho_0 - \theta)) \operatorname{div} \mathbf{u} + F(\theta + \rho_0, \mathbf{u}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \eta_0 \partial_t \mathbf{v} - \alpha \Delta \mathbf{v} - \beta \nabla \operatorname{div} \mathbf{v} + \nabla(P'(\eta_0)\rho) = \mathbf{G}(\theta + \rho_0, \mathbf{u}) + \tilde{\mathbf{G}}(\theta, \mathbf{u}) & \text{in } \mathbb{R}^N_+ \times (0,T), \\ \mathbf{v}|_{\partial \mathbb{R}^N_+} = 0, \quad (\rho, \mathbf{v})|_{t=0} = (0,0) & \text{in } \mathbb{R}^N_+. \end{cases} \tag{3.5}$$

Applying Theorem 2.1, we see the existence of solutions ρ and $\mathbf{v}$ of equations (3.5) satisfying the regularity condition:

$$\rho \in W^1_1((0,T), B^{s+1}_{q,1}(\mathbb{R}^N_+)), \quad \mathbf{v} \in L_1((0,T), B^{s+2}_{q,1}(\mathbb{R}^N_+)^N) \cap W^1_1((0,T), B^s_{q,1}(\mathbb{R}^N_+)^N)$$

as well as the estimate:

$$\begin{aligned} E_T(\rho, \mathbf{v}) \le C e^{\gamma T}(&\|(\eta_0 - \rho_0 - \theta) \operatorname{div} \mathbf{u}, F(\theta + \rho_0, \mathbf{u})\|_{L_1((0,T), B^{s+1}_{q,1}(\mathbb{R}^N_+))} \\ &+ \|(\mathbf{G}(\theta + \rho_0, \mathbf{u}), \tilde{\mathbf{G}}(\theta, \mathbf{u}))\|_{L_1((0,T), B^s_{q,1}(\mathbb{R}^N_+))}). \end{aligned} \tag{3.6}$$

Here, we notice that γ and C depend on ρ_* and $\|\tilde{\eta}_0\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)}$ but is independent of ω and T.

Now, we shall show that there exist constants $C > 0$ and $\omega > 0$ such that

$$\begin{aligned} &\|((\eta_0 - \rho_0 - \theta) \operatorname{div} \mathbf{u}, F(\theta + \rho_0, \mathbf{u}))\|_{L_1((0,T), B^{s+1}_{q,1}(\mathbb{R}^N_+))} \\ &\quad + \|(\mathbf{G}(\theta + \rho_0, \mathbf{u}), \tilde{\mathbf{G}}(\theta, \mathbf{u}))\|_{L_1((0,T), B^s_{q,1}(\mathbb{R}^N_+))} \le C(\omega^2 + \omega^3). \end{aligned} \tag{3.7}$$

If we show (3.7), then by (3.6) we have

$$E_T(\rho, \mathbf{v}) \le C e^{\gamma T}(\omega^2 + \omega^3).$$

Choose $\omega > 0$ and $T > 0$ so small that $Ce(\omega + \omega^2) \le 1/2$ and $\gamma T \le 1$. Then, we have

$$E_T(\rho, \mathbf{v}) < \omega/2, \tag{3.8}$$

which combined with (3.4), implies that $\eta = \eta_{\mathbf{a}} + \rho$ and $\mathbf{w} = \mathbf{w}_{\mathbf{a}} + \mathbf{v}$ satisfy equations (3.2) and $E_T(\eta, \mathbf{w}) < \omega$. Especially, ω is chosen so small that

$$\int_0^T \|\nabla \mathbf{w}(\cdot, \tau)\|_{B^{N/q}_{q,1}(\mathbb{R}^N_+)} \, d\tau \le C E_T(\eta, \mathbf{w}) \le C\omega \le c_1.$$

As a consequence, $(\eta, \mathbf{w}) \in S_{T,\omega}$. Thus, if we define the map Φ by $\Phi(\theta, \mathbf{u}) = (\eta, \mathbf{w})$, then Φ maps $S_{T,\omega}$ into $S_{T,\omega}$.

Now, we shall show (3.7). For notational simplicity, we omit $\mathbb{R}^N_+$ below. Notice that $B^{N/q}_{q,1}$ is a Banach algebra (cf. [9, Proposition 2.3]). By Lemma 2.4 and the assumption: $N/q \le s+1$, we see that $B^{s+1}_{q,1}$ is also a Banach algebra. In fact,

$$\begin{aligned}\|uv\|_{B^{s+1}_{q,1}} &\le \|(\nabla u)v\|_{B^s_{q,1}} + \|u\bar\nabla v\|_{B^s_{q,1}} \le C(\|\nabla u\|_{B^s_{q,1}}\|v\|_{B^{N/q}_{q,1}} + \|u\|_{B^{N/q}_{q,1}}\|\bar\nabla v\|_{B^s_{q,1}}) \\ &\le C\|u\|_{B^{s+1}_{q,1}}\|v\|_{B^{s+1}_{q,1}}.\end{aligned}$$

We first estimate $(\eta_0 - \rho_0 - \theta)\operatorname{div}\mathbf{u}$ and $F(\theta + \rho_0, \mathbf{u})$. By Lemma 2.4 and (3.1), we have

$$\|(\eta_0 - \rho_0)\operatorname{div}\mathbf{u}\|_{B^{s+1}_{q,1}} \le C\omega\|\mathbf{u}\|_{B^{s+2}_{q,1}}.$$

Since $B^{s+1}_{q,1}$ is a Banach algebra, we have

$$\|\theta\operatorname{div}\mathbf{u}\|_{B^{s+1}_{q,1}} \le C\|\theta\|_{B^{s+1}_{q,1}}\|\operatorname{div}\mathbf{u}\|_{B^{s+1}_{q,1}}.$$

Since $\theta|_{t=0} = 0$, here and in the sequel we use the following estimate:

$$\|\theta(\cdot,t)\|_{B^{s+1}_{q,1}} = \Big\|\int_0^t \partial_s\theta(\cdot,s)\,\mathrm{d}s\Big\|_{B^{s+1}_{q,1}(\mathbb{R}^N_+)} \le \|\partial_t\theta\|_{L_1((0,T),B^{s+1}_{q,1}(\mathbb{R}^N_+))}. \tag{3.9}$$

Thus, we have

$$\|\theta\operatorname{div}\mathbf{u}\|_{L_1((0,T),B^{s+1}_{q,1})} \le C\|\partial_t\theta\|_{L_1((0,T),B^{s+1}_{q,1})}\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})}.$$

We next estimate $F(\rho_0+\theta, \mathbf{u}) = (\rho_0+\theta)((\mathbb{I} - \mathbb{A}_\mathbf{u}) : \nabla\mathbf{u})$. Recall that $\mathbf{u}$ satisfies

$$\int_0^T \|\nabla\mathbf{u}(\cdot,\tau)\|_{B^{N/q}_{q,1}}\,\mathrm{d}\tau \le c_1.$$

Since $B^{N/q}_{q,1} \subset L_\infty$, we have

$$\sup_{t\in(0,T)}\Big\|\int_0^t \nabla\mathbf{u}(\cdot,\tau)\,\mathrm{d}\tau\Big\|_{L_\infty} \le C\int_0^T \|\nabla\mathbf{u}(\cdot,\tau)\|_{B^{N/q}_{q,1}}\,\mathrm{d}\tau \le Cc_1. \tag{3.10}$$

Choosing c_1 so small that $Cc_1 < 1$. Let $F(\ell)$ be a C^∞ function defined on $|\ell| \le Cc_1$ and $F(0) = 0$, and $\mathbb{I} - \mathbb{A}_\mathbf{u} = F(\int_0^t \nabla\mathbf{u}\,\mathrm{d}\ell)$. In fact, $F(\ell) = -\sum_{j=1}^\infty \ell^j$. Then, by Lemma 2.6 and (3.10), we have

$$\sup_{t\in(0,T)}\|F(\int_0^t \nabla\mathbf{u}\,\mathrm{d}\tau)\|_{B^{s+1}_{q,1}} \le C\int_0^T \|\nabla\mathbf{u}(\cdot,\tau)\|_{B^{s+1}_{q,1}}\,d\tau. \tag{3.11}$$

Since $B^{s+1}_{q,1}$ is a Banach algebra, using (3.11) we have

$$\|F(\rho_0+\theta,\mathbf{u})\|_{B^{s+1}_{q,1}} \le C(\|\rho_0\|_{B^{s+1}_{q,1}} + \|\theta(\cdot,t)\|_{B^{s+1}_{q,1}})\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})}\|\nabla\mathbf{u}(\cdot,t)\|_{B^{s+1}_{q,1}}.$$

Using (3.9), we have

$$\|F(\rho_0+\theta,\mathbf{u})\|_{L_1((0,T),B^{s+1}_{q,1})} \le C(\|\rho_0\|_{B^{s+1}_{q,1}} + \|\theta_t\|_{L_1((0,T),B^{s+1}_{q,1})})\|\mathbf{u}\|^2_{L_1((0,T),B^{s+2}_{q,1})}.$$

Summing up, we have proved that

$$\begin{aligned}
&\|(\eta_0-\rho_0-\theta)\operatorname{div}\mathbf{u}, F(\theta+\rho_0,\mathbf{u})\|_{L_1((0,T),B^{s+1}_{q,1})} \\
&\quad\le C\{\omega\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})}+\|\partial_t\theta\|_{L_1((0,T),B^{s+1}_{q,1})}\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})} \\
&\quad+(\|\eta_0\|_{B^{s+1}_{q,1}}+1)\|\mathbf{u}\|^2_{L_1((0,T),B^{s+2}_{q,1})}+\|\partial_t\theta\|_{L_1((0,T),B^{s+1}_{q,1})}\|\mathbf{u}\|^2_{L_1((0,T),B^{s+2}_{q,1})}\}.
\end{aligned}\tag{3.12}$$

Here and in the following, we use the estimate:

$$\|\rho_0\|_{B^{s+1}_{q,1}}\le\|\rho_0-\eta_0\|_{B^{s+1}_{q,1}}+\|\eta_0\|_{B^{s+1}_{q,1}}\le 1+\|\eta_0\|_{B^{s+1}_{q,1}}.$$

Next, we estimate $\mathbf{G}(\theta+\rho_0,\mathbf{u})$ and $\tilde{\mathbf{G}}(\theta,\mathbf{u})$. These terms can be estimated in a similiar method, and so we omit the prove here. Therefore we have

$$\begin{aligned}
&\|(\mathbf{G}(\theta+\rho_0,\mathbf{u})\|_{L_1((0,T),B^{s}_{q,1})} \\
&\quad\le C(\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})}(\|\rho_0\|_{B^{s+1}_{q,1}}+\|\partial_t\theta\|_{L_1((0,T),B^{s+1}_{q,1})})\|\partial_t\mathbf{u}\|_{L_1((0,T),B^{s}_{q,1})} \\
&\quad+\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})}(1+\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})})\|\mathbf{u}\|_{L_1((0,T),B^{s+2}_{q,1})}),
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
\|\tilde{\mathbf{G}}(\theta,\mathbf{u})\|_{L_1((0,T),B^{s}_{q,1})}&\le C(\rho_*,\|\tilde\eta_0\|_{B^{s+1}_{q,1}})\{\omega(\|\partial_t\mathbf{u}\|_{L_1((0,T),B^{s}_{q,1})}+\|\theta\|_{L_1((0,T),B^{s+1}_{q,1})}) \\
&\quad+\|\partial_t\theta\|_{L_1((0,T),B^{s+1}_{q,1})}(\|\theta\|_{L_1((0,T),B^{s+1}_{q,1})}+\|\partial_t\mathbf{u}\|_{L_1((0,T),B^{s}_{q,1})})\}.
\end{aligned}\tag{3.14}$$

Combining (3.12), (3.13), (3.14) and recalling that $E_T(\theta,\mathbf{u})\le\omega$, we have (3.7). And so, choosing $\omega>0$ and $T>0$ so small that $Ce(\omega+\omega^2)\le 1/2$ and $\gamma T\le 1$, we have (3.8). Here, C and γ depend on ρ_* and $\|\tilde\eta_0\|_{B^{s+1}_{q,1}}$, and so the smallness of ω and $T>0$ depends on ρ_* and $\|\tilde\eta_0\|_{B^{s+1}_{q,1}}$. Therefore, we see that Φ maps $S_{T,\omega}$ into itself.

We now prove that Φ is contraction map from $S_{T,\omega}$ into itself. To this end, pick up two elements $(\theta_i,\mathbf{u}_i)\in S_{T,\omega}$ $(i=1,2)$. Similiarly, we obtain

$$E_T(\eta_1-\eta_2,\mathbf{w}_1-\mathbf{w}_2)\le Ce^{\gamma T}(\omega+\omega^2)E_T(\theta_1-\theta_2,\mathbf{u}_1-\mathbf{u}_2).$$

Thus, choosing $\omega>0$ and $T>0$ so small that $Ce(\omega+\omega^2)\le 1/2$ and $\gamma T\le 1$, we have

$$E_T(\eta_1-\eta_2,\mathbf{w}_1-\mathbf{w}_2)\le(1/2)E_T(\theta_1-\theta_2,\mathbf{u}_1-\mathbf{u}_2),$$

which shows that Φ is a contraction map from $S_{T,\omega}$ into itself. Therefore, by the Banach fixed point theorem, Φ has a unique fixed point $(\eta,\mathbf{w})\in S_{T,\omega}$. In (3.2), setting $(\eta,\mathbf{w})=(\theta,\mathbf{u})$ and recalling $\rho=\rho_0+\theta$ and $\tilde{\mathbf{G}}(\theta,\mathbf{u})=(\eta_0-\rho_0-\theta)\partial_t\mathbf{u}-\nabla(P(\rho_0+\theta)-P(\rho_0)-P'(\eta_0)\theta)$, we see that θ and $\mathbf{u}$ satisfy equations:

$$\begin{cases}
\partial_t\theta+\eta_0\operatorname{div}\mathbf{u}=(\eta_0-\rho_0-\theta)\operatorname{div}\mathbf{u}+F(\rho_0+\theta,\mathbf{u}) & \text{in } \mathbb{R}^N_+\times(0,T),\\
\eta_0\partial_t\mathbf{u}-\alpha\Delta\mathbf{u}-\beta\nabla\operatorname{div}\mathbf{u}+\nabla(P'(\eta_0)\theta)=-\nabla P(\rho_0) & \\
\qquad\qquad+\mathbf{G}(\rho_0+\theta,\mathbf{u})-\tilde{\mathbf{G}}(\theta,\mathbf{u}) & \text{in } \mathbb{R}^N_+\times(0,T),\\
\mathbf{u}|_{\partial\mathbb{R}^N_+}=0,\quad(\eta,\mathbf{u})|_{t=0}=(0,\mathbf{u}_0) & \text{in } \mathbb{R}^N_+.
\end{cases}\tag{3.15}$$

Thus, setting $\rho=\rho_0+\theta$, from (3.15) it follows that ρ and $\mathbf{u}$ satisfy equations (1.5). Moreover, $(\rho,\mathbf{u})$ belongs to $S_{T,\omega}$, which completes the proof of Theorem 1.2.

Next, $y=X_{\mathbf{u}}(x,t)$ is a C^1 diffeomorphism from Ω onto itself for any $t\in(0,T)$, because $\mathbf{u}\in L_1((0,T),B^{s+2}_{q,1}(\Omega)^N)$. Let $x=X^{-1}_{\mathbf{u}}(y,t)$ be the inverse of $X_{\mathbf{u}}$. For any function $F\in B^s_{q,1}(\mathbb{R}^N_+)$, $1<q<\infty$, $s\in\mathbb{R}$, it follow from the chain rule that

$$\|F\circ X^{-1}_{\mathbf{u}}\|_{B^s_{q,1}(\mathbb{R}^N_+)}\le C\|F\|_{B^s_{q,1}(\mathbb{R}^N_+)}$$

with some constant $C > 0$ (cf. Amann [2, Theorem 2.1]). Let $(\rho, \mathbf{v}) = (\theta, \mathbf{u}) \circ X_{\mathbf{u}}^{-1}$ and $\mathbb{A}_{\mathbf{u}} = (\nabla_y X_{\mathbf{u}})^{-1}$. Let $\mathbb{A}_{\mathbf{u}}^\top = (A_{jk})$. There holds

$$\nabla_y(\rho, \mathbf{v}) = (\mathbb{A}_{\mathbf{u}}^\top \nabla_x(\theta, \mathbf{u})) \circ X_{\mathbf{u}}^{-1},$$
$$\partial_{y_j}\partial_{y_k}\mathbf{v} = \sum_{\ell,\ell'} A_{j\ell}\partial_{y_\ell}(A_{k\ell'}\partial_{y_{\ell'}}\mathbf{u})) \circ X_{\mathbf{u}}^{-1} \quad (j, k = 1, \ldots, N).$$

Hence, we rely on the relation:

$$\partial_t(\rho, \mathbf{v}) = \partial_t(\theta, \mathbf{u}) \circ X_{\mathbf{u}}^{-1} - ((\mathbf{u} \circ X_{\mathbf{u}}^{-1}) \cdot \nabla_y)(\rho, \mathbf{v}),$$

concerning the time derivative of ρ and $\mathbf{v}$. Therefore, by Theorem 1.2 and Lemma 2.4, we arrive at (1.3). This completes the proof of Theorem 1.1.

4 Acknowledgments

We would like to express our sincere gratitude to RIMS and the organizers Associate Professor Masahiro Suzuki and Lecturer Masashi Aiki for giving us this oppurtinty to give a talk in the workshop "Mathematical Analysis in Fluid and Gas Dynamics."

J.-C. Kuo is supported by JST SPRING, Grant Number JPMJSP2128 and Y. Shibata partially supported by JSPS KAKENHI Grant Number 22H01134.

MSC Classification of 2020: Primary: 76N10; Secondary: 35Q30.
keywords: Navier–Stokes equations; Free boundary problem; Maximal L_1-regularity

References

[1] H. Abidi and M. Paicu, *Existence globale pour un fluide inhomogène*, Ann. Inst. Fourier (Grenoble) **57** (3) (2007), 883–917

[2] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech. **2**(1) (2000), 16–98.

[3] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 343, Springer, Heidelberg, 2011. DOI: 10.1007/978-3-642-16830-7.

[4] F. Charve and R. Danchin, *A global existence result for the compressible Navier-Stokes equations in the critical L^p framework*, Arch. Rational Mech. Anal., **198** (2010), 233–271.

[5] G. Da Prato and P. Grisvard, *Sommes d'opérateurs linéaires et équations différentielles opŕationelles*, J. Math. Pures Appl. (9) **54** (3) (1974), 305–387.

[6] R. Danchin, M. Hieber, P. Mucha and P. Tolksdorf, *Free boundary problems via Da Prato-Grisvard theory*, Preprint: https://arxiv.org/abs/2011.07918.

[7] R. Danchin and P. Tolksdorf, *Critical regularity issues for the compressible Navier-Stokes system in nbounded domains*, Math. Ann. **387** (2023), 1903–1959.

[8] Y. Enomoto and Y. Shibata, *On the $\mathcal{R}$-sectoriality and the initial boundary value problem for the viscous compressible fluid flow*, Funk. Ekva. **56** (2013), 441–505. DOI: 10.1619/fesi.56.441

[9] B. Haspot, *Well-posedness in critical spaces for the system of compressible Navier-Stokes in larger spaces*, J. Differential Equations **251** (8) (2011), 2262–2295. DOI: 10.1016/j.jde.2011.06.013.

[10] J.-C. Kuo, *Resolvent estimate for the linearized system of the compressible Navier-Stokes equations in Besov spaces.* Master thesis of Waseda University, 2022.

[11] A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto. Univ., **20** (1980), 67–104.

[12] A. Matsumura and T. Nishida, *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Comm. Math. Phys. **89** (1983), 445–464.

[13] T. Muramatsu, *On Besov spaces and Sobolev spaces of generalized functions define on a general region*, Rubl. RIMS, Kyoto Univ. **9** (1974), 325–396.

[14] Y. Shibata, *New thought on Matsumura-Nishida theory in the L_p-L_q maximal regularity framework,* J. Math. Fluid Mech. **24**(3) Paper No. 66 (2022)

[15] Y. Shibata, *Spectral analysis approach to the maximal regularity for the Sotkes equations and free boundary problem for the Navier-Stokes equations*, RIMS Kôkyûroku. To appear.

[16] Y. Shibata and K. Watanabe, *Maximal L_1-regularity of the Navier-Stokes equations with free boundaary conditions via a generalized semigroup theory*, Preprint, arXiv:2311.04444 [math.AP] 8 Nov 2023.

[17] V. Solonnikov, *Solvability of the initial boundary value problem for the equations of motion of a viscous compressible fluid*, J. Sov. Math. **14** (1980), 1120–1132.

[18] G. Ströhmer, *About a certain class of parabolic-hyperbolic systems of differential equations*, Analysis **9** (1989), 1–39.

[19] G. Ströhmer, *About compressible viscous fluid flow in a bounded region*, Pacific J. Math., **143** (1990), 359–375.

[20] A. Tani, *On the first initial-boundary value problem of compressible viscous fluid motion*, Publ. Res. Inst. Math. Sci., **13** (1977), 193–253.

[21] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam, New York, 1978.

(J.-C. Kuo)
School of Science and Engineering, Waseda University,
3-4-5 Ohkubo Shinjuku-ku, Tokyo, 169-8555, Japan
Email address: kuojouchun@asagi.waseda.jp

(Y. Shibata)
Professor Emeritus of Waseda University,
3-4-5 Ohkubo Shinjuku-ku, Tokyo, 169-8555, Japan.,
Adjunct faculty member in the Department of Mechanical Engineering and Materials Science,
University of Pittsburgh, USA
Email address: yshibata325@gmail.com