

# On the homotopy properties of $p$ -subgroup posets

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## 1 Introduction

For a partially ordered set (simply, a poset)  $\mathcal{P}$ , the symbol  $\Delta(\mathcal{P})$  is the *order complex* of  $\mathcal{P}$ ; this is the simplicial complex whose  $k$ -dimensional simplices are the non-empty chains  $x_0 < x_1 < \cdots < x_k$  of  $\mathcal{P}$ . Also, by considering the topological space  $|\Delta(\mathcal{P})|$ , which is the geometric realization of the order complex  $\Delta(\mathcal{P})$ , we can discuss homotopy on  $|\Delta(\mathcal{P})|$ . Here we focus on nontrivial  $p$ -subgroup posets with  $p$  as a prime factor of the order of a finite group  $G$ , ordered by set inclusion. In particular,  $S_p(G)$  and  $A_p(G)$  consisting of nontrivial  $p$ -subgroup families are fundamental:

$$S_p(G) := \{P \mid P \text{ is a nontrivial } p\text{-subgroup of } G\},$$

$$A_p(G) := \{P \mid P \text{ is a nontrivial elementary abelian } p\text{-subgroup of } G\}$$

$\Delta(S_p(G))$  and  $\Delta(A_p(G))$  are respectively called a *Brown complex* and a *Quillen complex* of  $G$  at  $p$ . Such an order complex formed from a  $p$ -subgroup poset is called a  *$p$ -subgroup complex*. The purpose of this note is to review the research on homotopy types of  $p$ -subgroup complexes and to introduce the latest results.

## 2 Notations and preliminaries

Throughout this note, let  $G$  be a finite group and  $p$  be a prime number dividing its order. Let  $\mathcal{P}, \mathcal{Q}$  be finite posets. An order preserving map  $f : \mathcal{P} \rightarrow \mathcal{Q}$  induces naturally a continuous map  $|\Delta(f)| : |\Delta(\mathcal{P})| \rightarrow |\Delta(\mathcal{Q})|$ . Also, if two order preserving maps  $f, g : \mathcal{P} \rightarrow \mathcal{Q}$  satisfy  $f(x) \leq g(x)$  for any  $x \in \mathcal{P}$ , then  $|\Delta(f)|$  is homotopic to  $|\Delta(g)|$ . Note that in many combinatorics papers,  $|\Delta(f)|$  is often simply written as  $f$ . If a finite poset  $\mathcal{P}$  has a maximal or minimal element,  $|\Delta(\mathcal{P})|$  is immediately contractible. More generally, the following holds.

**Proposition 2.1.** Let  $\mathcal{P}$  be a finite poset and let  $x_0$  be an element of  $\mathcal{P}$ . If there is an order preserving self-map  $f : \mathcal{P} \rightarrow \mathcal{P}$  such that  $x \leq f(x) \leq x_0$  for any  $x \in \mathcal{P}$ , then  $|\Delta(\mathcal{P})|$  is contractible.

The above contractible is especially said to be **conically contractible**. Next, we write the *intercept* of the poset  $\mathcal{P}$  with the following symbols:

$$\mathcal{P}_{\leq x} := \{y \in \mathcal{P} \mid y \leq x\}.$$

$\mathcal{P}_{< x}, \mathcal{P}_{\geq x}, \mathcal{P}_{> x}$ , etc. are defined similarly. Under this symbol, the following theorem, which is fundamental in the homotopy theory of order complexes, is stated.

**Proposition 2.2.** Let  $\mathcal{P}, \mathcal{Q}$  be finite posets. If there is an order preserving map  $f : \mathcal{P} \rightarrow \mathcal{Q}$  and  $|\Delta(f^{-1}(\mathcal{Q}_{\leq y}))|$  (or  $|\Delta(f^{-1}(\mathcal{Q}_{\geq y}))|$ ) is contractible for any  $y \in \mathcal{Q}$ , then  $|\Delta(f)| : |\Delta(\mathcal{P})| \rightarrow |\Delta(\mathcal{Q})|$  is a homotopy equivalence.

This theorem was first discovered by Quillen and is called “Fibre Lemma”. The proof is given using the “Acyclic Carrier Theorem”. As a direct application of this proposition, the following is well known.

**Proposition 2.3.** Let  $\mathcal{P}$  be a finite poset and let

$$\mathcal{P}^< := \{x \in \mathcal{P} \mid |\Delta(\mathcal{P}_{< x})| \text{ is not contractible}\}.$$

Let  $\mathcal{Q}$  be a subposet of  $\mathcal{P}$  such that  $\mathcal{P}^< \subseteq \mathcal{Q} \subseteq \mathcal{P}$ . Then  $\mathcal{Q}$  is homotopy equivalent to  $\mathcal{P}$ , where the inclusion map induces a homotopy equivalence map.

Now, let  $\mathcal{P} = \mathcal{S}_p(G)$ . By definition,

$$\mathcal{S}_p(G)^< = \{P \in \mathcal{S}_p(G) \mid |\Delta(\mathcal{S}_p(G)_{< P})| \text{ is not contractible}\}.$$

Then we have the following proposition.

**Proposition 2.4.**  $\mathcal{A}_p(G) = \mathcal{S}_p(G)^<$

*Proof.* Let  $P \in \mathcal{S}_p(G)$  is **not** an elementary abelian  $p$ -subgroup, then there exists a nontrivial Frattini subgroup  $\Phi(P)$ , and so the map  $f : \mathcal{S}_p(G)_{< P} \rightarrow \mathcal{S}_p(G)_{< P}; Q \mapsto Q\Phi(P)$  is well-defined. Furthermore,  $Q \subseteq Q\Phi(P) \supseteq \Phi(P)$ . By Proposition 2.1,  $|\Delta(\mathcal{S}_p(G)_{< P})|$  is contractible. Therefore, we obtain  $\mathcal{A}_p(G) \supseteq \mathcal{S}_p(G)^<$ . Conversely, let  $P \in \mathcal{A}_p(G)$  be an elementary abelian  $p$ -subgroup, and  $P$  seems to be a vector space over  $\mathbb{Z}_p$ . Note that  $\mathcal{S}_p(G)_{< P} = \mathcal{S}_p(P)_{< P}$ , and this set is a bundle of families of proper subspaces of the vector space  $P$ . Hence, from the Solomon-Tits theorem,  $|\Delta(\mathcal{S}_p(G)_{< P})|$  is homotopy equivalent to some spherical bouquet or empty set. In any case, it is non-contractible. Thus,  $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)^<$ .  $\square$

**Corollary 2.5.**  $|\Delta(\mathcal{A}_p(G))|$  is homotopy equivalent to  $|\Delta(\mathcal{S}_p(G))|$ .

Needless to say, the Quillen complex  $\Delta(\mathcal{A}_p(G))$  is characterized by the homotopy term ‘contractivity’ in the Brown complex  $\Delta(\mathcal{S}_p(G))$ . The following question naturally arises.

**Problem 2.6.** Is there any other  $p$ -subgroup complex like this?

There are some known  $p$ -subgroup complexes on this problem. For example, there is the following  $p$ -subgroup poset by Bouc:

$$\mathcal{B}_p(G) := \{P \in \mathcal{S}_p(G) \mid P = O_p(N_G(P))\},$$

where the symbol  $O_p(N_G(P))$  denotes the maximal normal  $p$ -subgroup of the normalizer  $N_G(P)$  of  $P$  in  $G$ . As can be seen from its definition, it contains the set  $\text{Syl}_p(G)$  of all Sylow  $p$ -subgroups of a finite group  $G$ ,  $\mathcal{S}_p(G)$ . The order complex  $\Delta(\mathcal{B}_p(G))$  is called a  *$p$ -radical complex*, or *Bouc complex*. Now, let  $C_G(P)$  be the centralizer of  $P$  in  $G$ , then the following  $p$ -subgroup poset is given whose order complex is called a  *$p$ -centric complex*.

$$\mathcal{C}_p(G) := \{P \in \mathcal{S}_p(G) \mid \text{any } p\text{-element of } C_G(P) \text{ belongs to } P\}$$

We write the intersection of  $\mathcal{B}_p(G)$  and  $\mathcal{C}_p(G)$  by the symbol  $\mathcal{B}_p^{\text{cen}}(G)$ . That is,

$$\mathcal{B}_p^{\text{cen}}(G) := \mathcal{B}_p(G) \cap \mathcal{C}_p(G).$$

The geometric realization  $|\Delta(\mathcal{B}_p^{\text{cen}}(G))|$  of the order complex  $\Delta(\mathcal{B}_p^{\text{cen}}(G))$  is homotopy equivalent to the geometric realization  $|\Delta(\mathcal{C}_p(G))|$ , but it is not homotopy equivalent to  $|\Delta(\mathcal{B}_p(G))|$  in general ([11]).

### 3 Finite $T_0$ -spaces

A *finite space* is simply a finite set with a topology, and any finite space is homotopy equivalent to a finite  $T_0$ -space (namely, for any two distinct points, there exists a finite space which is an open neighborhood of one of the finite space that has one open neighborhood that does not contain the other). Hence, homotopy-theoretically, finite space theory is sufficient to target a finite  $T_0$ -space. Furthermore, since a finite  $T_0$ -space is in correspondence with a finite poset, both  $\mathcal{S}_p(G)$  and  $\mathcal{A}_p(G)$  are finite  $T_0$  spaces. A finite space  $X$  is weakly homotopy equivalent to the corresponding compact polyhedron  $|\Delta(X)|$ . Here, *weak homotopy*

*equivalence* means that the homotopy groups of every dimension are isomorphic. Let  $\alpha$  be a point in  $|\Delta(X)|$ , that is,  $\alpha = t_1x_1+t_2x_2+\cdots+t_rx_r$  where  $\sum_{i=1}^r t_i = 1, t_i > 0$  for every  $1 \leq i \leq r$  and  $x_1 < x_2 < \cdots < x_r$  is a chain of  $X$ . Then  $\mu_X : |\Delta(X)| \rightarrow X$  by  $\mu_X(\alpha) = x_1$  is a weak homotopy equivalence. We obtain the following commutative diagram:

$$\begin{array}{ccc} |\Delta(X)| & \xrightarrow{|\Delta(f)|} & |\Delta(Y)| \\ \mu_X \downarrow & & \downarrow \mu_Y \\ X & \xrightarrow{f} & Y, \end{array}$$

where  $f$  is a continuous map between finite  $T_0$  spaces  $X$  and  $Y$ , and  $|\Delta(f)|$  is also a continuous map between polyhedra  $|\Delta(X)|$  and  $|\Delta(Y)|$ . These were first discovered by McCord [7]. In the above diagram,  $f$  is a weak homotopy equivalence if and only if  $|\Delta(f)|$  is a homotopy equivalence. Let us consider an application to  $p$ -subgroup complexes. Let  $X = \mathcal{A}_p(G), Y = \mathcal{S}_p(G)$  and  $f = \iota$  (= inclusion map). Then  $\iota$  is a weak homotopy equivalence. Stong found that  $|\Delta(\iota)| : |\Delta(\mathcal{A}_p(G))| \rightarrow |\Delta(\mathcal{S}_p(G))|$  is a homotopy equivalence. However the homotopy type of the Quillen complex or the Brown complex is still unknown. His result is the following ([13]).

**Proposition 3.1.** A finite  $T_0$ -space  $\mathcal{S}_p(G)$  is contractible if and only if  $O_p(G)$  is non-trivial, where  $O_p(G)$  denotes the maximal normal  $p$ -subgroup of  $G$ .

Therefore, the Quillen conjecture can be rephrased as follows: If  $|\Delta(\mathcal{S}_p(G))|$  is contractible, then the finite  $T_0$ -space  $\mathcal{S}_p(G)$  is contractible. From a topological point of view, this problem deals with a difference between weak homotopy equivalence and homotopy equivalence. We consider a *finite  $T_0$ - $G$ -space* which is a finite  $T_0$ -space with a  $G$ -action. In particular, for a  $p$ -subgroup poset, we give the  $G$ -action by conjugation.

## 4 Main results

We shall recall the Bouc poset  $\mathcal{B}_p(G) = \{P \in \mathcal{S}_p(G) \mid P = O_p(N_G(P))\}$ . Notice that  $\mathcal{B}_p(G)$  contains  $\text{Syl}_p(G)$ . Then we obtain the following.

**Theorem 4.1.** If  $G$  is a nilpotent group, then  $|\Delta(\mathcal{B}_p(G))|$  is contractible.

*Proof.* By assumption,  $\text{Syl}_p(G)$  is the set consisting of a unique element. By Sylow's theorem, the Bouc poset  $\mathcal{B}_p(G)$  has the maximal element. The finite  $T_0$ -space  $\mathcal{B}_p(G)$  is contractible, and so  $|\Delta(\mathcal{B}_p(G))|$  is contractible.  $\square$

**Corollary 4.2.** Let  $p$  and  $q$  be prime numbers such that  $p > q$ . Let  $G$  be a finite group whose order is  $pq$ . Then  $|\Delta(\mathcal{B}_p(G))|$  is contractible.

*Proof.* Similarly the proof of the above theorem,  $\text{Syl}_p(G)$  is the set consisting of a unique element.  $\square$

**Theorem 4.3.** A finite  $T_0$ -space  $\mathcal{B}_p(G)$  is contractible if and only if  $O_p(G)$  is non-trivial.

*Proof.* If  $O_p(G)$  is non-trivial, the Bouc poset  $\mathcal{B}_p(G)$  has the minimal element. Thus a finite  $T_0$ -space  $\mathcal{B}_p(G)$  is contractible. Conversely, if a finite  $T_0$ -space  $\mathcal{B}_p(G)$  is contractible,  $\mathcal{B}_p(G)$  has the  $G$ -invariant core which consists of a unique element. Its element is the normal subgroup of  $G$ .  $\square$

Let  $X$  be a finite  $T_0$ - $G$ -space and let  $x$  and  $y$  be points of  $X$ . If  $x \in U_y$ , then  $gx \in gU_y = U_{gy}$ . Therefore a  $G$ -action on a finite  $T_0$ -space  $X$  preserves the order. Thus  $\Delta(X)$  is a  $G$ -simplicial complex (in short,  $G$ -complex). Let  $\mathbb{N}_0$  be the union set of natural numbers  $\{1, 2, 3, \dots\}$  and  $\{0\}$ . A  $CW$ -complex  $Z$  with a  $G$ -action is called a  $G$ - $CW$ -complex if it satisfies the following conditions:

- (i) The  $G$ -action determines a cellular map, that is, for any  $g \in G$ ,  $gZ^i \subset Z^i$  for each  $i \in \mathbb{N}_0$ , where  $Z^i$  denotes the union of cells of dimension  $\leq i$  and is called the  $i$ -skeleton of  $Z$ .
- (ii) Let  $e$  is a cell of  $Z$ . If  $g(e) = e$ , then  $g$  is trivial on  $\bar{e}$ , that is,  $Z^g \supset \bar{e}$ , where  $\bar{e}$  is the closure of  $e$ .

We obtained the following theorem in [3].

**Theorem 4.4.** Let  $X$  be a finite  $T_0$ - $G$ -space. Then  $|\Delta(X)|$  is a finite  $G$ - $CW$ -complex.

We remark that  $p$ -sugroup posets  $\mathcal{S}_p(G)$ ,  $\mathcal{A}_p(G)$  and  $\mathcal{B}_p(G)$  are  $G$ -posets by conjugation. Therefore their order complexes are  $G$ -complexes.  $|\Delta(\mathcal{S}_p(G))|$ ,  $|\Delta(\mathcal{A}_p(G))|$  and  $|\Delta(\mathcal{B}_p(G))|$  are all  $G$ - $CW$ -complexes.

Next we shall state the result about orbit space. We also got the following theorem in [3].

**Theorem 4.5.** Let  $X$  be a finite  $T_0$ - $G$ -space. Then  $|\Delta(X)|/G$  is homotopy equivalent to  $|\Delta(X/G)|$ .

Since the orbit map  $p : X \rightarrow X/G$  is continuous, it is an order-preserving map. It determines a simplicial map  $\Delta(p) : \Delta(X) \rightarrow \Delta(X/G)$  and also a continuous map  $|\Delta(p)| : |\Delta(X)| \rightarrow |\Delta(X/G)|$ . Noting that  $|\Delta(X/G)|$  is a  $G$ -space with a trivial  $G$ -action, we have a continuous map  $\tilde{p} : |\Delta(X)|/G \rightarrow |\Delta(X/G)|$  such that the following diagram commutes:

$$\begin{array}{ccc} |\Delta(X)| & & \\ \downarrow q & \searrow |\Delta(p)| & \\ |\Delta(X)|/G & \xrightarrow{\tilde{p}} & |\Delta(X/G)|, \end{array}$$

where  $q$  is the orbit map from  $|\Delta(X)| \rightarrow |\Delta(X)|/G$ . By McCord's result, there is a weak homotopy equivalence  $\mu_X : |\Delta(X)| \rightarrow X$ . Then  $\mu_X$  determines a continuous map  $\tilde{\mu}_X : |\Delta(X)|/G \rightarrow X/G$  such that the following diagram commutes:

$$\begin{array}{ccc} |\Delta(X)|/G & \xrightarrow{\tilde{p}} & |\Delta(X/G)| \\ & \searrow \tilde{\mu}_X & \downarrow \mu_{X/G} \\ & & X/G. \end{array}$$

Therefore  $\tilde{p}$  is a weak homotopy equivalence if and only if  $\tilde{\mu}_X$  is so. In general,  $\tilde{\mu}_X$  is not a weak homotopy equivalence. According to Symond's result [14],  $|\Delta(\mathcal{S}_p(G))|/G$  is contractible.

Let us have a terminology: a topological space is *homotopically contractible* if and only if all its homotopy groups are trivial.

**Theorem 4.6.** If  $G$  is a nilpotent group. If  $|\Delta(\mathcal{B}_p(G))|/G$  is homotopically contractible, it is contractible. Then  $|\Delta(\mathcal{B}_p(G)/G)|$  is also contractible.

*Proof.* Let  $\{*\}$  be a finite space consisting of a single point. Since  $\mathcal{B}_p(G)$  is contractible,  $\mathcal{B}_p(G)/G$  is also so. Hence we have a diagram:

$$\begin{array}{ccccc} |\Delta(\mathcal{B}_p(G))|/G & \xrightarrow{\tilde{p}} & |\Delta(\mathcal{B}_p(G)/G)| & \xrightarrow{|\Delta(f)|} & |\Delta(\{*\})| \\ & \searrow \tilde{\mu}_{\mathcal{B}_p(G)} & \downarrow \mu_{\mathcal{B}_p(G)/G} & & \downarrow \mu_{\{*\}} \\ & & \mathcal{B}_p(G)/G & \xrightarrow{f} & \{*\}, \end{array}$$

where  $f$  is a homotopy equivalence. By McCord's theorem, there exists a homotopy equivalence  $|\Delta(f)| : |\Delta(\mathcal{B}_p(G)/G)| \rightarrow |\Delta(\{*\})|$ . Therefore  $|\Delta(\mathcal{B}_p(G)/G)|$  is contractible. By assumption,  $\tilde{\mu}_{\mathcal{B}_p(G)}$  is a weak homotopy equivalence, and so  $\tilde{p}$  is a weak homotopy equivalence. Now, since both  $|\Delta(\mathcal{B}_p(G))/G|$  and  $|\Delta(\mathcal{B}_p(G)/G)|$  are  $CW$ -complexes,  $\tilde{p}$  is homotopy equivalence. Thus  $|\Delta(\mathcal{B}_p(G))/G|$  is contractible. Then it follows immediately from theorem 4.5,  $|\Delta(\mathcal{B}_p(G))/G|$  is also contractible.  $\square$

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