Existence of homogeneous Euler flows of degree $-\alpha \notin [-2, 0]$

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1 Introduction

This article is based on the paper [Abe24]. In this study, we investigate $(-\alpha)$-homogeneous solutions to the Euler equations in $\mathbb{R}^3\backslash\{0\}$ for $\alpha \geq 0$ and in $\mathbb{R}^3$ for $\alpha < 0$:

$$\begin{aligned} u \cdot \nabla u + \nabla p &= 0, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1.1}$$

We say that $u = (u^1, u^2, u^3)$ is $(-\alpha)$-*homogeneous* if there exists $\alpha \in \mathbb{R}$ such that $u(x) = \lambda^{\alpha} u(\lambda x)$ for all $\lambda > 0$ and $x = (x_1, x_2, x_3)$. We say that (u, p) is a $(-\alpha)$-homogeneous solution to (1.1) if $(-\alpha)$-homogeneous u and (-2α)-homogeneous p satisfy (1.1).

The well-known (-1)-homogeneous solutions to the Navier–Stokes equations are the Landau solutions [Lan44], [Squ51], [LL59, p.81], [Bat99, p.205], [TX98], [CK04]. They are explicit solutions, smooth away from the origin, and axisymmetric without swirls. Tian and Xin [TX98] showed that all axisymmetric (-1)-homogeneous solutions $u \in C^2(\mathbb{R}^3\backslash\{0\})$ to the Navier–Stokes equations are the Landau solutions. Šverák [Š11] demonstrated that all (-1)-homogeneous solutions $u \in C^2(\mathbb{R}^n\backslash\{0\})$ for $n = 3$ are the Landau solutions, as well as the nonexistence of (-1)-homogeneous solutions for $n \geq 4$ and their rigidity for $n = 2$ under the flux condition. The Landau solutions are relevant to the regularity of stationary solutions [Š11] and their asymptotic behavior as $|x| \to \infty$ [Š11], [Kv11], [MT12], [KMT12].

It is conjectured in the work of Šverák [Š11] that the Landau solutions are rigid among all smooth solutions in $\mathbb{R}^3\backslash\{0\}$ satisfying the following:

$$|u(x)| \leq \frac{C}{|x|}, \qquad x \in \mathbb{R}^3\backslash\{0\}.$$

Korolev and Šverák [Kv11] and Miura and Tsai [MT12], [Tsa18, 8.2] demonstrated that this conjecture holds for small constant C. Li et al. [LLY18b] discovered explicit axisymmetric (-1)-homogeneous solutions without swirls smooth away from the negative part of the x_3-axis, i.e., $u|_{\mathbb{S}^2} \in C^\infty(\mathbb{S}^2\backslash\{S\})$ for the South pole S. The work [LLY18b] also demonstrates the existence of axisymmetric (-1)-homogeneous solutions with swirls $u|_{\mathbb{S}^2} \in C^\infty(\mathbb{S}^2\backslash\{S\})$. The subsequent works [LLY18a] and [LLY19] show the existence of axisymmetric (-1)-homogeneous solutions with swirls smooth away from the x_3-axis, i.e., $u|_{\mathbb{S}^2} \in C^\infty(\mathbb{S}^2\backslash\{S \cup N\})$. Kwon and Tsai [KT21]

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explored the bifurcations of the Landau solutions in the class of axisymmetric and discrete homogeneous (self-similar) solutions.

Luo and Shvydkoy [LS15] and Shvydkoy [Shv18] investigated homogeneous solutions to the Euler equations. The work by Shvydkoy [Shv18] is motivated by Onsager's conjecture [Ons49], [Shv10], [DLS13], [Ise13], [CS14a]. We refer to the book of Bedrossian and Vicol [BV22, 6.1.2] for the introduction to Onsager's conjecture. The work [Shv18] demonstrates the nonexistence of $(-\alpha)$-homogeneous solutions to the Euler equations in the following cases.

Case 0: Irrotational flows $\nabla \times u = 0$. $(-\alpha)$-homogeneous solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ exist if and only if $\alpha \in \mathbb{Z}\backslash\{1\}$. They are given by spherical harmonics.
Case 1: $\alpha = 1$. No (-1)-homogeneous solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ exist.
Case 2: $\alpha > 1$. For $1 < \alpha \leq 2$, no $(-\alpha)$-homogeneous solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ exist other than the irrotational solution among the following:

(A) Beltrami flows $(\nabla \times u) \times u = 0$ or
(B) Axisymmetric flows.

Case 3: $\alpha < 1$. In classes (A) for $\alpha < 1$ and (B) for $0 \leq \alpha < 1$, no $(-\alpha)$-homogeneous solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ exist other than the irrotational solutions.

These rigidity results are based on the homogeneous solution's equations on the sphere [Š11], [Shv18] and do not assume their continuity at $x = 0$ for $\alpha < 0$. We include them [Shv18] in the main statements of this study ((i) and (ii) of Theorems 1.1, 1.4, and 1.5, except (ii) of Theorem 1.4 for $-2 \leq \alpha < 0$).

On the existence side, only explicit homogeneous solutions to (1.1) are known [LS15], [Shv18] (Remarks 1.6). This study aims to show the existence of axisymmetric homogeneous solutions. We use the cylindrical coordinates (r, ϕ, z) defined by the following:

$$x_1 = r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = z,$$

and the associated orthogonal flame

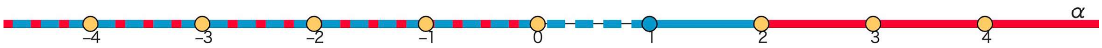
$$e_r = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad e_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For axisymmetric $u = u^r e_r + u^\phi e_\phi + u^z e_z$, we denote the poloidal component by $u^P = u^r e_r + u^z e_z$ and the toroidal component by $u^\phi e_\phi$. We say that u is axisymmetric without swirl if $u^\phi = 0$.

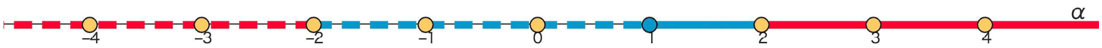
1.1 Statements of the main results

We consider continuously differentiable $(-\alpha)$-homogeneous solutions $(u.p) \in C^1(\mathbb{R}^3\backslash\{0\})$ for $\alpha > 2$ in $\mathbb{R}^3\backslash\{0\}$ and continuous $(-\alpha)$-homogeneous solutions $(u.p) \in C(\mathbb{R}^3)$ for $\alpha < 0$ in $\mathbb{R}^3$ satisfying (1.1) in the distributional sense. We say that a $(-\alpha)$-homogeneous solution $(u.p) \in C(\mathbb{R}^3)$ for $\alpha < 0$ is a Beltrami flow in $\mathbb{R}^3$ if the Bernoulli function $\Pi = p + |u|^2/2$ vanishes.

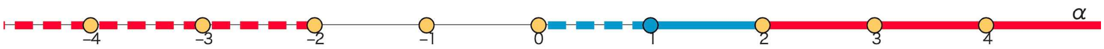
Theorem 1.1. *The following holds for rotational Beltrami $(-\alpha)$-homogeneous solutions to (1.1):*
(i) For $1 \leq \alpha \leq 2$, no solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ exist.
(ii) For $\alpha < 1$, no solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ exist.
(iii) For $\alpha > 2$, axisymmetric solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ such that $u^P, p \in C^2(\mathbb{R}^3\backslash\{0\})$ and $u^\phi e_\phi \in C^1(\mathbb{R}^3\backslash\{0\})$ exist.
(iv) For $\alpha < 0$, axisymmetric solutions $(u, p) \in C(\mathbb{R}^3)$ such that $u^P, p \in C^1(\mathbb{R}^3\backslash\{r = 0\}) \cap C(\mathbb{R}^3)$ and $u^\phi e_\phi \in C(\mathbb{R}^3)$ exist.



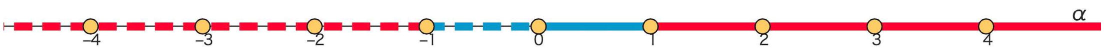
(a) Beltrami $(-\alpha)$-homogeneous solutions in Theorem 1.1.



(b) Axisymmetric $(-\alpha)$-homogeneous solutions without swirls in Theorem 1.4.



(c) Axisymmetric $(-\alpha)$-homogeneous solutions with a non-constant Bernoulli function in Theorem 1.5.



(d) 2D reflection symmetric $(-\alpha)$-homogeneous solutions in Theorem 1.7.

Figure 1: The existence and nonexistence ranges of $(-\alpha)$-homogeneous solutions to (1.1) in Theorems 1.1, 1.4, 1.5, and 1.7. The yellow dots represent irrotational solutions. The blue dot and line represent the nonexistence range (i). The blue dashed line represents the nonexistence range (ii). The red line represents the existence range (iii). The dashed red line represents the existence range (iv).

Remarks 1.2. (i) A vector field $u \in C^1$ is a Beltrami flow if there exists a proportionality factor φ such that

$$\nabla \times u = \varphi u, \quad \nabla \cdot u = 0. \tag{1.2}$$

The factor φ is a first integral of u, i.e., $u \cdot \nabla\varphi = 0$, and streamlines of u are constrained on the

level sets of φ. It is known [EPS12], [EPS15] that there exists a smooth Beltrami flow with a constant factor $\varphi \equiv$ const. in $\mathbb{R}^3$ having arbitrary knotted and linked vortexlines and decaying by the order $u = O(|x|^{-1})$ as $|x| \to \infty$.
(ii) It is known [Nad14], [CC15], [CW16] that the locally square integrable Beltrami flows in $\mathbb{R}^3$ do not exist if $u = o(|x|^{-1})$ as $|x| \to \infty$, cf. [EPS12], [EPS15].
(iii) It is also known [EPS16] that the smooth Beltrami flows in a domain do not exist if the proportionality factor $\varphi \in C^{2+\gamma}$ $(0 < \gamma < 1)$ admits a level set diffeomorphic to a sphere, e.g., radial or having local extrema.
(iv) Constantin et al. [CDG] demonstrate that the axisymmetric Beltrami flows in a hollowed-out periodic cylinder are translation invariant if the poloidal component u^P has no stagnation points in the cross-section, cf. [HN17], [HN19].
(v) There exist asymptotically constant axisymmetric Beltrami flows in $\mathbb{R}^3$ with a nonconstant factor $\varphi \not\equiv$ const. whose level set is a ball [Mof69], a solid torus [Tur89], and nested tori [Abe22].
(vi) Axisymmetric Beltrami $(-\alpha)$-homogeneous solutions $u \in C^1(\mathbb{R}^3\backslash\{0\})$ for $\alpha > 2$ in Theorem 1.1 (iii) possess the axisymmetric stream function $\psi(z, r)$ and the proportionality factor

$$\varphi = C\left(1 + \frac{1}{\alpha - 2}\right)|\psi|^{\frac{1}{\alpha-2}}. \tag{1.3}$$

This solution is not square integrable at $x = 0$ and decaying faster than $o(|x|^{-1})$ as $|x| \to \infty$, cf. [Nad14], [CC15], [CW16]. The level sets of the proportionality factor φ are nested surfaces created by the rotation of multifoils, cf. [EPS16] (Remark 1.12). The solution $u \in C(\mathbb{R}^3)$ for $\alpha < 0$ in Theorem 1.1 (iv) is growing as $|x| \to \infty$.

A simple class of rotational flows with a nonconstant Bernoulli function is as follows:

(C) Radially irrotational flows $\nabla \times u \cdot x = 0$.

We remark that the tangentially irrotational homogeneous flows $(\nabla\times u)\times x = 0$ are irrotational (Remark 2.14). The radially irrotational flows include axisymmetric flows without swirls. On the contrary, we demonstrate the following:

Theorem 1.3. *All radially irrotational $(-\alpha)$-homogeneous solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ for $\alpha \in \mathbb{R}$ with a nonconstant Bernoulli function to (1.1) are axisymmetric without swirls.*

The existence and nonexistence ranges of axisymmetric $(-\alpha)$-homogeneous solutions without swirls are split into $\alpha \in \mathbb{R}\backslash[-2, 2]$ and $\alpha \in [-2, 2]$.

Theorem 1.4. *The following holds for rotational axisymmetric $(-\alpha)$-homogeneous solutions without swirls to (1.1):*
(i) For $1 \leq \alpha \leq 2$, no solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ exist.
(ii) For $-2 < \alpha < 1$, no solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ exist. For $\alpha = -2$, no solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ exist provided that $\nabla \times u \cdot e_\phi/r$ vanishes on the z-axis.
(iii) For $\alpha > 2$, solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ exist.
(iv) For $\alpha < -2$, solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\}) \cap C(\mathbb{R}^3)$ exist.

We state a general existence result on axisymmetric $(-\alpha)$-homogeneous solutions with a nonconstant Bernoulli function.

Theorem 1.5. *The following holds for rotational axisymmetric $(-\alpha)$-homogeneous solutions to (1.1):*
(i) For $1 \leq \alpha \leq 2$, no solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ exist.
(ii) For $0 \leq \alpha < 1$, no solutions $(u, p) \in C^2(\mathbb{R}^3\backslash\{0\})$ exist.
(iii) For $\alpha > 2$, solutions $(u, p) \in C^1(\mathbb{R}^3\backslash\{0\})$ such that $u^P, p \in C^2(\mathbb{R}^3\backslash\{0\})$ and $u^\phi e_\phi \in C^1(\mathbb{R}^3\backslash\{0\})$ exist.
(iv) For $\alpha < -2$, solutions $(u, p) \in C(\mathbb{R}^3)$ such that $u^P, p \in C^1(\mathbb{R}^3\backslash\{r = 0\}) \cap C(\mathbb{R}^3)$ and $u^\phi e_\phi \in C(\mathbb{R}^3)$ exist.

Remarks 1.6. (i) The explicit rotational axisymmetric $(-\alpha)$-homogeneous solution with a nonconstant Bernoulli function (u, p) exists for $\alpha \leq 0$ [Shv18, p.2521, (13)]:

$$
\begin{aligned}
u(x) &= b^2 \frac{x_3}{x_1^2 + x_2^2} K^{-\frac{\alpha}{2}} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} - \frac{b}{x_1^2 + x_2^2} K^{\frac{1-\alpha}{2}} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} + a^2 K^{-\frac{\alpha}{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
K(x) &= \left(a^2(x_1^2 + x_2^2) - b^2 x_3^2\right)_+, \quad a^2 + b^2 = 1, \quad \alpha p = 0, \quad \alpha \leq 0.
\end{aligned}
$$

Here, $s_+ = \max\{s, 0\}$ for $s \in \mathbb{R}$. For $b = 0$, $u = r^{-\alpha} e_z$ is without swirl and belongs to $C^1(\mathbb{R}^2)$ for $\alpha < -1$ and $C^2(\mathbb{R}^2)$ for $\alpha \leq -2$. For $\alpha = -2$, the toroidal component of vorticity is $\nabla \times u \cdot e_\phi / r = -2$, cf. Theorem 1.4 (ii). For $b \neq 0$, u is with swirls, belongs to $C^1(\mathbb{R}^3)$ for $\alpha < -2$, and is supported in the wedged region $\{a^2 r^2 - b^2 z^2 > 0\}$. In particular, $u|_{\mathbb{S}^2}$ is compactly supported in $\{\theta_0 < \theta < \pi - \theta,\ 0 \leq \phi \leq 2\pi\}$ on $\mathbb{S}^2$ for $\theta_0 = \arctan|b/a|$, cf. Theorem 1.5 (iv). Here, θ is the geodesic radial coordinate on $\mathbb{S}^2$ (Section 2). This solution is as follows:

(D) Geodesic flows $(u \cdot \nabla u) \times u = 0$.

Namely, streamlines are rays. The solutions of (1.1) with a constant pressure are geodesic flows. It is demonstrated in the work of Shvydkoy [Shv18, Proposition 5.3] that all axisymmetric $(-\alpha)$-homogeneous solutions $u \in C^1(\mathbb{R}^3\backslash\{0\})$ with a constant pressure p are this solution or the irrotational solution for $\alpha = 2$. We remark on the existence of compactly supported inhomogeneous axisymmetric solutions with swirls in $\mathbb{R}^3$ [Gav19], [CLV19], [DVEPS21] and compactly supported vortex patch solutions in $\mathbb{R}^2$ [GSPS]. Baldi [Bal] discusses the streamline geometry of compactly supported inhomogeneous axisymmetric solutions with swirls.
(ii) The two-dimensional (2D) $(-\alpha)$-homogeneous solutions $u = (u^1, u^2, 0)$ can exist for all $\alpha \in \mathbb{R}$. Luo and Shvydkoy [LS15], [Shv18, 2.2] found several explicit solutions and investigated the streamlines of $(-\alpha)$-homogeneous solutions based on the stream function's Hamiltonian PDE. The pressure p of 2D $(-\alpha)$-homogeneous solutions is constant on the circle $r = 1$. In fact, for any $\alpha \in \mathbb{R}$,

$$u(x)=\frac{a}{r^{\alpha+1}}\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix},\quad p(x)=-\left(\frac{a^2}{2\alpha}\right)\frac{1}{r^{2\alpha}},\quad a\in\mathbb{R},$$

is a radially symmetric $(-\alpha)$-homogeneous solution (a circular flow) to (1.1) in $\mathbb{R}^3\backslash\{r=0\}$ for $\alpha\geq 0$ and in $\mathbb{R}^3$ for $\alpha<0$, cf. Theorem 1.5 (i) and (ii). All 2D radially symmetric $(-\alpha)$-homogeneous solution for $\alpha\in\mathbb{R}\backslash\{1\}$ is this solution (Theorem A.1). The work by Shvydkoy [Shv18, Proposition 4.1] demonstrates that all $(-\alpha)$-homogeneous solutions $(u,p)\in C^1(\mathbb{R}^3\backslash\{0\})$ to (1.1) are 2D radially symmetric solutions for $\alpha\leq -1$, provided that

(E) Tangential flows $u\cdot x=0$.

Guo et al. [GHPWar], [GPW23] demonstrated that the radially symmetric 1-homogeneous solution is stable in the axisymmetric Euler equations via the Euler–Coriolis equations.

Noncircular streamlines appear for the following:

(F) 2D reflection symmetric flows $u=(u^1,u^2,0)$,

$$\begin{aligned}u^1(x_1,x_2)&=u^1(x_1,-x_2),\\ u^2(x_1,x_2)&=-u^2(x_1,-x_2),\\ p(x_1,x_2)&=p(x_1,-x_2).\end{aligned}$$

Irrotational 2D reflection symmetric $(-\alpha)$-homogeneous solutions $u=(u^1,u^2,0)\in C^1(\mathbb{R}^2\backslash\{0\})$ to (1.1) exist if and only if $\alpha\in\mathbb{Z}$ (Theorem A.2). They are constant multiples of the following:

$$\begin{aligned}u&=\frac{1}{r^2}\begin{pmatrix}x_1\\ x_2\\ 0\end{pmatrix},\quad \alpha=1,\\ u&=\begin{pmatrix}\partial_2\psi\\ -\partial_1\psi\\ 0\end{pmatrix},\quad \alpha\in\mathbb{Z}\backslash\{1\},\end{aligned}\tag{1.4}$$

for the stream function

$$\psi(x_1,x_2)=\frac{\sin(n\phi)}{r^n},\quad n=\alpha-1.\tag{1.5}$$

Figure 2 shows the level sets of ψ for $n=\pm1$, ±2, and ±3.

We consider 2D reflection symmetric $(-\alpha)$-homogeneous solutions $u\in C^1(\mathbb{R}^2\backslash\{0\})$ for $\alpha\geq 0$ in $\mathbb{R}^2\backslash\{0\}$ and $u\in C(\mathbb{R}^2)$ for $\alpha<0$ in $\mathbb{R}^2$ satisfying (1.1) in the distributional sense. The existence and nonexistence ranges of 2D reflection symmetric $(-\alpha)$-homogeneous solutions are split into $\alpha\in\mathbb{R}\backslash[-1,1]$ and $\alpha\in[-1,1]$, cf. Theorem 1.4.

Theorem 1.7. *The following holds for rotational 2D reflection symmetric* $(-\alpha)$*-homogeneous solutions* $u=(u^1,u^2,0)$ *and* $r^{2\alpha}p=const.$ *to (1.1):*
(i) For $0\leq\alpha\leq 1$*, no solutions* $u\in C^1(\mathbb{R}^2\backslash\{0\})$ *exist.*
(ii) For $-1\leq\alpha<0$*, no solutions* $u\in C^2(\mathbb{R}^2\backslash\{0\})$ *exist.*
(iii) For $\alpha>1$*, solutions* $u\in C^2(\mathbb{R}^2\backslash\{0\})$ *exist.*
(iv) For $\alpha<-1$*, solutions* $u\in C^1(\mathbb{R}^2\backslash\{0\})\cap C(\mathbb{R}^2)$ *exist.*

The stream function level sets $\{\psi=\pm C\}$ for $C>0$ of the rotational 2D reflection symmetric $(-\alpha)$-homogeneous solutions for $\alpha>1$ in Theorem 1.7 (iii) are unions of the Jordan curves sharing the origin (multifoils). For $\alpha=2$, the stream function level sets of the irrotational 2D reflection symmetric (-2)-homogeneous solution consist of the Jordan curves $\{\psi=C\}$ in the upper half plane and the Jordan curves $\{\psi=-C\}$ in the lower half plane ($n=1$ in Figure 2). We show the existence of rotational $(-\alpha)$-homogeneous solutions for $1<\alpha<2$ whose stream function level sets are homeomorphic to those of the irrotational 2D reflection symmetric (-2)-homogeneous solution.

Theorem 1.8. *For* $1<\alpha<2$*, there exist rotational 2D reflection symmetric* $(-\alpha)$*-homogeneous solutions* $u\in C^2(\mathbb{R}^2\backslash\{0\})$ *to (1.1) whose stream function level sets are homeomorphic to those of the irrotational* (-2)*-homogeneous solution.*

Remarks 1.9. (i) Choffrut and Šverák [Cv12] investigated a local one-to-one correspondence between smooth 2D steady states $u=(u^1,u^2,0)$ and co-adjoint orbits of the nonstationary problem in an annulus for steady states whose stream function ψ and vorticity ω have no critical points and satisfy nondegeneracy conditions. The stream function and vorticity of $(-\alpha)$-homogeneous solutions in Theorem 1.7 (iii) and (iv) are the following:

$$\begin{aligned}\psi(x_1,x_2)&=\frac{w(\phi)}{r^{\alpha-1}},\\ \omega&=c\psi|\psi|^{\frac{2}{\alpha-1}},\end{aligned}\tag{1.6}$$

for some function $w(\phi)$ on $[-\pi,\pi]$ and a positive constant $c>0$. Their gradients are the following:

$$|\nabla\psi|^2=\frac{|\alpha-1|^2w^2+|w'|^2}{r^{2\alpha}},\quad \nabla\omega=\frac{\alpha+1}{\alpha-1}|\psi|^{\frac{2}{\alpha-1}}\nabla\psi.$$

For $\alpha>1$, ψ has no critical points in $\mathbb{R}^2\backslash\{0\}$ because w and w' do not vanish at the same point (Remarks B.3 (iii)). The vorticity ω has critical points on, e.g., $\{x_2=0\}$. For $\alpha<-1$, both ψ and ω have critical points at the origin. We remark that Choffrut and Székelyhidi [CS14b] demonstrated the existence of merely bounded steady states near a given smooth steady state in $\mathbb{T}^d$ for $d\geq 2$ based on the convex integration.
(ii) Hamel and Nadirashvili [HN23, Theorem 1.8] established rigidity theorems for the 2D Euler equations in bounded annuli, exteriors of disk, punctured disks, and punctured planes. It is shown that all solutions of the 2D Euler equations in a punctured plane $u=(u^1,u^2,0)\in C^2(\mathbb{R}^2\backslash\{0\})$ satisfying

$$
\begin{aligned}
&|u| > 0 \quad \text{in } \mathbb{R}^2\backslash\{0\}, \\
&\liminf_{r\to\infty} |u| > 0, \\
&u\cdot e_r = o\left(\frac{1}{r}\right) \quad \text{as } r\to\infty, \\
&\int_{\{r=\varepsilon\}} |u\cdot e_r| \mathrm{d}H \to 0 \quad \text{as } \varepsilon\to 0,
\end{aligned} \tag{1.7}
$$

are circular flows, i.e., $u \cdot e_r = 0$. The vector field $(1.4)_2$ is a noncircular irrotational $(-\alpha)$-homogeneous solution to the 2D Euler equation in $\mathbb{R}^2\backslash\{0\}$, violating the conditions $(1.7)_2$ and $(1.7)_4$ for $n > 0$ and $(1.7)_3$ for $n < 0$. The solutions in Theorem 1.7 (iii) are examples of the noncircular rotational $(-\alpha)$-homogeneous solutions for $\alpha > 1$ in $\mathbb{R}^2\backslash\{0\}$.
(iii) It is shown in the work of Hamel and Nadirashvili [HN19, Theorem 1.1] that all solutions of the 2D Euler equations in the plane $u = (u^1, u^2, 0) \in C^2(\mathbb{R}^2)$ satisfying

$$
\begin{aligned}
&\sup_{\mathbb{R}^2} |u| < \infty, \\
&\inf_{\mathbb{R}^2} |u| > 0,
\end{aligned}
$$

are shear flows, i.e., $u = (u^1(x_2), 0, 0)$, for some function u^1 with a constant strict sign by a suitable rotation. (Koch and Nadirashvili [KN] discuss analyticity of streamlines and Hamel and Nadirashvili [HN17] discuss a rigidity theorem in a strip). The vector field $(1.4)_2$ for $n < 0$ is an irrotational $(-\alpha)$-homogeneous solution to the 2D Euler equation in $\mathbb{R}^2$. This solution is constant (a shear flow) for $n = -1$ and has a stagnation point at $x = 0$ and is growing as $|x| \to \infty$ for $n \leq -2$ (Figure 2). The solutions $u \in C^1(\mathbb{R}^2\backslash\{0\}) \cap C(\mathbb{R}^2)$ in Theorem 1.7 (iv) are examples of the nonshear rotational $(-\alpha)$-homogeneous solutions for $\alpha < -1$ in $\mathbb{R}^2$.
(iv) It is a conjecture [Šve, chapter 34], [Shn13] that vorticity of the 2D nonstationary Euler equation is generically weakly compact but not strongly compact as $t \to \infty$. Glatt-Holtz et al. [GHvV15, 2.2] discuss the relationship between the compactness of vorticity and coherent structures at the end state. The behavior of solutions around shear and circular flows are investigated in perturbative regimes. We refer to the important works [BM15], [BGM19], [DM], [MZ], [IJ20], [IJ22], [IJ] on the nonlinear asymptotic stability of the 2D Euler equations.

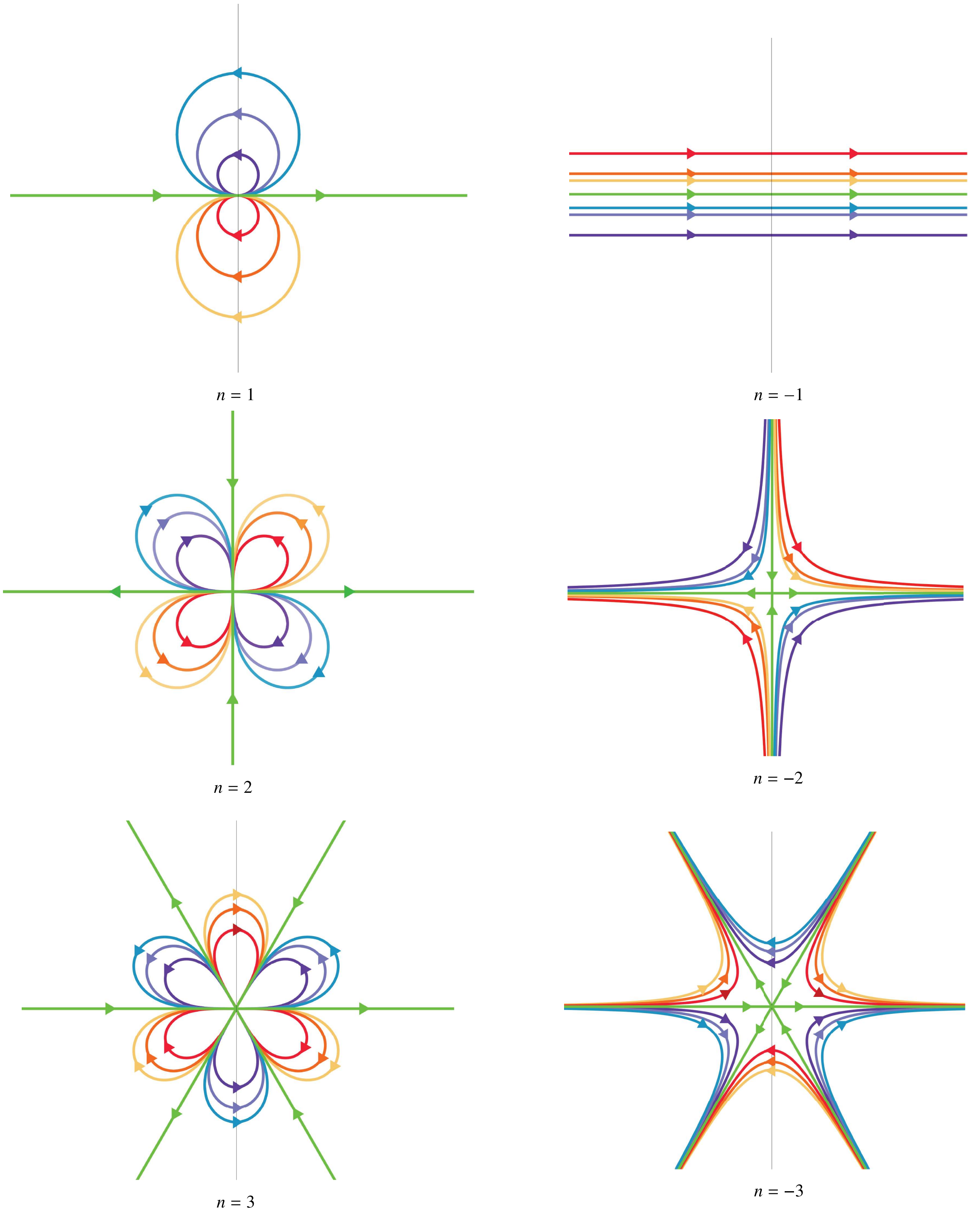


Figure 2: The level sets of $\psi(x_1, x_2) = \sin(n\phi)/r^n$ for $n = \pm 1$ (dipole), ± 2 (quadrupole), and ± 3 (hexapole). The sets $\psi^{-1}(k)$ are represented in purple ($k = 1$), blue ($k = 1/2$), light blue ($k = 1/3$), green ($k = 0$), yellow ($k = -1/3$), orange ($k = -1/2$), and red ($k = -1$).

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