A priori estimates for solutions to equations of motion of an inextensible hanging string

Tatsuo Iguchi and Masahiro Takayama*

Department of Mathematics Faculty of Science and Technology, Keio University

1 Introduction

We are concerned with the motion of a homogeneous and inextensible string of finite length L under the action of the gravity and a tension of the string. Suppose that one end of the string is fixed and another one is free. Let s be the arc length of the string measured from the free end of the string so that the string is descried as a curve

$$\boldsymbol{x}(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t)), \qquad s \in [0,L]$$

at time t. We can assume without loss of generality that the fixed end of the string is placed at the origin in $\mathbb{R}^3$. Let ρ be a constant density of the string, $\boldsymbol{g}$ the acceleration of gravity vector, and $\tau(s,t)$ a scalar tension of the string at the point $\boldsymbol{x}(s,t)$ at time t. See Figure 1.1. Then, the motion of the string is described by the equations

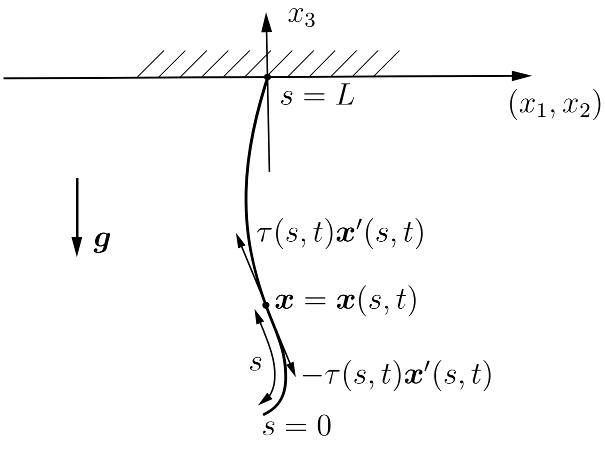

Figure 1.1: Hanging String

$$\begin{cases} \rho\ddot{\boldsymbol{x}} - (\tau\boldsymbol{x}')' = \rho\boldsymbol{g} & \text{in} \quad (0,L)\times(0,T), \\ |\boldsymbol{x}'| = 1 & \text{in} \quad (0,L)\times(0,T), \end{cases}$$

where $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}'$ denote the derivatives of $\boldsymbol{x}$ with respect to t and s, respectively, so that $\boldsymbol{x}'$ is a unit tangential vector of the string. For a derivation of these equations, we refer,

*Corresponding author

for example, to Reeken [11] and Yong [18]. The first equation is the equation of motion. The peculiarity of this problem is that we do not assume any elasticity of the string so that the tension τ in the first equation is also an unknown quantity of the problem. In other words, we do not assume any constitutive equation for the tension τ. However, we impose the second equation, which describes the fact that the string is inextensible. We also note that the tension is caused by the inextensibility of the string as we will see below. The boundary conditions at the both ends of the string are given by

$$\begin{cases} \boldsymbol{x} = \boldsymbol{0} & \text{on} \quad \{s = L\} \times (0, T), \\ \tau = 0 & \text{on} \quad \{s = 0\} \times (0, T). \end{cases}$$

The first boundary condition represents that the one end of the string is fixed at the origin, and the second one represents that another end is free. In the case $\boldsymbol{g} \neq \boldsymbol{0}$, by making the change of the variables $s \to Ls$, $t \to \sqrt{L/g}t$, $\boldsymbol{x} \to L\boldsymbol{x}$, $\tau \to \rho g L \tau$, and $\boldsymbol{g}/g \to \boldsymbol{g}$ with $g = |\boldsymbol{g}|$, we may assume that $\rho = 1$, $L = 1$, and $|\boldsymbol{g}| = 1$. Similarly, in the case $\boldsymbol{g} = \boldsymbol{0}$, by making the same change of the variables as above with any positive constant g, we may assume that $\rho = 1$ and $L = 1$. Therefore, in the following we consider the equations

$$\begin{cases} \ddot{\boldsymbol{x}} - (\tau \boldsymbol{x}')' = \boldsymbol{g} & \text{in} \quad (0, 1) \times (0, T), \\ |\boldsymbol{x}'| = 1 & \text{in} \quad (0, 1) \times (0, T), \end{cases} \tag{1.1}$$

under the boundary conditions

$$\begin{cases} \boldsymbol{x} = \boldsymbol{0} & \text{on} \quad \{s = 1\} \times (0, T), \\ \tau = 0 & \text{on} \quad \{s = 0\} \times (0, T). \end{cases} \tag{1.2}$$

Here, $\boldsymbol{g}$ is a constant unit vector or the zero vector. Finally, we impose the initial conditions of the form

$$(\boldsymbol{x}, \dot{\boldsymbol{x}})|_{t=0} = (\boldsymbol{x}_0^{\text{in}}, \boldsymbol{x}_1^{\text{in}}) \quad \text{in} \quad (0, 1). \tag{1.3}$$

This is the initial boundary value problem that we are going to consider in this paper. Here, we remark that the problem (1.1) and (1.2) also arises in a minimization problem of the action function $J(\boldsymbol{x}) = \int_0^T \int_0^1 \bigl(\frac{1}{2}|\dot{\boldsymbol{x}}(s,t)|^2 + \boldsymbol{g} \cdot \boldsymbol{x}(s,t)\bigr) \mathrm{d}s\mathrm{d}t$ under the constraints $|\boldsymbol{x}(s,t)| \equiv 1$ and $\boldsymbol{x}(1,t) = \boldsymbol{0}$. In this case, the tension τ appears as a Lagrangian multiplier. For more details on this variational principle, we refer, for example, to Şengül and Vorotnikov [15] and the references therein.

As was explained above, the tension τ is also an unknown quantity. On the other hand, we assume that the string is inextensible so that we impose the constraint $|\boldsymbol{x}'| = 1$, which causes a tension of the string. In other words, by using the constraint we can derive an equation for the tension τ as follows. Let $(\boldsymbol{x}, \tau)$ be a solution to (1.1) and (1.2). Then, we see that τ satisfies the following two-point boundary value problem

$$\begin{cases} -\tau'' + |\boldsymbol{x}''|^2 \tau = |\dot{\boldsymbol{x}}'|^2 & \text{in} \quad (0, 1) \times (0, T), \\ \tau = 0 & \text{on} \quad \{s = 0\} \times (0, T), \\ \tau' = -\boldsymbol{g} \cdot \boldsymbol{x}' & \text{on} \quad \{s = 1\} \times (0, T), \end{cases} \tag{1.4}$$

where we regard the time t as a parameter. This is a well-known fact and is easily verified; see, for example, Preston [8, Section 2.1] and Şengül and Vorotnikov [15, Section 2.4]. In fact, by differentiating the constraint $|\boldsymbol{x}'|^2 = 1$ with respect to s and t, we have $\boldsymbol{x}' \cdot \boldsymbol{x}'' = 0$, $\boldsymbol{x}' \cdot \boldsymbol{x}''' + |\boldsymbol{x}''|^2 = 0$, $\boldsymbol{x}' \cdot \dot{\boldsymbol{x}}' = 0$, and $\boldsymbol{x}' \cdot \ddot{\boldsymbol{x}}' + |\dot{\boldsymbol{x}}'|^2 = 0$. Therefore, differentiating the first equation in (1.1) with respect to s and then taking an inner product with $\boldsymbol{x}'$, we obtain the first equation in (1.4). Taking an inner product of the first equation in (1.1) with $\boldsymbol{x}'$, taking its trace on $s = 0$, and using the first boundary condition in (1.2), we obtain the last boundary condition in (1.4). It is easy to see that for each fixed time t, the two-point boundary value problem (1.4) can be solved uniquely, so that τ is determined by $\boldsymbol{x}'(\cdot, t)$ and $\dot{\boldsymbol{x}}'(\cdot, t)$. Unlike standard theories of nonlinear wave equaions, in our problem the tension τ depends nonlocally in space and time on $\boldsymbol{x}'$. Particularly, we need an information of the curvature vector $\boldsymbol{x}''(\cdot, t)$ and the deformation velocity $\dot{\boldsymbol{x}}'(\cdot, t)$ of the tangential vector of the string to determine the tension τ.

For the well-posedness of the initial boundary value problem, standard analysis on hyperbolic systems requires a positivity of the tension τ. However, the positivity fails necessarily at the free end $s = 0$ due to the boundary condition on τ. Taking these into account, in place of assuming a strict positivity of τ, we impose the following stability condition

$$\frac{\tau(s,t)}{s} \geq c_0 > 0 \tag{1.5}$$

for $(s, t) \in (0, 1) \times (0, T)$. If we consider a linearized problem around the rest state, then the corresponding stability condition is reduced to $-\boldsymbol{g} \cdot \boldsymbol{x}'(1, t) \geq c_0 > 0$ for $t \in (0, T)$. This last condition can be easily understood geometrically; see Figures 1.2 and 1.3. As

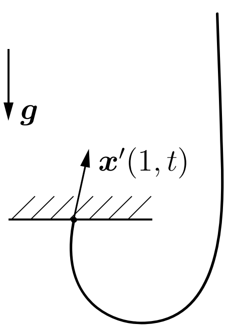

Figure 1.2: The case $-\boldsymbol{g} \cdot \boldsymbol{x}'(1, t) > 0$

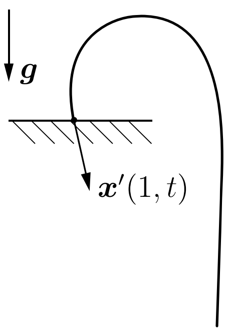

Figure 1.3: The case $-\boldsymbol{g} \cdot \boldsymbol{x}'(1, t) < 0$

we will see in Section 4, under the condition $\int_0^1 s|\boldsymbol{x}''(s,t)|^2 \mathrm{d}s \lesssim 1$, the tension τ can be bounded from below as

$$\frac{\tau(s,t)}{s} \gtrsim -\boldsymbol{g} \cdot \boldsymbol{x}'(1,t) + \int_0^1 s|\dot{\boldsymbol{x}}'(s,t)|^2 \mathrm{d}s \exp\left(- \int_0^1 s|\boldsymbol{x}''(s,t)|^2 \mathrm{d}s \right), \tag{1.6}$$

if the right-hand side is non-negative. This reveals a nonlinear stabilizing effect of the problem, and moreover, ensures the stability condition even in the case $\boldsymbol{g} = \boldsymbol{0}$ if $\dot{\boldsymbol{x}}(s, t) \not\equiv$

$\mathbf{0}$; see Subsection 4.3. Our main objective is to show the local well-posedness of the initial boundary value problem (1.1)–(1.3) in appropriate weighted Sobolev spaces under the stability condition (1.5). Toward this goal, in this paper we will derive a priori estimates for the solution $(\boldsymbol{x}, \tau)$ to the problem, which are given in Theorem 2.1. We also prove a uniqueness of solutions to the problem in the class where a priori estimates would be obtained. The uniqueness is given in Theorems 2.3 and 2.4.

Even if a priori estimates for the solution $(\boldsymbol{x}, \tau)$ would be obtained, it is not straightforward to construct an existence theory and we need more technical calculations than what will be done in this paper. Therefore, we postpone this existence part in our future work. Here, we just give a brief comment relative to the existence of a solution: In the derivation of the two-point boundary value problem (1.4), we use essentially the constraint $|\boldsymbol{x}'| = 1$ so that it is natural to expect that (1.4) contains an information of the constraint. In view of this, we will consider the initial boundary value problem to hyperbolic equations

$$\begin{cases} \ddot{\boldsymbol{x}} - (\tau \boldsymbol{x}')' = \boldsymbol{g} & \text{in} \quad (0,1) \times (0,T), \\ \boldsymbol{x} = \mathbf{0} & \text{on} \quad \{s=1\} \times (0,T), \end{cases} \tag{1.7}$$

for $\boldsymbol{x}$ under the initial condition (1.3), coupled with the two-point boundary value problem (1.4) for τ, in place of the problem (1.1)–(1.3). One may think that a boundary condition on the free end $s = 0$ is missing for the well-posedness of the problem for $\boldsymbol{x}$. However, it is not the case because the tension is degenerate at $s = 0$. For more details, see Takayama [17]. We will use the initial boundary value problem to the hyperbolic and elliptic coupled system (1.7), (1.4), and (1.3) to construct the solution $(\boldsymbol{x}, \tau)$. In order to show the equivalence of the problems, we need to show that the solution $(\boldsymbol{x}, \tau)$ to the transformed system (1.7), (1.4), and (1.3) satisfies the constraint $|\boldsymbol{x}'| = 1$ under appropriate conditions on the initial data. Here, we note that if there exists a smooth solution to the original problem (1.1)–(1.3), then the initial data have to satisfy the constraints $|\boldsymbol{x}_0^{\mathrm{in}\prime}| = 1$ and $\boldsymbol{x}_0^{\mathrm{in}\prime} \cdot \boldsymbol{x}_1^{\mathrm{in}\prime} = 0$ in $(0,1)$. Conversely, we will show in Theorem 2.5 that if the initial data satisfy these constrains, then any regular solution $(\boldsymbol{x}, \tau)$ to the transformed system (1.7), (1.4), and (1.3) satisfying the stability condition (1.5) satisfies the constraint $|\boldsymbol{x}'| = 1$. This will be carried out by using an energy estimate.

Contrary to the studies on elastic strings, there are few results on the well-posedness of the initial boundary value problem (1.1)–(1.3) to the motion of an inextensible straing. Reeken [12, 13] considered the motion of an inextensible string of *infinite* length having one end fixed at the point $(0,0,\infty)$ in a gravity field. For technical reasons he assumed that the acceleration of gravity vector $\boldsymbol{g}$ is not constant. To be precise, he assumed that $\boldsymbol{g} = \boldsymbol{g}(s) \in C^\infty([0,\infty))$ is constant for $s \in [0,l]$ and grows linearly beyond $s = l$ for some positive l. Under this non-physical condition, he proved the existence locally in time and uniqueness of the solution provided that the initial data are sufficiently close to a trivial stationary solution in some weighted Sobolev spaces. The method that he used to solve the original problem (1.1)–(1.3) is quite different from solving the transformed problem (1.7), (1.4), and (1.3). He applied the hard implicit function theorem, which is also known as the Nash–Moser theorem, to construct the solution so that higher regularity must be imposed on the initial data and that a loss of derivatives was allowed. Preston [8] considered the motion of an inextensible string of finite length in the case without any external forces, that is, in the case $\boldsymbol{g} = \boldsymbol{0}$. Under this particular situation, he proved the existence locally in time and uniqueness of the solution for arbitrary initial data in some weighted Sobolev spaces. Although the weighted Sobolev spaces used by Preston [8] may seem to be different from those used by Reeken [12, 13], their norms are equivalent so that their weighted Sobolev spaces are identical; see Proposition 3.5. In order to solve the original problem (1.1)–(1.3), he used the transformed problem (1.7), (1.4), and (1.3). To be precise, in order to construct a solution he introduced a discretized problem with respect to s and used uniform estimates for discrete solutions. Moreover, since the constraint $|\boldsymbol{x}'| = 1$ could not be achieved if we use the discretization method described above, he guaranteed it by using the spherical coordinate such as $\boldsymbol{x}'(s,t) = (\cos\theta(s,t), \sin\theta(s,t))$ in the two-dimensional case. Şengül and Vorotnikov [15] considered exactly the same initial boundary value problem (1.1)–(1.3) as ours and proved the existence of an admissible Young measure solution after transforming the problem into a system of conservation laws with a discontinuous flux. We note that the existence of such a generalized Young measure solution does not imply the classical well-posedness of the problem. To our knowledge, these are only results on the existence of a solution to the initial boundary value problem (1.1)–(1.3), so that its well-posedness has not been resolved so far. We aim to show the well-posedness of the problem (1.1)–(1.3) in the weighted Sobolev spaces used by Reeken [12, 13] and Preston [8].

As related topics on the motion of an inextensible string, Preston [9] studied the geodesics on an infinite-dimensional manifold of inextensible curves in the L^2-metric and proved that the geodesics are determined by (1.1) and (1.2) with $\boldsymbol{g} = \boldsymbol{0}$. Similarly, Preston and Saxton [10] studied the geodesic on this manifold in the H^1-metric and Shi and Vorotnikov [16] studied the gradient flow of a potential energy on this manifold in the L^2-metric. Moreover, there are several results on the rotations of an inextensible hanging string about a vertical axis with one free end under the action of the gravity. We can observe stable configurations, in which its shape is not changing with time, when we force to rotate the string from the upper fixed end. These configurations are related to the angular velocity of the rotation. A representative result on this problem was given by Kolodner [6], who proved that the corresponding nonlinear eigenvalue problem with a constant angular velocity ω has exactly n non-trivial solutions if and only if ω satisfies $\omega_n < \omega \leq \omega_{n+1}$ with $\omega_n \equiv \sigma_n\sqrt{|\boldsymbol{g}|/4L}$, where σ_n is the n-th zero of the Bessel function $J_0(z)$, $\boldsymbol{g}$ is the acceleration of gravity vector, and L is the length of the string. For more results on the rotating string, see references in Amore, Boyd, and Márquez [1]. The study of the motion of an inextensible string has applications: see Grothaus and Marheineke [3] to textile industry; and Connell and Yue [2], Lee, Huang, and Sung [7], and Ryu, Park, Kim, and Sung [14] to flapping dynamics of a flag.

The contents of this paper are as follows. In Section 2 we begin with introducing a weighted Sobolev space X^m, which plays an important role in the problem, and then state our main results in this paper: a priori estimates for solutions in Theorem 2.1, uniqueness of solutions in Theorems 2.3 and 2.4, and the equivalence of the original problem (1.1)–(1.3) and the transformed problem (1.7), (1.4), and (1.3) in Theorem 2.5. In Section 3 we explain that the weighted Sobolev space X^m that we will use in this paper arise naturally from the standard theory of hyperbolic systems. We also prove the weighted Sobolev space used by Preston [8] and that by by Reeken [12, 13] are the same. In Section 4 we analyze Green's function related to the two-point boundary value problem (1.4) to derive precise pointwise estimates for the solution in terms of norms of the weighted Sobolev space X^m for the coefficients. In Section 5 we analyze a linearized system to the problem (1.1), (1.2), and (1.4), and derive an energy estimate for the solution. Finally, in Section 6 we prove Theorem 2.1.

Notation. For $1 \le p \le \infty$, we denote by L^p the Lebesgue space on the open interval $(0,1)$. For non-negative integer m, we denote by H^m the L^2 Sobolev space of order m on $(0,1)$. The norm of a Banach space B is denoted by $\|\cdot\|_B$. The inner product in L^2 is denoted by $(\cdot,\cdot)_{L^2}$. We put $\partial_t = \frac{\partial}{\partial t}$ and $\partial_s = \frac{\partial}{\partial s}$. The norm of a weighted L^p space with a weight s^α is denoted by $\|s^\alpha u\|_{L^p}$, so that $\|s^\alpha u\|_{L^p}^p = \int_0^1 s^{\alpha p}|u(s)|^p \mathrm{d}s$ for $1 \le p < \infty$. It is sometimes denoted by $\|\sigma^\alpha u\|_{L^p}$, too. This would cause no confusion. $[P,Q] = PQ - QP$ denotes the commutator. We denote by $C(a_1, a_2, \ldots)$ a positive constant depending on $a_1, a_2, \ldots$. $f \lesssim g$ means that there exists a non-essential positive constant C such that $f \le Cg$ holds. $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$ hold. $a_1 \vee a_2 = \max\{a_1, a_2\}$.

Acknowledgement
T. I. is partially supported by JSPS KAKENHI Grant Number JP23K22404.

2 Main results

In order to state our main results, we first introduce function spaces that we are going to use in this paper. For a non-negative integer m we define a weighted Sobolev space X^m as a set of all function $u = u(s) \in L^2$ equipped with a norm $\|\cdot\|_{X^m}$ defined by

$$\|u\|_{X^m}^2 = \begin{cases} \|u\|_{H^k}^2 + \displaystyle\sum_{j=1}^{k} \|s^j \partial_s^{k+j} u\|_{L^2}^2 & \text{for} \quad m = 2k, \\ \|u\|_{H^k}^2 + \displaystyle\sum_{j=1}^{k+1} \|s^{j-\frac{1}{2}} \partial_s^{k+j} u\|_{L^2}^2 & \text{for} \quad m = 2k+1. \end{cases}$$

This weighted Sobolev space X^m is essentially the same one introduced by Reeken [12, 13]. For a function $u = u(s,t)$ depending also on time t, we introduce norms $|\!|\!|\cdot|\!|\!|_m$ and $|\!|\!|\cdot|\!|\!|_{m,*}$ by

$$|\!|\!|u(t)|\!|\!|_m^2 = \sum_{j=0}^{m} \|\partial_t^j u(t)\|_{X^{m-j}}^2, \qquad |\!|\!|u(t)|\!|\!|_{m,*}^2 = \sum_{j=0}^{m-1} \|\partial_t^j u(t)\|_{X^{m-j}}^2.$$

The first norm $|\!|\!|\cdot|\!|\!|_m$ will be used to evaluate $\boldsymbol{x}$, whereas the second norm $|\!|\!|\cdot|\!|\!|_{m,*}$ will be used to evaluate τ. However, in the critical case on the regularity index m, we need to use a weaker norm than $|\!|\!|\cdot|\!|\!|_{m,*}$. For $\epsilon > 0$, we introduce norms $\|\cdot\|_{X_\epsilon^k}$ for $k = 1,2,3$ as

$$\|u\|_{X_\epsilon^k}^2 = \begin{cases} \|s^\epsilon u\|_{L^\infty}^2 + \|s^{\frac{1}{2}+\epsilon} u'\|_{L^2}^2 & \text{for} \quad k = 1, \\ \|u\|_{L^\infty}^2 + \|s^\epsilon u'\|_{L^2}^2 + \|s^{1+\epsilon} u''\|_{L^2}^2 & \text{for} \quad k = 2, \\ \|u\|_{L^\infty}^2 + \|s^\epsilon u'\|_{L^\infty}^2 + \|s^{\frac{1}{2}+\epsilon} u''\|_{L^2}^2 + \|s^{\frac{3}{2}+\epsilon} u'''\|_{L^2}^2 & \text{for} \quad k = 3, \end{cases}$$

and put

$$|||u(t)|||_{3,*,\epsilon}^2 = \|u(t)\|_{X_\epsilon^3}^2 + \|\partial_t u(t)\|_{X_\epsilon^2}^2 + \|\partial_t^2 u(t)\|_{X_\epsilon^1}^2.$$

The following theorem is one of main theorems in this paper and gives a priori estimates for the solution $(\boldsymbol{x}, \tau)$ to the problem (1.1)–(1.3).

Theorem 2.1. *For any integer $m \geq 4$ and any positive constants M_0 and c_0, there exist a sufficiently small positive time T and a large constant C such that if the initial data satisfy*

$$\begin{cases} \|\boldsymbol{x}_0^{\rm in}\|_{X^m} + \|\boldsymbol{x}_1^{\rm in}\|_{X^{m-1}} \leq M_0, \\ \frac{\tau_0^{\rm in}(s)}{s} \geq 2c_0 \quad \textit{for} \quad 0 < s < 1, \end{cases} \tag{2.1}$$

where $\tau_0^{\rm in}(s) = \tau(s, 0)$ is the initial tension, then any regular solution $(\boldsymbol{x}, \tau)$ to the initial boundary value problem (1.1)–(1.3) satisfies the stability condition (1.5), $|||\boldsymbol{x}(t)|||_m \leq C$, and

$$\begin{cases} C^{-1}s \leq \tau(s,t) \leq Cs, \quad \sum_{j=1}^{m-3} |\partial_t^j \tau(s,t)| \leq Cs, \quad |\partial_t^{m-2}\tau(s,t)| \leq Cs^{\frac{1}{2}}, \\ |||\tau'(t)|||_{m-1,*} \leq C \qquad \textit{in the case } m \geq 5, \\ |||\tau'(t)|||_{3,*,\epsilon} \leq C(\epsilon) \qquad \textit{in the case } m = 4 \end{cases}$$

for $0 \leq t \leq T$ and $\epsilon > 0$, where the constant $C(\epsilon)$ depends also on ϵ.

Remark 2.2. (1) *Since $\tau(\cdot, t)$ is uniquely determined from $(\boldsymbol{x}(\cdot, t), \dot{\boldsymbol{x}}(\cdot, t))$ as a solution of the two-point boundary value problem (1.4), the initial tension $\tau_0^{\rm in}$ is also uniquely determined from the initial data $(\boldsymbol{x}_0^{\rm in}, \boldsymbol{x}_1^{\rm in})$. Moreover, by Lemma 4.3, under the condition $\|\boldsymbol{x}_0^{\rm in}\|_{X^m} \leq M_0$, we have*

$$\frac{\tau_0^{\rm in}(s)}{s} \gtrsim -\boldsymbol{g} \cdot \boldsymbol{x}_0^{\rm in\prime}(1) + \int_0^1 s|\boldsymbol{x}_1^{\rm in\prime}(s)|^2 {\rm d}s \exp\left(- \int_0^1 s|\boldsymbol{x}_0^{\rm in\prime\prime}(s)|^2 {\rm d}s \right).$$

Therefore, if the initial string is in fact hanging from the fixed end $s = 1$, that is, if $-\boldsymbol{g} \cdot \boldsymbol{x}_0^{\rm in\prime}(1) > 0$, then the second condition in (2.1) is satisfied. Moreover, even in the case $\boldsymbol{g} = \boldsymbol{0}$, if the initial deformation velocity $\boldsymbol{x}_1^{\rm in}$ is not identically zero, then the second condition in (2.1) is satisfied, too.

(2) *Lemma 4.3 also implies that if $\boldsymbol{x}_1^{\rm in} = \boldsymbol{0}$, then we have $\frac{\tau_0^{\rm in}(s)}{s} \simeq -\boldsymbol{g} \cdot \boldsymbol{x}_0^{\rm in\prime}(1)$. Therefore, if, in addition, $-\boldsymbol{g} \cdot \boldsymbol{x}_0^{\rm in\prime}(1) < 0$, then the initial tension $\tau_0^{\rm in}$ is negative everywhere except at the free end $s = 0$, so that the equation of motion in (1.1) becomes elliptic in space and time. As a result, the initial boundary value problem becomes ill-posed.*

(3) *The requirement $m \geq 4$ corresponds to the quasilinear regularity in the sense that $m = 4$ is the minimal integer regularity index m that ensures the embedding*

$$C^0([0,T]; X^m) \cap C^1([0,T]; X^{m-1}) \hookrightarrow C^1([0,1] \times [0,T]);$$

see Remark 3.2. Therefore, $m = 4$ is a critical regularity index in the classical sense.

We then consider the uniqueness of the solution $(\boldsymbol{x}, \tau)$ to the initial boundary value problem (1.1)–(1.3). To this end, we need to specify a class that the solutions belong to. Here, we consider the solutions satisfying

$$\boldsymbol{x}' \in L^\infty(0,T;X^2) \cap W^{1,\infty}(0,T;X^1). \tag{2.2}$$

We note that if $\boldsymbol{x} \in L^\infty(0,T;X^4) \cap W^{1,\infty}(0,T;X^3)$, then it also satisfies (2.2); see Remark 3.2. Under the conditions (2.2), the solutions satisfy also $\boldsymbol{x}' \in W^{2,\infty}(0,T;L^2)$. In view of these and the boundary condition $\boldsymbol{x}|_{s=1} = \boldsymbol{0}$, we may assume without loss of generality that $\boldsymbol{x}, \boldsymbol{x}' \in C^0([0,T];X^1) \cap C^1([0,T];L^2)$. Therefore, the initial conditions (1.3) can be understood in the classical sense.

Theorem 2.3. *The solution to the initial boundary value problem* (1.1)–(1.3) *is unique in the class* (2.2) *satisfying the stability condition* (1.5).

In the case $\boldsymbol{g} = \boldsymbol{0}$, if the initial deformation velocity $\boldsymbol{x}_1^{\mathrm{in}}$ is identically zero, then the initial boundary value problem (1.1)–(1.3) has a trivial solution $(\boldsymbol{x}(s,t), \tau(s,t)) = (\boldsymbol{x}_0^{\mathrm{in}}(s), 0)$. Since this solution does not satisfy the stability condition (1.5), we cannot apply directly Theorem 2.3 to ensure the uniqueness of solutions in this case. Nevertheless, by Lemma 4.4 we see that this trivial solution is the only one that does not satisfy the stability condition (1.5) in the case $\boldsymbol{g} = \boldsymbol{0}$. As a result, we have the following uniqueness theorem without assuming a priori the stability condition.

Theorem 2.4. *In the case $\boldsymbol{g} = \boldsymbol{0}$, the solution to the initial boundary value problem* (1.1)–(1.3) *is unique in the class* (2.2).

The following theorem ensures the equivalence of the original problem (1.1)–(1.3) and the transformed problem (1.7), (1.4), and (1.3).

Theorem 2.5. *Let $(\boldsymbol{x}, \tau)$ be a solution to the transformed problem* (1.7), (1.4), *and* (1.3) *in the class* (2.2) *satisfying the stability condition* (1.5). *Suppose that the initial data satisfy $|\boldsymbol{x}_0^{\mathrm{in}}{}'(s)| \equiv 1$ and $\boldsymbol{x}_0^{\mathrm{in}}{}'(s) \cdot \boldsymbol{x}_1^{\mathrm{in}}{}'(s) \equiv 0$. Then, we have $|\boldsymbol{x}'(s,t)| \equiv 1$.*

In this paper we only give the proof of Theorem 2.1.

3 Weighted Sobolev space X^m

3.1 Characterization of X^m

The weighted Sobolev space X^m introduced in Section 2 is characterized as follows. Let D be the unit disc in $\mathbb{R}^2$ and $H^m(D)$ the standard L^2 Sobolev space of order m on D and we define $H^m_{\mathrm{rad}}(D)$ as a set of all radial function $w = w(r) \in H^m(D)$, where $r = \sqrt{x^2+y^2}$. For a function u defined in the open interval $(0,1)$, we define $u^\sharp(x,y) = u(x^2+y^2)$ which is a function on D.

Lemma 3.1 ([17, Proposition 3.2])**.** *Let m be a non-negative integer. The map $X^m \ni u \mapsto u^\sharp \in H^m_{\mathrm{rad}}(D)$ is bijective and it holds that $\|u\|_{X^m} \simeq \|u^\sharp\|_{H^m(D)}$ for any $u \in X^m$.*

Remark 3.2. (1) *The embedding* $X^1 \hookrightarrow L^\infty(0,1)$ *does not hold. A counter-example is given by* $u(s) = \log(\log(\frac{\mathrm{e}}{s}))$.

(2) *Unlike the standard Sobolev spaces,* $u \in X^{m+1}$ *does not necessarily imply* $u' \in X^m$. *A counter-example is given by* $u(s) = \int_0^s \log(\log(\frac{\mathrm{e}}{\sigma}))\mathrm{d}\sigma$, *which is in* X^3. *However, in view of the embedding* $X^2 \hookrightarrow L^\infty(0,1)$ *we easily check that its first derivative* u' *is not in* X^2.

(3) *Alternatively, we have* $\|su'\|_{X^m} \leq \|u\|_{X^{m+1}}$ *and* $\|u'\|_{X^m} \leq \|u\|_{X^{m+2}}$.

Moreover, the transformation $\sharp$ defined above has the following property.

Lemma 3.3 ([17, Proposition 3.5])**.** *For a function* $u \in X^2$, *it holds that* $\{(su')'\}^\sharp = \frac{1}{4}\Delta u^\sharp$, *where* Δ *is the two-dimensional Laplacian.*

Remark 3.4. *For a general function* $u = u(s)$ *defined on* $(0,1)$, $\{(su')'\}^\sharp = \frac{1}{4}\Delta u^\sharp$ *does not necessarily hold. A counter-example is given by* $u(s) = \log s$. *Indeed, in this case, we have* $\{(su')'\}^\sharp = 0$ *and* $\frac{1}{4}\Delta u^\sharp = \pi\delta_0$, *where* δ_0 *is the Dirac delta function at the origin.*

In the remainder of this subsection, we explain that the weighted Sobolev space X^m that we will use in this paper arise naturally from the standard theory of hyperbolic systems. In the case where $\boldsymbol{g}$ is a unit constant vector, the problem (1.1)–(1.2) has a trivial stationary solution $(\boldsymbol{x}_{\mathrm{s}}(s), \tau_{\mathrm{s}}(s)) = ((1-s)\boldsymbol{g}, s)$. Linearizing (1.1)–(1.2) around this stationary solution and picking up only the highest order terms, we obtain

$$\begin{cases} \partial_t^2 \boldsymbol{x} = (s\boldsymbol{x}')' & \text{in} \quad (0,1)\times(0,T), \\ \boldsymbol{x} = \boldsymbol{0} & \text{on} \quad \{s=1\}\times(0,T). \end{cases} \tag{3.1}$$

By introducing a new quantity $\boldsymbol{x}^\sharp(x,y,t) = \boldsymbol{x}(x^2+y^2,t)$, the linearized problem is transformed equivalently into

$$\begin{cases} \partial_t^2 \boldsymbol{x}^\sharp = \frac{1}{4}\Delta \boldsymbol{x}^\sharp & \text{in} \quad D\times(0,T), \\ \boldsymbol{x}^\sharp = \boldsymbol{0} & \text{on} \quad \partial D\times(0,T), \end{cases} \tag{3.2}$$

where we use Lemma 3.3. It is well-known that the initial boundary value problem corresponding to (3.2) is well-posed in the class $\bigcap_{j=0}^m C^j([0,T]; H_{\mathrm{rad}}^{m-j}(D))$. Therefore, Lemma 3.1 implies that the initial boundary value problem corresponding to (3.1) is also well-posed in the class $\bigcap_{j=0}^m C^j([0,T]; X^{m-j})$. Thus, we are naturally led to the weighted Sobolev space X^m.

3.2 The weighted Sobolev spaces N_m by Preston and X^m by Reeken

For a non-negative integer m, we let N_m be the weighted Sobolev space introduced by Preston [8], namely, a set of all function $u = u(s) \in L^2$ equipped with a norm $\|u\|_{N_m}$ defined by

$$\|u\|_{N_m}^2 = \sum_{j=0}^m \int_0^1 s^j |\partial_s^j(s)| \mathrm{d}s.$$

Then, we have the following equivalence.

Proposition 3.5. *For any non-negative integer m, it holda that $N_m = X^m$ and that the norms $\|\cdot\|_{N_m}$ and $\|\cdot\|_{X^m}$ are equivalent.*

Proof. Obviously, we have $\|u\|_{N_m} \le \|u\|_{X^m}$. Therefore, it is sufficiently to show $\|u\|_{X^m} \lesssim \|u\|_{N_m}$ in the case $m \ge 2$.

Let $r > -1$. Then, we see that

$$|u(1)|^2 = \int_0^1 (s^{r+1}|u(s)|^2)'\mathrm{d}s = \int_0^1 \{(r+1)s^r|u(s)|^2 + 2s^{r+1}u(s)u'(s)\}\mathrm{d}s,$$

so that

$$\begin{aligned}\int_0^1 s^r|u(s)|^2\mathrm{d}s &= \frac{1}{r+1}\left(|u(1)|^2 - 2\int_0^1 s^{r+1}u(s)u'(s)\mathrm{d}s\right) \\ &\le \frac{1}{r+1}\left(|u(1)|^2 + \left(\int_0^1 s^r|u(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^1 s^{r+2}|u'(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\right).\end{aligned}$$

This implies

$$\int_0^1 s^r|u(s)|^2\mathrm{d}s \le \frac{2}{r+1}|u(1)|^2 + \frac{4}{(r+1)^2}\int_0^1 s^{r+2}|u'(s)|^2\mathrm{d}s.$$

Using this inductively, we obtain

$$\int_0^1 s^r|u(s)|^2\mathrm{d}s \lesssim \sum_{i=0}^{l}|\partial_s^i u(1)|^2 + \int_0^1 s^{r+2(l+1)}|\partial_s^{l+1}(s)|^2\mathrm{d}s \tag{3.3}$$

for $l = 0, 1, 2, \ldots$. We note also that by the standard Sobolev embedding theorem we have

$$\sum_{j=1}^{m-1}|\partial_s^j u(1)| \lesssim \|u\|_{N_m}.$$

We first consider the case $m = 2k$ with $k \ge 1$. For any $j \in \{1, 2, \ldots, k\}$, it follows from (3.3) with $r = 0$ and $l = j-1$ that

$$\int_0^1 |\partial_s^j u(s)|^2\mathrm{d}s \lesssim \sum_{i=0}^{j-1}|\partial_s^{j+i}u(1)|^2 + \int_0^1 s^{2j}|\partial_s^{2j}u(s)|^2\mathrm{d}s \lesssim \|u\|_{N_m}^2.$$

Similarly, for any $j \in \{1, 2, \ldots, k-1\}$, it follows from (3.3) with $r = 2j$ and $l = k-j-1$ that

$$\int_0^1 s^{2j}|\partial_s^{k+j}u(s)|^2\mathrm{d}s \lesssim \sum_{i=0}^{k-j-1}|\partial_s^{k+j+i}u(1)|^2 + \int_0^1 s^{2k}|\partial_s^{2k}u(s)|^2\mathrm{d}s \lesssim \|u\|_{N_m}^2.$$

These estimates give $\|u\|_{X^m} \lesssim \|u\|_{N_m}$.

We then consider the case $m = 2k+1$ with $k \geq 1$. For any $j \in \{1, 2, \ldots, k\}$, it follows from (3.3) with $r = 0$ and $l = j-1$ that

$$\int_0^1 |\partial_s^j u(s)|^2 \mathrm{d}s \lesssim \sum_{i=0}^{j-1} |\partial_s^{j+i} u(1)|^2 + \int_0^1 s^{2j} |\partial_s^{2j} u(s)|^2 \mathrm{d}s \lesssim \|u\|_{N_m}^2.$$

Similarly, for any $j \in \{1, 2, \ldots, k\}$, it follows from (3.3) with $r = 2j-1$ and $l = k-j$ that

$$\int_0^1 s^{2j-1} |\partial_s^{k+j} u(s)|^2 \mathrm{d}s \lesssim \sum_{i=0}^{k-j} |\partial_s^{k+j+i} u(1)|^2 + \int_0^1 s^{2k+1} |\partial_s^{2k+1} u(s)|^2 \mathrm{d}s \lesssim \|u\|_{N_m}^2.$$

These estimates give $\|u\|_{X^m} \lesssim \|u\|_{N_m}$. This completes the proof. □

4 Two-point boundary value problem

In this paper, we evaluate the solution $\boldsymbol{x}$ of (1.1) in the weighted Sobolev space X^m by using the energy method. This requires to evaluate the solution τ of the two-point boundary value problem (1.4) also in a weighted Sobolev space. To this end, we express the solution τ by using Green's function of the problem and evaluate it through precise pointwise estimates of Green's function.

In view of (1.4) we will consider the two-point boundary value problem

$$\begin{cases} -\tau'' + |\boldsymbol{x}''|^2 \tau = h \quad \text{in} \quad (0,1), \\ \tau(0) = 0, \quad \tau'(1) = a, \end{cases} \tag{4.1}$$

where $\boldsymbol{x}(s)$ and $h(s)$ are given functions and a is a constant.

4.1 Green's function

As is well-known, Green's function to the boundary value problem (4.1) can be constructed as follows. Let φ and ψ be unique solutions to the initial value problems

$$\begin{cases} -\varphi'' + |\boldsymbol{x}''|^2 \varphi = 0 \quad \text{in} \quad (0,1), \\ \varphi(0) = 0, \quad \varphi'(0) = 1, \end{cases} \tag{4.2}$$

and

$$\begin{cases} -\psi'' + |\boldsymbol{x}''|^2 \psi = 0 \quad \text{in} \quad (0,1), \\ \psi(1) = 1, \quad \psi'(1) = 0, \end{cases} \tag{4.3}$$

respectively. The Wronskian $W(s; \varphi, \psi) = \varphi(s)\psi'(s) - \varphi'(s)\psi(s)$ is a non-zero constant since the uniqueness of solutions to the boundary value problem (4.1) is easily verified. Particularly, we have $W(s; \varphi, \psi) \equiv -\varphi'(1)$. A sharp estimate for $\varphi'(1)$ will be given below; see Lemma 4.1. In terms of these fundamental solutions, Green's function to the boundary value problem (4.1) is given by

$$G(s,r)=\begin{cases}\dfrac{\varphi(s)\psi(r)}{\varphi'(1)} & \text{for}\quad 0\le s\le r,\\ \dfrac{\psi(s)\varphi(r)}{\varphi'(1)} & \text{for}\quad r\le s\le 1.\end{cases}$$

Particularly, the unique solution to the problem (4.1) can be expressed as

$$\tau(s)=a\frac{\varphi(s)}{\varphi'(1)}+\frac{\psi(s)}{\varphi'(1)}\int_0^s\varphi(\sigma)h(\sigma)\mathrm{d}\sigma+\frac{\varphi(s)}{\varphi'(1)}\int_s^1\psi(\sigma)h(\sigma)\mathrm{d}\sigma. \tag{4.4}$$

We proceed to evaluate these fundamental solutions φ and ψ.

Lemma 4.1. *Let φ be a unique solution to* (4.2)*. Then, for any $s\in[0,1]$ we have*

$$\begin{cases}1\le\varphi'(s)\le\exp(\|\sigma^{\frac12}\boldsymbol{x}''\|_{L^2}^2),\\ s\le\varphi(s)\le s\exp(\|\sigma^{\frac12}\boldsymbol{x}''\|_{L^2}^2).\end{cases}$$

Proof. It is sufficient to show the first estimate because the second one can easily follow from the first one by integrating it over $[0,s]$ and by using the initial condition $\varphi(0)=0$.

We first show that $\varphi(s)>0$ for all $s\in(0,1]$. In view of the initial conditions at $s=0$, we have $\varphi(s)>0$ for $0<s\ll1$. Now, suppose that there exists $s_*\in(0,1]$ such that $\varphi(s_*)=0$. We can assume without loss of generality that $\varphi(s)>0$ for $0<s<s_*$, so that $\varphi'(s_*)\le0$. Then, we have $\varphi''(s)=|\boldsymbol{x}''(s)|^2\varphi(s)\ge0$ for $0<s<s_*$. This implies that $\varphi'(s)$ is non-decreasing in the interval $[0,s_*]$, so that $\varphi'(s_*)\ge\varphi'(0)=1$. This contradicts with $\varphi'(s_*)\le0$. Therefore, $\varphi(s)>0$ holds for all $s\in(0,1]$. Particularly, $\varphi'(s)$ is non-decreasing in the whole interval $[0,1]$, so that we obtain $\varphi'(s)\ge\varphi'(0)=1$ for all $s\in[0,1]$.

We proceed to show the upper bound of $\varphi'(s)$. Since $\varphi'(s)$ is a non-decreasing function, we have $\varphi(s)=\int_0^s\varphi'(\sigma)\mathrm{d}\sigma\le s\varphi'(s)$. Therefore, we see that

$$\varphi'(s)=1+\int_0^s\varphi''(\sigma)\mathrm{d}\sigma=1+\int_0^s|\boldsymbol{x}''(\sigma)|^2\varphi(\sigma)\mathrm{d}\sigma\le1+\int_0^s\sigma|\boldsymbol{x}''(\sigma)|^2\varphi'(\sigma)\mathrm{d}\sigma,$$

which together with Gronwall's inequality yields $\varphi'(s)\le\exp(\int_0^s\sigma|\boldsymbol{x}''(\sigma)|^2\mathrm{d}\sigma)$. This gives the desired estimate. □

Lemma 4.2. *Let ψ be a unique solution to* (4.3)*. Then, for any $s\in[0,1]$ and any $\alpha\ge0$ we have*

$$\begin{cases}1\le\psi(s)\le\exp(\|\sigma^{\frac12}\boldsymbol{x}''\|_{L^2}^2),\\ 0\ge s^\alpha\psi'(s)\ge-\|\sigma^{\frac{\alpha}{2}}\boldsymbol{x}''\|_{L^2}^2\exp(\|\sigma^{\frac12}\boldsymbol{x}''\|_{L^2}^2).\end{cases}$$

Proof. We first show that $\psi(s)>0$ for all $s\in[0,1]$. In view of the initial condition $\psi(1)=1$, we have $\psi(s)>0$ for $0\le1-s\ll1$. Now, suppose that there exists $s_*\in[0,1)$ such that $\psi(s_*)=0$. We can assume without loss of generality that $\psi(s)>0$

for $s_* < s \le 1$, so that $\psi'(s_*) \ge 0$. Then, we have $\psi''(s) = |\boldsymbol{x}''(s)|^2\psi(s) \ge 0$ for $s_* < s \le 1$. This implies that $\psi'(s)$ is non-decreasing in the interval $[s_*, 1]$, so that $\psi'(s) \le \psi'(1) = 0$ for all $s \in [s_*, 1]$. This implies that $\psi(s)$ is non-increasing in the interval $[s_*, 1]$, so that $\psi(s_*) \ge \psi(1) = 1$. This contradicts with $\psi(s_*) = 0$. Therefore, $\psi(s) > 0$ holds for all $s \in [0, 1]$. Particularly, $\psi''(s) \ge 0$ holds for all $s \in [0, 1]$, which implies in turn that $\psi'(s) \le 0$ and $\psi(s) \ge 1$ for all $s \in [0, 1]$.

We then show the upper bound of $\psi(s)$. Noting that $\psi(s)$ is a non-increasing function and that $\psi'(1) = 0$, we see that

$$\psi'(s) = -\int_s^1 \psi''(\sigma)\mathrm{d}\sigma = -\int_s^1 |\boldsymbol{x}''(\sigma)|^2\psi(\sigma)\mathrm{d}\sigma \ge -\int_s^1 |\boldsymbol{x}''(\sigma)|^2\mathrm{d}\sigma\psi(s), \tag{4.5}$$

which together with Gronwall's inequality and $\psi(1) = 1$ yields

$$\psi(s) \le \exp\left(\int_s^1 \int_\sigma^1 |\boldsymbol{x}''(\tilde{\sigma})|^2\mathrm{d}\tilde{\sigma}\mathrm{d}\sigma\right) \le \exp\left(\int_0^1 \sigma|\boldsymbol{x}''(\sigma)|^2\mathrm{d}\sigma\right).$$

This shows the desired upper bound.

We finally show the lower bound of $\psi'(s)$. It follows from (4.5) that

$$s^\alpha\psi'(s) \ge -\int_s^1 \sigma^\alpha|\boldsymbol{x}''(\sigma)|^2\mathrm{d}\sigma\psi(s),$$

which together with the upper bound of $\psi(s)$ gives the desired one. □

4.2 Preliminaries

In view of (1.4) we first consider the case where $h(s)$ is non-negative.

Lemma 4.3. *Let τ be a unique solution to the boundary value problem* (4.1). *Suppose that $h(s) \ge 0$ and $a + \|\sigma h\|_{L^1}\exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2) \ge 0$. Then, for any $s \in [0, 1]$ we have*

$$\begin{cases} s\{a + \|\sigma h\|_{L^1}\exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\}\exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2) \le \tau(s) \le s(a + \|h\|_{L^1}), \\ a - (a + \|h\|_{L^1})\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2 \le \tau'(s) \le a + \|h\|_{L^1}. \end{cases}$$

Proof. We remind that the solution τ is expressed by Green's function as (4.4). Under the assumptions, by Lemmas 4.1 and 4.2 we see that

$$\begin{aligned} \tau(s) &\ge \frac{\varphi(s)}{\varphi'(1)}\left\{a + \exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\int_0^s \varphi(\sigma)h(\sigma)\mathrm{d}\sigma + \int_s^1 \psi(\sigma)h(\sigma)\mathrm{d}\sigma\right\} \\ &\ge \frac{\varphi(s)}{\varphi'(1)}\left\{a + \exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\int_0^s \sigma h(\sigma)\mathrm{d}\sigma + \exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\int_s^1 \sigma h(\sigma)\mathrm{d}\sigma\right\} \\ &= \frac{\varphi(s)}{\varphi'(1)}\{a + \exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\|\sigma h\|_{L^1}\} \\ &\ge s\exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\{a + \exp(-\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2)\|\sigma h\|_{L^1}\}, \end{aligned}$$

which implies the lower bound of $\tau(s)$. Integrating the equation for τ over $[s,1]$, we have

$$\tau'(s) + \int_s^1 |\boldsymbol{x}''(\sigma)|^2 \tau(\sigma) \mathrm{d}\sigma = a + \int_s^1 h(\sigma) \mathrm{d}\sigma.$$

Since the positivity of $\tau(s)$ is already guaranteed, this implies the upper bound of $\tau'(s)$, and then that of $\tau(s)$. As for the lower bound of $\tau'(s)$, we see that

$$\tau'(s) \geq a - \int_s^1 |\boldsymbol{x}''(\sigma)|^2 \tau(\sigma) \mathrm{d}\sigma \geq a - (a + \|h\|_{L^1}) \int_s^1 \sigma |\boldsymbol{x}''(\sigma)|^2 \mathrm{d}\sigma.$$

This gives the desired estimate. □

4.3 Stability condition in the case $\boldsymbol{g} = \mathbf{0}$

As an application of Lemma 4.3, in the case $\boldsymbol{g} = \mathbf{0}$, we prove that the trivial solution the problem (1.1)–(1.3) is the only one that does not satisfy the stability condition (1.5).

Lemma 4.4. *Let $\boldsymbol{g} = \mathbf{0}$ and $(\boldsymbol{x}, \tau)$ be a solution to the problem* (1.1)–(1.3) *in the class* (2.2)*. Suppose that the stability condition* (1.5) *is not satisfied, that is,*

$$\inf_{(s,t) \in (0,1) \times (0,T)} \frac{\tau(s,t)}{s} \leq 0. \tag{4.6}$$

Then, we have $\boldsymbol{x}(s,t) \equiv \boldsymbol{x}_0^{\rm in}(s)$ and $\tau(s,t) \equiv 0$. Particularly, $\boldsymbol{x}_1^{\rm in}(s) \equiv \mathbf{0}$ must hold.

Proof. By Lemma 4.3 we see that $\frac{\tau(s,t)}{s} \geq \|\sigma^{\frac{1}{2}} \dot{\boldsymbol{x}}'(t)\|_{L^2}^2 \exp(-2\|\sigma^{\frac{1}{2}} \boldsymbol{x}''(t)\|_{L^2}^2)$ for any $(s,t) \in (0,1] \times [0,T]$. Then, continuity of the right-hand side with respect to t together with (4.6) yields that there exists a $t_0 \in [0,T]$ such that $\dot{\boldsymbol{x}}'(s,t_0) \equiv \mathbf{0}$, which together with the boundary condition (1.2) implies $\dot{\boldsymbol{x}}(s,t_0) \equiv \mathbf{0}$. On the other hand, we see easily that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 |\dot{\boldsymbol{x}}(s,t)|^2 \mathrm{d}s = 2 \int_0^1 \ddot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} \mathrm{d}s = 2 \int_0^1 (\tau \boldsymbol{x}')' \cdot \dot{\boldsymbol{x}} \mathrm{d}s = -2 \int_0^1 \tau \boldsymbol{x}' \cdot \dot{\boldsymbol{x}}' \mathrm{d}s = 0,$$

where we used $\boldsymbol{x}' \cdot \dot{\boldsymbol{x}}' = 0$, which comes from the second equation in (1.1). Therefore, for any $t \in [0,T]$, it holds that $\|\dot{\boldsymbol{x}}(t)\|_{L^2} = \|\dot{\boldsymbol{x}}(t_0)\|_{L^2} = 0$. This implies $\dot{\boldsymbol{x}}(s,t) \equiv \mathbf{0}$, and hence we have $\boldsymbol{x}(s,t) \equiv \boldsymbol{x}_0^{\rm in}(s)$ and $\boldsymbol{x}_1^{\rm in}(s) \equiv \mathbf{0}$. Moreover, by Lemma 4.3 we obtain $0 \leq \tau(s,t) \leq s\|\dot{\boldsymbol{x}}'(t)\|_{L^2}^2 = 0$, which implies $\tau(s,t) \equiv 0$. □

4.4 Estimate of solutions

We proceed to give estimate for the solution τ to the problem (4.1) without assuming the non-negativity of $h(s)$ and a. Such estimates will be used to evaluate the derivatives of τ with respect to t.

Lemma 4.5. *For any $M > 0$ there exists a constant $C = C(M) > 0$ such that if $\|\sigma^{\frac{1}{2}} \boldsymbol{x}''\|_{L^2} \leq M$, then the solution τ to the boundary value problem* (4.1) *satisfies*

$$\begin{cases} |\tau(s)| \leq C(|a|s + \|\sigma^\alpha h\|_{L^1} s^{1-\alpha}), \\ s^\alpha |\tau'(s)| \leq C(|a|s^\alpha + \|\sigma^\alpha h\|_{L^1}) \end{cases}$$

for any $s \in [0,1]$ and any $\alpha \in [0,1]$.

Proof. It follows from Lemmas 4.1 and 4.2 that $\varphi(s) \simeq s$, $\varphi'(s) \simeq 1$, $\psi(s) \simeq 1$, and $s|\psi'(s)| \lesssim 1$. Since the solution $\tau(s)$ is expressed as (4.4), we have

$$|\tau(s)| \lesssim |a|s + \int_0^s \sigma|h(\sigma)|\mathrm{d}\sigma + s\int_s^1 |h(\sigma)|\mathrm{d}\sigma \lesssim |a|s + s^{1-\alpha}\int_0^1 \sigma^\alpha|h(\sigma)|\mathrm{d}\sigma.$$

This gives the first estimate of the lemma. In view of

$$\tau'(s) = a\frac{\varphi'(s)}{\varphi'(1)} + \frac{\psi'(s)}{\varphi'(1)}\int_0^s \varphi(\sigma)h(\sigma)\mathrm{d}\sigma + \frac{\varphi'(s)}{\varphi'(1)}\int_s^1 \psi(\sigma)h(\sigma)\mathrm{d}\sigma, \tag{4.7}$$

we see that

$$s^\alpha|\tau'(s)| \lesssim |a|s^\alpha + s^\alpha|\psi'(s)|\int_0^s \sigma|h(\sigma)|\mathrm{d}\sigma + s^\alpha\int_s^1 |h(\sigma)|\mathrm{d}\sigma \lesssim |a|s^\alpha + \int_0^1 \sigma^\alpha|h(\sigma)|\mathrm{d}\sigma.$$

This gives the second estimate of the lemma. □

In general, if $h(s)$ has a singularity at $s=0$, then so is $\tau'(s)$. To evaluate the singularity in terms of L^p-norm, the above pointwise estimate does not give a sharp one. Next, we will derive a sharp L^p estimate for $\tau'(s)$. To this end, we prepare the following calculus inequality.

Lemma 4.6. *Let $1 \le p \le \infty$ and $\alpha + \frac{1}{p} > 0$, and put $H(s) = \int_s^1 h(\sigma)\mathrm{d}\sigma$. Then, we have*

$$\|s^\alpha H\|_{L^p} \le \frac{1}{(\alpha+\frac{1}{p})^{\frac{1}{p}}}\|s^{\alpha+\frac{1}{p}}h\|_{L^1}.$$

Proof. The case $p=\infty$ is trivial, so that we assume $p<\infty$. We may also assume without loss of generality that $h(s)$ is non-negative. By integration by parts, we see that

$$\begin{aligned}
\|s^\alpha H\|_{L^p}^p &= \int_0^1 s^{\alpha p}\left(\int_s^1 h(\sigma)\mathrm{d}\sigma\right)^p \mathrm{d}s \\
&= \left[\frac{1}{\alpha p+1}s^{\alpha p+1}\left(\int_s^1 h(\sigma)\mathrm{d}\sigma\right)^p\right]_0^1 + \frac{p}{\alpha p+1}\int_0^1 s^{\alpha p+1}h(s)\left(\int_s^1 h(\sigma)\mathrm{d}\sigma\right)^{p-1}\mathrm{d}s \\
&= \frac{p}{\alpha p+1}\int_0^1 s^{\alpha+\frac{1}{p}}h(s)\left(s^{\alpha+\frac{1}{p}}\int_s^1 h(\sigma)\mathrm{d}\sigma\right)^{p-1}\mathrm{d}s \\
&\le \frac{p}{\alpha p+1}\left(\int_0^1 s^{\alpha+\frac{1}{p}}h(s)\mathrm{d}s\right)^p.
\end{aligned}$$

Therefore, we obtain the desired estimate. □

Lemma 4.7. *For any $M>0$ there exists a constant $C=C(M)>0$ such that if $\|\sigma^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2} \le M$, then the solution τ to the boundary value problem (4.1) satisfies*

$$\|s^\alpha\tau'\|_{L^p} \le C(|a| + \|s^{\alpha+\frac{1}{p}}h\|_{L^1})$$

for any $p\in[1,\infty]$ and any $\alpha \ge 0$ satisfying $\alpha+\frac{1}{p} \le 1$.

Proof. The case $p=\infty$ has already been proved in Lemma 4.5, so that we assume $p<\infty$. It follows from (4.7) and Lemmas 4.1 and 4.2 that

$$\begin{aligned} s^\alpha|\tau'(s)| &\lesssim |a|s^\alpha + s^\alpha|\psi'(s)|\int_0^s \sigma|h(\sigma)|\mathrm{d}\sigma + s^\alpha\int_s^1 |h(\sigma)|\mathrm{d}\sigma \\ &\lesssim |a|s^\alpha + s^{1-\frac{1}{p}}|\psi'(s)|\int_0^s \sigma^{\alpha+\frac{1}{p}}|h(\sigma)|\mathrm{d}\sigma + s^\alpha\int_s^1 |h(\sigma)|\mathrm{d}\sigma. \end{aligned}$$

Here, in view of (4.5) we have $|\psi'(s)| \lesssim \int_s^1 |\boldsymbol{x}''(\sigma)|^2\mathrm{d}\sigma$, so that by Lemma 4.6 we get

$$\|s^{1-\frac{1}{p}}\psi'\|_{L^p} \lesssim \|s|\boldsymbol{x}''|^2\|_{L^1} = \|s^{\frac{1}{2}}\boldsymbol{x}''\|_{L^2}^2 \lesssim 1.$$

Therefore, by Lemma 4.6 again we obtain

$$\|s^\alpha\tau'\|_{L^p} \lesssim |a| + \left(1+\frac{1}{(\alpha+\frac{1}{p})^{\frac{1}{p}}}\right)\|s^{\alpha+\frac{1}{p}}h\|_{L^1}.$$

Since $(\alpha+\frac{1}{p})^{\frac{1}{p}} \geq (\frac{1}{p})^{\frac{1}{p}} \geq \exp(\exp(-1))$, we obtain the desired estimate. □

5 Energy estimate for a linearized system

In this section we derive an energy estimate for solutions to a linearized system for (1.1), (1.2), and (1.4). We denote variations of $(\boldsymbol{x},\tau)$ by $(\boldsymbol{y},\nu)$ in the linearization. Then, the linearized system has the form

$$\begin{cases} \ddot{\boldsymbol{y}} + (\tau\boldsymbol{y}')' + (\nu\boldsymbol{x}')' = \boldsymbol{f} & \text{in} \quad (0,1)\times(0,T), \\ \boldsymbol{x}'\cdot\boldsymbol{y}' = f & \text{in} \quad (0,1)\times(0,T), \\ \boldsymbol{y} = \boldsymbol{0} & \text{on} \quad \{s=1\}\times(0,T), \end{cases} \tag{5.1}$$

and

$$\begin{cases} -\nu'' + |\boldsymbol{x}''|^2\nu = 2\dot{\boldsymbol{x}}'\cdot\dot{\boldsymbol{y}}' - 2(\boldsymbol{x}''\cdot\boldsymbol{y}'')\tau + h & \text{in} \quad (0,1)\times(0,T), \\ \nu = 0 & \text{on} \quad \{s=0\}\times(0,T), \\ \nu' = -\boldsymbol{g}\cdot\boldsymbol{y}' & \text{on} \quad \{s=1\}\times(0,T), \end{cases} \tag{5.2}$$

where $\boldsymbol{f}$, f, and h can be regarded as given functions. As for $\boldsymbol{x}$, we assume that

$$\|\boldsymbol{x}'(t)\|_{X^2} + \|\dot{\boldsymbol{x}}'(t)\|_{X^1} \leq M, \tag{5.3}$$

$$\|\boldsymbol{x}(t)\|_{X^3} + \|\dot{\boldsymbol{x}}(t)\|_{X^2} \leq M_1 \tag{5.4}$$

for $0\leq t\leq T$. We are going to evaluate the functional $E(t)$ defined by

$$E(t) = \|\dot{\boldsymbol{y}}(t)\|_{X^1}^2 + \|\boldsymbol{y}(t)\|_{X^2}^2.$$

Proposition 5.1. *For any positive constants M_1, M, and c_0 and any $\epsilon \in (0, \frac{1}{2})$, there exist positive constants $C_1 = C(M_1, c_0)$ and $C_2(\epsilon) = C(M, M_1, c_0, \epsilon)$ such that if $(\boldsymbol{x}, \tau)$ is a solution to the problem* (1.1)–(1.2) *satisfying* (5.3), (5.4)*, and the stability condition* (1.5)*, then for any solution $(\boldsymbol{y}, \nu)$ to* (5.1)–(5.2) *we have*

$$E(t) \le C_1 \mathrm{e}^{C_2(\epsilon)t} \left(E(0) + S_1(0) + C_2(\epsilon) \int_0^t S_2(t')\mathrm{d}t' \right),$$

where

$$\begin{cases} S_1(t) = \|\boldsymbol{f}\|_{L^2}^2 + \|sh\|_{L^1}^2, \\ S_2(t) = \|\dot{\boldsymbol{f}}\|_{L^2}^2 + \|s^{\frac{1}{2}-\epsilon}\dot{f}\|_{L^2}^2 + |\dot{f}|_{s=1}|^2 + \|s\dot{h}\|_{L^1}^2 + \|s^{\frac{1}{2}+\epsilon}h\|_{L^2}^2. \end{cases} \tag{5.5}$$

We omit the proof of this proposition.

6 A priori estimates of solutions

In this section we prove Theorem 2.1; see Subsection 6.4. For this purpose, we prepare some lemmas in Subsections 6.1–6.3. We omit proofs of these lemmas.

6.1 Estimates for the tension τ

We derive estimates for the tension τ. In the case $m \ge 5$ we obtain the following lemma.

Lemma 6.1. *Let M be a positive constant and m and j integers such that $m \ge 5$ and $0 \le j \le m-2$. There exists a positive constant $C = C(M, m)$ such that if $\boldsymbol{x}$ satisfies*

$$\sum_{l=0}^{j+1} \|\partial_t^l \boldsymbol{x}(t)\|_{X^{m-l}} \le M,$$

then the solution τ to the boundary value problem (1.4) *satisfies the following estimates:*

$$\begin{cases} \|\partial_t^j \tau'(t)\|_{L^\infty \cap X^{m-1-j}} \le C & \text{in the case } j \le m-3, \\ \|\partial_t^{m-2} \tau'(t)\|_{X^1} \le C & \text{in the case } j = m-2. \end{cases}$$

In the case $m = 4$ we cannot expect that the estimates for the tension τ obtained in Lemma 6.1 hold. In this critical case, we obtain weaker estimates for the tension τ, which are given in the following lemma.

Lemma 6.2. *Let M be a positive constant and j an integer such that $0 \le j \le 2$. For any $\epsilon > 0$, there exists a positive constant $C(\epsilon) = C(M, \epsilon)$ such that if $\boldsymbol{x}$ satisfies*

$$\sum_{l=0}^{j+1} \|\partial_t^l \boldsymbol{x}(t)\|_{X^{4-l}} \le M, \tag{6.1}$$

then the solution τ to the boundary value problem (1.4) *satisfies the following estimates:*

$$\begin{cases} \|\partial_t^j \tau'(t)\|_{X_\epsilon^{3-j}} \le C(\epsilon) & \text{in the case } j = 0, 1, \\ \|\partial_t^2 \tau'(t)\|_{X_\epsilon^1} \le C(\epsilon) & \text{in the case } j = 2. \end{cases}$$

In addition to (6.1) *with $j = 2$, if we assume $\|\partial_t \boldsymbol{x}'(t)\|_{L^\infty} \le M$, then we have $|\!|\!|\tau'(t)|\!|\!|_{3,*} \le C$, where $C = C(M) > 0$.*

6.2 Estimates for initial values

In this subsection we evaluate the initial value $|||\boldsymbol{x}(0)|||_m$ in terms of the initial data $\|\boldsymbol{x}(0)\|_{X^m}$ and $\|\dot{\boldsymbol{x}}(0)\|_{X^{m-1}}$. Although it is sufficient to evaluate $\partial_t^j \boldsymbol{x}$ only at time $t = 0$, we will evaluate them at general time t.

Lemma 6.3. *Let M be a positive constant and m an integer such that $m \geq 4$. There exists a positive constant $C = C(M, m)$ such that if $(\boldsymbol{x}, \tau)$ is a solution to* (1.1) *and* (1.2) *satisfying $\|\boldsymbol{x}(t)\|_{X^m} + \|\partial_t \boldsymbol{x}(t)\|_{X^{m-1}} \leq M$, then we have $|||\boldsymbol{x}(t)|||_m \leq C$.*

6.3 Estimates for the position vector $\boldsymbol{x}$

Lemma 6.4. *For any integer $m \geq 4$ and any positive constants M_1, M_2, and c_0, there exists a positive constant $C_2 = C_2(M_1, M_2, c_0, m)$ such that if $(\boldsymbol{x}, \tau)$ is a regular solution to* (1.1) *and* (1.2) *satisfying the stability condition* (1.5) *and*

$$
\begin{cases} |||\boldsymbol{x}(t)|||_{m-1} \leq M_1, \\ \|\partial_t^{m-1}\boldsymbol{x}(t)\|_{X^1}^2 + \|\partial_t^{m-2}\boldsymbol{x}(t)\|_{X^2}^2 \leq M_2, \end{cases} \tag{6.2}
$$

then we have $|||\boldsymbol{x}(t)|||_m \leq C_2$.

6.4 Proof of Theorem 2.1

We are ready prove Theorem 2.1. We are going to show that for any regular solution $(\boldsymbol{x}, \tau)$ to the problem, if the initial data satisfy (2.1), then the estimates in (6.2) hold in fact for $0 \leq t \leq T$ by choosing appropriately the positive constants M_1, M_2, and the positive time T. In the following, we simply denote the constants $C_0 = C(M_0, c_0, m)$, $C_1 = C(M_1, c_0, m)$, and $C_2 = C(M_2, M_1, c_0, m)$. These constants may change from line to line.

Suppose that the initial data $(\boldsymbol{x}_0^{\rm in}, \boldsymbol{x}_1^{\rm in})$ satisfy (2.1) and that $(\boldsymbol{x}, \tau)$ is a regular solution to the problem (1.1)–(1.3). By Lemmas 6.3, 6.1, and 6.2, we have

$$
\begin{cases} |||\boldsymbol{x}(0)|||_m + |||\tau'(0)|||_{m-2,*} \leq C_0, \\ C_0^{-1}s \leq \tau(s,0) \leq C_0 s, \quad \sum_{j=1}^{m-3} |\partial_t^j \tau(s,0)| \leq C_0 s. \end{cases}
$$

Suppose also that the solution $(\boldsymbol{x}, \tau)$ satisfies (6.2) for $0 \leq t \leq T$, where the constants M_0, M_1, and time T will be defined later. Then, by Lemmas 6.4, 6.1, and 6.2, we have

$$
\begin{cases} |||\boldsymbol{x}(t)|||_m \leq C_2, \\ C_1^{-1}s \leq \tau(s,t) \leq C_1 s, \quad \sum_{j=1}^{m-3} |\partial_t^j \tau(s,t)| \leq C_2 s, \quad |\partial_t^{m-2}\tau(s,t)| \leq C_2 s^{\frac{1}{2}}, \\ |||\tau'(t)|||_{m-1,*} \leq C_2 \qquad \text{in the case } m \geq 5, \\ |||\tau'(t)|||_{3,*,\epsilon} \leq C_2 \qquad \text{in the case } m = 4 \text{ with } \epsilon = \frac{1}{4} \end{cases}
$$

for $0 \leq t \leq T$. Here, we note that there is no special reason on the choice $\epsilon = \frac{1}{4}$ and that we can choose ϵ arbitrarily such that $0 < \epsilon < \frac{1}{2}$. Put $\boldsymbol{y} = \partial_t^{m-2}\boldsymbol{x}$ and $\nu = \partial_t^{m-2}\tau$. Then, we see that $(\boldsymbol{y},\nu)$ satisfies the linearized system (5.1) and (5.2) with $(\boldsymbol{f},f,h)$ given by

$$\begin{aligned}
\boldsymbol{f} &= \{\partial_t^{m-2}(\tau\boldsymbol{x}') - (\partial_t^{m-2}\tau)\boldsymbol{x}' - \tau\partial_t^{m-2}\boldsymbol{x}'\}', \\
f &= -\frac{1}{2}\{\partial_t^{m-2}(\boldsymbol{x}'\cdot\boldsymbol{x}') - 2\boldsymbol{x}'\cdot\partial_t^{m-2}\boldsymbol{x}'\}, \\
h &= \{\partial_t^{m-2}(\dot{\boldsymbol{x}}'\cdot\dot{\boldsymbol{x}}') - 2\dot{\boldsymbol{x}}'\cdot\partial_t^{m-2}\dot{\boldsymbol{x}}'\} \\
&\quad - \{\partial_t^{m-2}(\tau\boldsymbol{x}''\cdot\boldsymbol{x}'') - (\partial_t^{m-2}\tau)\boldsymbol{x}''\cdot\boldsymbol{x}'' - 2\tau\boldsymbol{x}''\cdot\partial_t^{m-2}\boldsymbol{x}''\}.
\end{aligned}$$

Therefore, by Proposition 5.1 we obtain the energy estimate

$$E(t) \le C_1\mathrm{e}^{C_2t}\left(E(0) + S_1(0) + C_2\int_0^t S_2(t')\mathrm{d}t'\right), \tag{6.3}$$

where $E(t) = \|\dot{\boldsymbol{y}}(t)\|_{X^1}^2 + \|\boldsymbol{y}(t)\|_{X^2}^2 = \|\partial_t^{m-1}\boldsymbol{x}(t)\|_{X^1}^2 + \|\partial_t^{m-2}\boldsymbol{x}(t)\|_{X^2}^2$, and $S_1(t)$ and $S_2(t)$ are defined by (5.5).

Lemma 6.5. *It holds that $E(0)+S_1(0) \le C_0$ and $S_2(t) \le C_2$.*

We also omit the proof of this lemma. This lemma and (6.3) implies $E(t) \le C_1\mathrm{e}^{C_2t}(C_0 + C_2t)$. On the other hand, it is easy to see that $\|\!|\boldsymbol{x}(t)|\!\|_{m-1} \le \|\!|\boldsymbol{x}(0)|\!\|_{m-1} + \int_0^t \|\!|\boldsymbol{x}(t')|\!\|_m\mathrm{d}t' \le C_0 + C_2t$ and that $\frac{\tau(s,t)}{s} \ge \frac{\tau(s,0)}{s} - \frac{1}{s}\int_0^t |\partial_t\tau(s,t')|\mathrm{d}t' \ge 2c_0 - C_2t$. Summarizing the above estimates, we have shown

$$\begin{cases}
\|\partial_t^{m-1}\boldsymbol{x}(t)\|_{X^1}^2 + \|\partial_t^{m-2}\boldsymbol{x}(t)\|_{X^2}^2 \le C_1\mathrm{e}^{C_2t}(C_0+C_2t), \\
\|\!|\boldsymbol{x}(t)|\!\|_{m-1} \le C_0 + C_2t, \\
\frac{\tau(s,t)}{s} \ge 2c_0 - C_2t.
\end{cases}$$

Now, we define the constants M_1 and M_2 by $M_1 = 2C_0$ and $M_2 = 4C_0C_1$ and then choose the time T so small that $C_2T \le \min\{C_0, c_0, \log 2\}$. Then, by the standard argument we see that the solution $(\boldsymbol{x},\tau)$ satisfies in fact (6.2) for $0 \le t \le T$ and the estimates in Theorem 2.1 follows from Lemmas 6.4, 6.1, and 6.2. The proof of Theorem 2.1 is complete. □

References

[1] Amore, P., Boyd, J. P., Márquez, A.: The heterogeneous helicoseir. Physica D **446**, 133669 (2023).

[2] Connell, B. S. H., Yue, D. K. P.: Flapping dynamics of a flag in a uniform stream. J. Fluid Mech. **581**, 33–67 (2007).

[3] Grothaus, M.; Marheineke, N.: On a nonlinear partial differential algebraic system arising in the technical textile industry: analysis and numerics. IMA J. Numer. Anal. **36**, 1783–1803 (2016).

[4] Iguchi, T., Takayama, M.: A priori estimates for solutions to equations of motion of an inextensible hanging string. Math. Ann. **390**, 1919–1971 (2024).

[5] Iguchi, T., Takayama, M.: Well-posedness of the initial boundary value problem for degenerate hyperbolic systems with a localized term and its application to the linearized system for the motion of an inextensible hanging string. arXiv:2404.05292.

[6] Kolodner, I. I.: Heavy rotating string—a nonlinear eigenvalue problem. Comm. Pure Appl. Math. **8**, 395–408 (1955).

[7] Lee, J. H., Huang, W.-X., Sung, H. J.: Flapping dynamics of a flexible flag in a uniform flow. Fluid Dynam. Res. **46**, 055517 (2014).

[8] Preston, S. C.: The motion of whips and chains. J. Differential Equations **251**, 504–550 (2011).

[9] Preston, S. C.: The geometry of whips. Ann. Global Anal. Geom. **41**, 281–305 (2012).

[10] Preston, S. C., Saxton, R.: An H^1 model for inextensible strings. Discrete Contin. Dyn. Syst. **33**, 2065–2083 (2013).

[11] Reeken, M.: The equation of motion of a chain. Math. Z. **155**, 219–237 (1977).

[12] Reeken, M.: Classical solutions of the chain equation I. Math. Z. **165**, 143–169 (1979).

[13] Reeken, M.: Classical solutions of the chain equation II. Math. Z. **166**, 67–82 (1979).

[14] Ryu, J., Park, S. G., Kim, B., Sung, H. J.: Flapping dynamics of an inverted flag in a uniform flow. J. Fluids Struct. **57**, 159–169 (2015).

[15] Şengül, Y., Vorotnikov, D.: Generalized solutions for inextensible string equations. J. Differential Equations **262**, 3610–3641 (2017).

[16] Shi, W., Vorotnikov, D.: The gradient flow of the potential energy on the space of arcs. Calc. Var. **58**, Paper No. 59 (2019).

[17] Takayama, M.: Initial-boundary value problem for the degenerate hyperbolic equation of a hanging string. Osaka J. Math. **55**, 547–565 (2018)

[18] Yong, D.: Strings, chains, and ropes. SIAM Rev. **48**, 771–781 (2006).

Tatsuo Iguchi
Department of Mathematics
Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
E-mail: iguchi@math.keio.ac.jp

Masahiro Takayama
Department of Mathematics
Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
E-mail: masahiro@math.keio.ac.jp