New proof of the Local Existence of a Classical Solution for Quasi-Linear Hyperbolic Systems

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Abstract

We study the local existence of the classical solutions to the quasilinear system. This proof has already been given in several books. However, there seemed to be some gaps that we was not able to fill in. The purpose in this paper is to provide the new proof.

In this article, we give a proof of the local existence of the classical solutions to the quasilinear system

$$u_t + A(x,t,u)u_x = h(x,t,u) \tag{1}$$

with initial data

$$u(x,0) = \bar{u}(x), \qquad x \in [a,b], \tag{2}$$

where we assume

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(A) Each matrix $A(x,t,u)$ has n real distinct eigenvalues. The functions $A : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ and $h : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable with respect to all variables and their derivatives are locally Lipschitz continuous with respect to all variables. In addition, we assume $\bar{u}$ is continuously differentiable.

Remark 1. *In [1], it is not assumed that the derivatives of A and h are locally Lipschitz continuous. However, this assumption seems be needed in the proof.*

For simplicity, we further assume that $A(x,t,u)$ is a diagonal matrix, i.e.,

$$A(x,t,u) = \begin{pmatrix} \lambda_1(x,t,u) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(x,t,u) \end{pmatrix}.$$

Then we have the following theorem.

Theorem 2. *Suppose that A, h, and $\bar{u}$ satisfy the assumption (A). Then there exist constants Λ, $T > 0$ and a continuously differentiable function u which is the unique classical solution of* (1), (2) *on the domain*

$$\mathcal{D} = \mathcal{D}_{\Lambda,T} := \{(x,t) : t \in [0,T],\ a + \Lambda t \le x \le b - \Lambda t\}. \tag{3}$$

Proof. To get the solution, we consider the sequence $u^{(\nu)} : \mathcal{D} \to \mathbb{R}^n$ such that $u^{(0)}(x,0) = \bar{u}(x)$ and, for $\nu \ge 1$, $u^{(\nu)} = (u_1^{(\nu)}, \dots, u_n^{(\nu)})$ is defined inductively by the solution of the semilinear problem

$$u_{i,t}^{(\nu)} + \lambda_i(x,t,u^{(\nu-1)}) u_{i,x}^{(\nu)} = h_i(x,t,u^{(\nu-1)}), \quad u^{(\nu)}(x,0) = \bar{u}(x), \quad i = 1, \dots, n, \qquad (4)_\nu$$

where h_i is the i-th component of h.

The proof consists of 5 steps. The first three steps follow the argument in [1]. The essential part is Step 4.

Step 1. We set

$$C_0 := \max_{x \in [a,b]} |\bar{u}(x)| \tag{5}$$

and choose Λ in (3) to be

$$\Lambda = \max\{|\lambda_i(x,t,u)| :\ t \in [0,1],\ x \in [a,b],\ |u| \le C_0 + 1,\ i = 1, \dots, n\}.$$

Under the choice of Λ, the set $\mathcal{D}$ becomes a domain of determinacy for $(4)_\nu$ provided that

$$T \le 1, \qquad |u^{(\nu-1)}(x,t)| \le C_0 + 1 \quad \text{for } (x,t) \in \mathcal{D}. \qquad (6)_\nu$$

With the assumption (A), one can choose constants C_1 and C_2 such that

$$|\bar{u}'(x)| \le C_1, \quad |h(x,0,\bar{u}(x)) - A(x,0,\bar{u}(x))\bar{u}'(x)| \le C_1 \qquad \text{for all } x \in [a,b] \tag{7}$$

and

$$|h_x| \le C_2, \quad |h_t| \le C_2, \quad |h_u| + |A_x| \le C_2, \quad |h_u| + |A_t| \le C_2, \quad |A_u| \le C_2 \tag{8}$$

for all $t \in [0,1]$, $x \in [a,b]$, $|u| \le C_0 + 1$.

Let $\bar{Y}$ be the solution of the ODE

$$Y' = C_2(1+nY)^2, \qquad Y(0) = C_1.$$

Then $\bar{Y}$ can be solved explicitly by

$$\bar{Y}(t) = \frac{1}{n}\left(\frac{1}{(1+nC_1)^{-1} - nC_2 t} - 1\right) \quad \text{for all } t \in \left[0, \frac{1}{nC_2(1+nC_1)}\right).$$

We now choose $T > 0$ small enough such that

$$\int_0^T n\bar{Y}(t)dt \le 1. \tag{9}$$

Step 2. We now prove by induction that $(6)_\nu$ holds together with

$$|u_x^{(\nu)}(x,t)| \le n\bar{Y}(t), \quad |u_t^{(\nu)}(x,t)| \le n\bar{Y}(t) \qquad \text{for all } (x,t) \in \mathcal{D},\ \nu \in \mathbb{N} \cup \{0\}. \tag*{$(10)_\nu$}$$

If $(6)_\nu$ and $(10)_\nu$ holds for all $\nu \ge 0$, then we will prove the following:

(a) $\mathcal{D}$ serves as a universal domain of determinacy for all problems $(4)_\nu$.

(b) The Lipschitz constant of the functions $u^{(\nu)}$ is uniformly bounded on $\mathcal{D}$.

$(10)_\nu$ is obvious for $\nu = 0$. Suppose that $(10)_{\nu-1}$ is true. Then, by (5) and (9), we have

$$\begin{aligned} |u^{(\nu-1)}(x,t)| &\le |u^{(\nu-1)}(x,0)| + \int_0^t |u_t^{(\nu-1)}(x,s)|ds \\ &\le |\bar{u}(x)| + \int_0^t n\bar{Y}(s)ds \le C_0 + 1 \qquad \text{for } (x,t) \in \mathcal{D}, \end{aligned} \tag{11}$$

which proves $(6)_\nu$. It follows from [1, Theorem 3.6] (see also [2, Theorem 6]) that problem $(4)_\nu$ admits a classical solution $u^{(\nu)}$ on $\mathcal{D}$. Moreorver, if we write $\mathfrak{v}^{(\nu)} := u_x^{(\nu)}$, $\mathfrak{w}^{(\nu)} := u_t^{(\nu)}$, and $\mathfrak{v}_i^{(\nu)}$, $\mathfrak{w}_i^{(\nu)}$ denote the i-th component of $\mathfrak{v}^{(\nu)}$, $\mathfrak{w}^{(\nu)}$ respectively, then they will be *broad solutions* (see [1, p. 48] for its definition) for the following systems

$$\begin{aligned} \mathfrak{v}_{i,t}^{(\nu)} + \lambda_i\left(x,t,u^{(\nu-1)}(x,t)\right)\mathfrak{v}_{i,x}^{(\nu)} &= h_{i,x} + h_{i,u}\cdot u_x^{(\nu-1)} - \left(\lambda_{i,x} + \lambda_{i,u}\cdot u_x^{(\nu-1)}\right)\mathfrak{v}^{(\nu)}, \quad i = 1,\dots,n, \\ \mathfrak{w}_{i,t}^{(\nu)} + \lambda_i\left(x,t,u^{(\nu-1)}(x,t)\right)\mathfrak{w}_{i,x}^{(\nu)} &= h_{i,t} + h_{i,u}\cdot u_t^{(\nu-1)} - \left(\lambda_{i,t} + \lambda_{i,u}\cdot u_t^{(\nu-1)}\right)\mathfrak{w}^{(\nu)}, \quad i = 1,\dots,n, \end{aligned}$$

which implies

$$\frac{d}{d\tau}\left\{\mathfrak{v}_i^{(\nu)}(x_i^{(\nu-1)}(\tau;x,t),\tau)\right\} = h_{i,x} + h_{i,u}\cdot u_x^{(\nu-1)} - \left(\lambda_{i,x} + \lambda_{i,u}\cdot u_x^{(\nu-1)}\right)\mathfrak{v}^{(\nu)}, \quad i = 1,\dots,n, \tag{12}$$

$$\frac{d}{d\tau}\left\{\mathfrak{w}_i^{(\nu)}(x_i^{(\nu-1)}(\tau;x,t),\tau)\right\} = h_{i,t} + h_{i,u}\cdot u_t^{(\nu-1)} - \left(\lambda_{i,t} + \lambda_{i,u}\cdot u_t^{(\nu-1)}\right)\mathfrak{w}^{(\nu)}, \quad i = 1,\dots,n, \tag{13}$$

where $\tau \mapsto x_i^{(\nu-1)}(\tau;x,t)$ denotes the i-th characteristic curve related to $u^{(\nu)}$ passing through (x,t); more precisely, $x_i^{(\nu-1)}(\tau;x,t)$ is the solution of

$$\frac{dX}{d\tau} = \lambda_i\left(X,\tau,u^{(\nu-1)}(X,\tau)\right), \qquad X(t) = x. \tag{14}$$

Define

$$Y(\tau) := \max\{|\mathfrak{v}_i^{(\nu)}(X,\tau)|, |\mathfrak{w}_i^{(\nu)}(X,\tau)| : \ X \in [a+\Lambda\tau, b-\Lambda\tau], \ i = 1,\dots,n\}.$$

In view of (7)-(8), (11)-(13), and the induction hypothesis, we obtain that

$$Y'(\tau) \le C_2[1 + n\bar{Y}(\tau) + n^2\bar{Y}(\tau)Y(\tau)] < \bar{Y}'(\tau), \qquad Y(0) \le C_1,$$

provided that $Y(\tau) \le \bar{Y}(\tau)$. A simple comparison argument gives that $Y(\tau) \le \bar{Y}(\tau)$ for all $\tau \in [0,T]$. Thus, by induction, we complete the proof of $(10)_\nu$.

Step 3. Next, we show the uniform convergence of the sequence $u^{(\nu)}$ on $\mathcal{D}$. For simplicity, we write

$$\mathfrak{u}^\nu = u^{(\nu)} - u^{(\nu-1)}, \quad \lambda_i^{\nu-1}(x,t) = \lambda_i\left(x,t,u^{(\nu-1)}(x,t)\right), \quad h_i^{\nu-1}(x,t) = h_i\left(x,t,u^{(\nu-1)}(x,t)\right). \tag{15}$$

From (4), we get that

$$\mathfrak{u}_{i,t}^\nu + \lambda_i^{\nu-1}\mathfrak{u}_{i,x}^\nu = h_i^{\nu-1} - h_i^{\nu-2} - (\lambda_i^{\nu-1} - \lambda_i^{\nu-2})u_{i,x}^{(\nu-1)}, \quad i = 1,\dots,n,$$

and then

$$\frac{d}{d\tau}\left\{\mathfrak{u}_i^\nu(x_i^{(\nu-1)}(\tau;x,t),\tau)\right\} = h_i^{\nu-1} - h_i^{\nu-2} - (\lambda_i^{\nu-1} - \lambda_i^{\nu-2})u_{i,x}^{(\nu-1)}, \quad i = 1,\dots,n. \tag{16}$$

From the assumption (A), there exists a constant C_3 such that

$$|A(x,t,u) - A(x,t,u')| \le C_3|u-u'|, \qquad |h(x,t,u) - h(x,t,u')| \le C_3|u-u'|$$

provided that $(x,t) \in \mathcal{D}$, $|u|, |u'| \le C_0 + 1$. On the other hand, the bounds (10) say that there exists a constant C_4 such that $|u_x^{(\nu)}| \le C_4$ for $(x,t) \in \mathcal{D}$ and $\nu \in \mathbb{N} \cup \{0\}$. We now consider the function

$$Z_\nu(\tau) := \max\{|\mathfrak{u}_i^\nu(X,\tau)| : \ X \in [a+\Lambda\tau, b-\Lambda\tau], \ i = 1,\dots,n\}.$$

Then (16) yields that

$$Z_\nu'(\tau) \le C_3 n Z_{\nu-1}(\tau) + [C_3 n Z_{\nu-1}(\tau)]C_4, \quad Z_\nu(0) = 0. \tag{17}$$

Then the uniform convergence of the series $\sum Z_\nu(\tau)$, for $\tau \in [0,T]$, follows directly from the following lemma. Therefore, the sequence $u^{(\nu)}$ is uniformly convergent on $\mathcal{D}$.

Lemma 3. *Let $\{Z_\nu(\tau)\}_{\nu\geq 0}$ be a sequence of continuous, non-negative functions satisfying $Z_0(\tau) \leq \bar{Z}e^{\alpha\tau}$. Suppose that $\alpha > 0$ and $\beta, \bar{Z} \geq 0$. If $Z_\nu(\tau)$, $\nu \geq 1$, satisfy the following recurrent inequality*

$$Z_\nu(\tau) \leq \int_0^\tau [\alpha Z_\nu(\eta) + \beta Z_{\nu-1}(\eta)]d\eta \qquad \text{for } \nu \geq 1, \tag{18}$$

then, for $\nu \geq 1$,

$$Z_\nu(\tau) \leq \frac{(\beta\tau)^\nu e^{\alpha\tau}}{\nu!}\bar{Z} \quad \text{and} \quad \sum_{\nu=0}^{\infty} Z_\nu(\tau) = e^{(\alpha+\beta)\tau}\bar{Y}. \tag{19}$$

Proof. Let

$$W_\nu(\tau) := \int_0^\tau [\alpha Z_\nu(\eta) + \beta Z_{\nu-1}(\eta)]d\eta.$$

From (18), we deduce

$$W'_\nu(\tau) = \alpha Z_\nu(\tau) + \beta Z_{\nu-1}(\tau) \leq \alpha W_\nu(\tau) + \beta Z_{\nu-1}(\tau).$$

Multiplying $e^{-\alpha\tau}$ and integrating the resultant inequality on $[0,\tau]$, we have

$$W_\nu(\tau) \leq \beta \int_0^\tau e^{\alpha(\tau-\eta)} Z_{\nu-1}(\eta)d\eta,$$

which together with (18) gives that

$$Z_\nu(\tau) \leq \beta \int_0^\tau e^{\alpha(\tau-\eta)} Z_{\nu-1}(\eta)d\eta.$$

Then one can easily obtain (19) by induction on ν. □

Step 4. Here we observe [1]. The first term on the right hand side of [1, (3.126)] is $v_x^{\nu-1} = u_{xx}^{\nu-1}$, not $v^{\nu-1}$. Since we consider classical solutions, $u_{xx}^{\nu-1}$ does not make sense. Therefore, we need to modify the argument. To do this, we prove that the subsequence of $u_x^{(\nu)}$ converges uniformly by the Arzela-Ascoli theorem. This step is the essential part of this paper.

We first prove that $u_x^{(\nu)}$ is equicontinous on $\mathcal{D}$. Since $u_x^{(\nu)}$ and $u_t^{(\nu)}$ are uniformly bounded on $\mathcal{D}$ by (10), there exists a $C_5 > 0$ independent of ν such that

$$|u^{(\nu)}(x,t) - u^{(\nu)}(x',t')| < C_5\delta_1 \quad \text{if } |(x,t)-(x',t')| < \delta_1 \text{ and } (x,t),(x',t') \in \mathcal{D} \tag{20}$$

for all $\delta_1 > 0$.

For any fixed $\varepsilon > 0$, we choose δ_2 such that $0 < \delta_2 \leq \delta_1/C_6 \leq \varepsilon/(4nC_7)$, where C_6, C_7 are constants determined later. Moreover, let (x,t) and (x',t') be any two fixed points in $\mathcal{D}$ such that $|(x,t)-(x',t')| < \delta_2$ and $t \geq t'$.

By (14), we have

$$\frac{d}{d\tau}x_i^{(\nu-1)}(t-\tau;x,t) = -\lambda_i^{\nu-1}\left(x_i^{(\nu-1)}(t-\tau;x,t), t-\tau\right) =: -\lambda_i, \tag{21}$$

and

$$\frac{d}{d\tau}x_i^{(\nu-1)}\left(t'-\frac{t'}{t}\tau;x',t'\right)=-\frac{t'}{t}\lambda_i^{\nu-1}\left(x_i^{(\nu-1)}\left((t'-\frac{t'}{t}\tau;x',t'\right),t'-\frac{t'}{t}\tau\right)=:-\frac{t'}{t}\lambda_i'. \quad (22)$$

For any fixed ν, we use $x_i(\tau)$ and $x_i'(\tau)$ to denote $x_i^{(\nu-1)}(\tau;x,t)$ and $x_i^{(\nu-1)}(\tau'(\tau);x',t')$ for simplicity, where $\tau'(\tau):=\frac{t'}{t}\tau$. Substracting (22) from (21), we get that

$$\frac{d}{d\tau}(x_i-x_i')(t-\tau)=(\lambda_i'-\lambda_i)+\frac{t'-t}{t}\lambda_i'. \quad (23)$$

The Lipschitz continuity of λ_i and (20) yields that

$$|\lambda_i'-\lambda_i|\le C_8\left(|(x_i-x_i')(t-\tau)|+|t-t'|\right), \quad (24)$$

for some constant C_8. Applying the differential inequality on (23)-(24), we have that there exists a constant chosen as C_6 such that

$$|(x_i(\tau),\tau)-(x_i'(\tau),\tau'(\tau))|\le C_6\delta_2\le\delta_1 \quad \text{for } \tau\in[0,t]. \quad (25)$$

And (25) holds for any two points (x,t), $(x',t')\in\mathcal{D}$ such that $|(x,t)-(x',t')|<\delta_2$ if T is a sufficiently small constant independent of ν and δ_2.

We now prove by induction that, if there exists a $\delta_3>0$ such that

$$|\bar u'(x)-\bar u'(x')|\le\frac{\varepsilon}{2} \quad \text{for } |x-x'|<\delta_3 \text{ and } x,x'\in[a,b], \quad (26)$$

then, for every $\nu\ge 0$, we have

$$|u_x^{(\nu)}(x,t)-u_x^{(\nu)}(x',t')|<\varepsilon \quad \text{if } |(x,t)-(x',t')|<\delta_2\le\frac{\delta_3}{C_6} \text{ and } (x,t),(x',t')\in\mathcal{D} \quad (27)_\nu$$

provided that T is small enough. (26) follows directly from the assumption (A), because $\bar u'(x)$ is uniformly continuous in $\mathcal{D}$. In addition, it is easy to see that if $(27)_\nu$ holds, then we also have

$$|u_x^{(\nu)}(x,t)-u_x^{(\nu)}(x',t')|<([C_7]+1)\varepsilon \quad \text{if } |(x,t)-(x',t')|<C_7\delta_2 \text{ and } (x,t),(x',t')\in\mathcal{D}, \quad (28)_\nu$$

where $[\cdot]$ denotes the greatest integer function. $(27)_\nu$ for the case $\nu=0$ is obvious. We assume that $(27)_{\nu-1}$ holds. To show that $(27)_\nu$ is also true, we write $v^\nu(x,t)=u_x^{(\nu)}(x,t)$ for simplicity. From $(4)_\nu$, we have

$$v_{i,t}^\nu+\lambda_i^{\nu-1}v_{i,x}^\nu=h_{i,x}^{\nu-1}+h_{i,u}^{\nu-1}\cdot v^{\nu-1}-\lambda_{i,x}^{\nu-1}v_i^\nu-\left(\lambda_{i,u}^{\nu-1}\cdot v^{\nu-1}\right)v_i^\nu, \quad (29)$$

where $\lambda_i^{\nu-1}$ and $h_i^{\nu-1}$ are defined in (15). We obtain from (29) that

$$\frac{d}{d\tau}v_i^\nu(x_i(\tau),\tau)=\left\{h_{i,x}^{\nu-1}+h_{i,u}^{\nu-1}\cdot v^{\nu-1}-\lambda_{i,x}^{\nu-1}v_i^\nu-\left(\lambda_{i,u}^{\nu-1}\cdot v^{\nu-1}\right)v_i^\nu\right\}(x_i(\tau),\tau). \quad (30)$$

Since $\tau'(\tau) = \frac{t'}{t}\tau$ and

$$(x_i'(\tau), \tau'(\tau)) = \left(x_i\left(\frac{t'}{t}\tau\right), \frac{t'}{t}\tau\right) = (x_i(\tau'), \tau'),$$

we also have

$$\begin{aligned}\frac{d}{d\tau}v_i^\nu(x_i'(\tau), \tau'(\tau)) &= \frac{t'}{t}\frac{d}{d\tau'}v_i^\nu(x_i(\tau'), \tau') \\ &= \frac{t'}{t}\left\{h_{i,x}^{\nu-1} + h_{i,u}^{\nu-1}\cdot v^{\nu-1} - \lambda_{i,x}^{\nu-1}v_i^\nu - \left(\lambda_{i,u}^{\nu-1}\cdot v^{\nu-1}\right)v_i^\nu\right\}(x_i'(\tau), \tau'(\tau)).\end{aligned} \tag{31}$$

Let

$$w^\nu(\tau) := v^\nu(x_i(\tau), \tau) - v^\nu(x_i'(\tau), \tau'(\tau)). \tag{32}$$

By (30)-(32), we get that

$$\begin{aligned}\frac{d}{d\tau}w_i^\nu(\tau) = &\Big\{\left[h_{i,x}^{\nu-1} + h_{i,u}^{\nu-1}\cdot v^{\nu-1} - \lambda_{i,x}^{\nu-1}v_i^\nu - \left(\lambda_{i,u}^{\nu-1}\cdot v^{\nu-1}\right)v_i^\nu\right](x_i(\tau), \tau) \\ &- \left[h_{i,x}^{\nu-1} + h_{i,u}^{\nu-1}\cdot v^{\nu-1} - \lambda_{i,x}^{\nu-1}v_i^\nu - \left(\lambda_{i,u}^{\nu-1}\cdot v^{\nu-1}\right)v_i^\nu\right](x_i'(\tau), \tau'(\tau))\Big\} \\ &+ \frac{t-t'}{t}\left\{h_{i,x}^{\nu-1} + h_{i,u}^{\nu-1}\cdot v^{\nu-1} - \lambda_{i,x}^{\nu-1}v_i^\nu - \left(\lambda_{i,u}^{\nu-1}\cdot v^{\nu-1}\right)v_i^\nu\right\}(x_i'(\tau), \tau'(\tau)) \\ =:&\ I_1 + I_2.\end{aligned} \tag{33}$$

The C^1 continuities of h, λ, and (6) give that there exists a constant chosen as C_7 such that

$$|I_2| \leq \frac{C_7|t-t'|}{t} \leq \frac{C_7\delta_2}{t} \leq \frac{\varepsilon}{4nt}. \tag{34}$$

On the other hand, we write I_1 as

$$I_1 = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8, \tag{35}$$

where

$$\begin{aligned}J_1 &:= \left\{h_{i,x}^{\nu-1}(x_i(\tau), \tau) - h_{i,x}^{\nu-1}(x_i'(\tau), \tau'(\tau))\right\}, \\ J_2 &:= \left\{h_{i,u}^{\nu-1}(x_i(\tau), \tau) - h_{i,u}^{\nu-1}(x_i'(\tau), \tau'(\tau))\right\}\cdot v^{\nu-1}(x_i(\tau), \tau), \\ J_3 &:= h_{i,u}^{\nu-1}(x_i'(\tau), \tau'(\tau))\cdot w^{\nu-1}(\tau), \\ J_4 &:= -\left\{\lambda_{i,x}^{\nu-1}(x_i(\tau), \tau) - \lambda_{i,x}^{\nu-1}(x_i'(\tau), \tau'(\tau))\right\}v_i^\nu(x_i(\tau), \tau), \\ J_5 &:= -\lambda_{i,x}^{\nu-1}(x_i'(\tau), \tau'(\tau))w_i^\nu(\tau), \\ J_6 &:= -\left\{\left(\lambda_{i,u}^{\nu-1}(x_i(\tau), \tau) - \lambda_{i,u}^{\nu-1}(x_i'(\tau), \tau'(\tau))\right)\cdot v^{\nu-1}(x_i(\tau), \tau)\right\}v_i^\nu(x_i(\tau), \tau), \\ J_7 &:= -\left\{\lambda_{i,u}^{\nu-1}(x_i'(\tau), \tau'(\tau))\cdot w^{\nu-1}(\tau)\right\}v_i^\nu(x_i(\tau), \tau), \\ J_8 &:= -\left\{\lambda_{i,u}^{\nu-1}(x_i'(\tau), \tau'(\tau))\cdot v^{\nu-1}(x_i'(\tau), \tau'(\tau))\right\}w_i^\nu(\tau).\end{aligned}$$

Applying (6), (20), (25), $(28)_{\nu-1}$, and the assumption (A), we obtain that

$$|J_1|, |J_2|, |J_4|, |J_6| \leq C_9(\delta_2 + \delta_1) \leq C_{10}\varepsilon, \qquad |J_3|, |J_7| \leq C_{10}\varepsilon. \tag{36}$$

for some constants C_9, C_{10}. The uniform boundedness of $u^{(\nu)}$ and $u_x^{(\nu)}$ gives that

$$|J_5|, |J_8| \le C_{11}|w_i^\nu|, \tag{37}$$

for some constant C_{11}. Using the differential inequality on (33)-(37) together with (26), we have

$$|u_{i,x}^{(\nu)}(x,t) - u_{i,x}^{(\nu)}(x',t')| = |w_i^{(\nu)}(t)| \le \frac{\varepsilon}{n} \quad \text{for } t \in [0,T]$$

if T is small enough, which is valid for $i = 1, \ldots, n$. Therefore, $(27)_\nu$ is also true. By induction, we get the equicontinuity of $u_x^{(\nu)}$.

Step 5. We are in a position to prove the local existence of classical solution for problem (1), (2). Since $u^{(\nu)}$ is uniformly convergent on $\mathcal{D}$, we let

$$u(x,t) := \lim_{\nu\to\infty} u^{(\nu)}(x,t) \quad \text{for } (x,t) \in \mathcal{D}. \tag{38}$$

On the other hand, since $u_x^{(\nu)}$ is uniformly bounded and equicontinuous on $\mathcal{D}$, the Arzela-Ascoli theorem says that there exists a subsequence $\{\nu_k\}$ of $\mathbb{N}$ such that $u_x^{(\nu_k)}$ is uniformly convergent on $\mathcal{D}$. Thus, we get that

$$u_x(x,t) = \lim_{k\to\infty} u_x^{(\nu_k)}(x,t) \tag{39}$$

exists and is continuous on $\mathcal{D}$. From (4) and the Lipschitz continuity of h_i, λ_i, we find that

$$\begin{aligned} \lim_{k\to\infty} u_t^{(\nu_k)}(x,t) &= h_i\left(x,t,\lim_{k\to\infty} u^{(\nu_k)}\right) - \lambda_i\left(x,t,\lim_{k\to\infty} u^{(\nu_k)}\right)\lim_{k\to\infty} u_x^{(\nu_k)}(x,t) \\ &= h_i(x,t,u) - \lambda_i(x,t,u)u_x. \end{aligned} \tag{40}$$

Since the convergence in (40) is uniform with respect to x and t, we have

$$\lim_{k\to\infty} u_t^{(\nu_k)}(x,t) = \left(\lim_{k\to\infty} u^{(\nu_k)}(x,t)\right)_t = u_t(x,t) \tag{41}$$

exists is continuous on $\mathcal{D}$. Combining (40) and (41), we prove that u is a classical solution for problem (1), (2) on $\mathcal{D}$. Its uniqueness follows directly from [2, Theorem 1]. □

Remark 4. *If $u_x^{(\nu)}$ converges uniformly, we can deduce from* (30)

$$v_i(x_i(t),t) - v_i(x_i(0),0) = \int_0^t \{h_{i,x} + h_{i,u}\cdot v - \lambda_{i,x}v_i - (\lambda_{i,u}\cdot v)\, v_i\}\,(x_i(\tau),\tau)d\tau. \tag{42}$$

However, since we prove the uniform convergence of only the subsequence, we cannot obtain (42).

We next consider the characteristic curve $x_i^{(\nu)}(t;\xi)$, which is the solution of

$$\frac{dX}{dt} = \lambda_i\left(X,t,u^{(\nu)}(X,t)\right), \quad X(0) = \xi.$$

Then, we cannot prove the uniform convergence of the derivative with respect to initial data, i.e.,

$$\lim_{\nu\to\infty} x_{i,\xi}^{(\nu)}(t;\xi) = x_{i,\xi}(t;\xi) \quad \textit{uniformly}. \tag{43}$$

We do not know whether (42) *and* (43) *hold or not.*

References

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