Model Completeness of Theories of Hrushovski's Pseudoplanes Associated to Irrational Numbers

Hirotaka Kikyo
Graduate School of System Informatics
Kobe University

Abstract

Let M be the generic structure of an amalgamation class defined using Hrushovski's unbounded log-like function associated to an irrational number. Then the theory of M is model complete.

1 Introduction

We essentially use notation and terminology from Kikyo [12], Baldwin-Shi [3] and Wagner [16]. We also use some terminology from graph theory [4].

Suppose A is a graph. $V(A)$ denotes the set of vertices of A, and $E(A)$ the set of edges of A. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. We usually follow this convention. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [4].

We say that X is *connected* in A if X is a connected graph in the graph theoretical sense [4]. A maximal connected substructure of A is a *connected component* of A.

Let A, B, C be graphs such that $A \subseteq C$ and $B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$. We also write $X - Y$ in general for the relative compliment of Y in X also known as the set difference of X and Y. If $A \cap B = \emptyset$, $E(A,B)$ denotes the set of edges xy such that $x \in A$ and $y \in B$. We put $e(A,B) = |E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working.

Let D be a graph and A, B, and C substructures of D. We write $D = B \otimes_A C$ if $D = BC$, $B \cap C = A$, and $E(D) = E(B) \cup E(C)$. $E(D) = E(B) \cup E(C)$ means that there are no edges between $B - A$ and $C - A$. D is called a *free amalgam of B and C over A*. If A is empty, we write $D = B \otimes C$, and D is also called a *free amalgam of B and C*.

Definition 1.1 Let α be a real number such that $0 < \alpha < 1$.

(1) For a finite graph A, we define a predimension function δ_α by $\delta_\alpha(A) = |A| - e(A)\alpha$.

(2) Let A and B be substructures of a common graph. Put $\delta_\alpha(A/B) = \delta_\alpha(AB) - \delta_\alpha(B)$.

Definition 1.2 Let A and B be graphs with $A \subseteq B$, and suppose A is finite.

$A <_\alpha B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta_\alpha(A) < \delta_\alpha(X)$.

We say that A is *closed* in B if $A <_\alpha B$. We also say that B is a *strong extension* of A.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs A such that $\emptyset <_\alpha A$.

Some facts about $<_\alpha$ appear in [3, 16, 17]. Some proofs are given in [12].

Fact 1.3 *Let A and B be disjoint substructures of a common graph. Then $\delta_\alpha(A/B) = \delta_\alpha(A) - e(A,B)\alpha$.*

Fact 1.4 *If $A <_\alpha B \subseteq D$ and $C \subseteq D$ then $A \cap C <_\alpha B \cap C$.*

Fact 1.5 *Let $D = B \otimes_A C$.*

(1) $\delta_\alpha(D/A) = \delta_\alpha(B/A) + \delta_\alpha(C/A)$.

(2) If $A <_\alpha C$ then $B <_\alpha D$.

(3) If $A <_\alpha B$ and $A <_\alpha C$ then $A <_\alpha D$.

Let B, C be graphs and $g : B \to C$ a graph embedding. g is a *closed embedding* of B into C if $g(B) <_\alpha C$. Let A be a graph with $A \subseteq B$ and $A \subseteq C$. g is a *closed embedding over* A if g is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

Definition 1.6 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <_\alpha)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph M is a *generic structure* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B <_\alpha M$.

(2) If $A \subseteq M$ then $A \in \mathbf{K}$.

(3) For any $A, B \in \mathbf{K}$, if $A <_\alpha M$ and $A <_\alpha B$ then there is a closed embedding of B into M over A.

Let A be a finite structure of M. There is a smallest B satisfying $A \subseteq B <_\alpha M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called the *closure* of A in M.

Fact 1.7 ([3, 16, 17]) *Let $(\mathbf{K}, <_\alpha)$ be an amalgamation class. Then there is a generic structure of $(\mathbf{K}, <_\alpha)$. Let M be a generic structure of $(\mathbf{K}, <_\alpha)$. Then any isomorphism between finite closed substructures of M can be extended to an automorphism of M.*

Definition 1.8 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <_\alpha)$ has the *free amalgamation property* if whenever $D = B \otimes_A C$ with $B, C \in \mathbf{K}$, $A <_\alpha B$ and $A <_\alpha C$ then $D \in \mathbf{K}$.

By Fact 1.5 (2), we have the following.

Fact 1.9 *Let* $\mathbf{K}$ *be a subclass of* $\mathbf{K}_\alpha$. *If* $(\mathbf{K}, <_\alpha)$ *has the free amalgamation property then it has the amalgamation property.*

Definition 1.10 Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \leq 1$. We assume that f is piecewise smooth. $f'_+(x)$ denotes the right-hand derivative at x. We have $f(x+h) \leq f(x) + f'_+(x)h$ for $h > 0$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta_\alpha(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f, <_\alpha)$ has a countably categorical theory [17].

A graph X is *normal to* f if $\delta(X) \geq f(|X|)$. A graph A belongs to $\mathbf{K}_f$ if and only if U is normal to f for any substructure U of A.

2 Hrushovski's Log-like Functions

Definition 2.1 ([7]) Let α be a positive real number with $1 > \alpha > 0$. We define x_n, e_n, k_n, d_n for integers $n \geq 1$ by induction as follows: Put $x_1 = 2$ and $e_1 = 1$. Assume that x_n and e_n are defined. Let r_n be a smallest rational number r such that $r = k/d > \alpha$ with $d \leq e_n$ where k and d are positive integers. Let k_n/d_n be a reduced fraction with $k_n/d_n = r_n$. Finally, let $x_{n+1} = x_n + k_n$, and $e_{n+1} = e_n + d_n$.

Let $a_0 = (0,0)$, and $a_n = (x_n, x_n - e_n\alpha)$ for $n \geq 1$. Let f_α be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ whose graph on interval $[x_n, x_{n+1}]$ with $n \geq 0$ is a line segment connecting a_n and a_{n+1}. We call f_α a *Hrushovski's log-like function* associated to α.

Fact 2.2 ([7]) *Let* f_α *be a Hrushovski's log-like function and* $\{x_i\}$, $\{e_i\}$, $\{k_i\}$, $\{d_i\}$ *sequences in the definition of* f_α.

Suppose C *is an extension of* B *by* x *points and* z *edges,* $|B| \geq x_n$ *and* $x/z \geq k_n/d_n$ *for some* n*, and* B *is normal to* f_α*. Then* C *is normal to* f_α*.*

Fact 2.3 ([7]) *Let $D = B \otimes_A C$. If $\delta_\alpha(A) < \delta_\alpha(B)$, $\delta_\alpha(A) < \delta_\alpha(C)$, and A, B, C are normal to f_α then D is normal to f_α.*

Fact 2.4 ([7]) *Let α be a real number with $1 > \alpha > 0$. Then f_α is strictly increasing and concave, and $(\mathbf{K}_{f_\alpha}, <_\alpha)$ has the free amalgamation property. Therefore, there is a generic structure of $(\mathbf{K}_{f_\alpha}, <_\alpha)$. Any one point structure is closed in any structure in $\mathbf{K}_{f_\alpha}$. If α is rational then f_α is unbounded.*

The following is easy.

Lemma 2.5 *Let $f = f_\alpha$ be the Hrushovski's log-like function with $1 > \alpha > 0$.*

(1) Suppose $A \in \mathbf{K}_f$. Then any point in A is closed in A. Note that any point in closed in a tree with respect to $<_\alpha$.

(2) Let $C = A \otimes_p B$ with p a point. If $A, B \in \mathbf{K}_f$ then $C \in \mathbf{K}_f$.

(3) Any finite forests belong to $\mathbf{K}_f$.

Lemma 2.6 *Suppose $2/3 > \alpha > 1/2$.*

(1) The first few terms of the sequences defining f_α are given by the following chart:

x_i	2	3	4	6
e_i	1	2	3	6
k_i	1	1	2	k_4
d_i	1	1	3	d_4

(2) Let A be a graph which is normal to f_α and a, b vertices of A with distance at least 3 in A. Let B be an extension of A by a path from a to b of length at least 3. Then B is normal to f_α.

Proof. (1) is straightforward and (2) is by Fact 2.2 and (1). □

3 Minimal Intrinsic Extensions

Definition 3.1 Suppose A is a substructure of B. B is called a *minimal intrinsic extension* of A with respect to a real number α if $\delta_\alpha(B/A) \leq 0$ but whenever X is a proper substructure of B with $A \subseteq X$ then $A <_\alpha X$. B is called a *biminimal intrinsic extension* of A with respect to a real number α if $\delta_\alpha(B/A) \leq 0$ but whenever X is a proper substructure of B then $A \cap X <_\alpha X$.

Definition 3.2 Let α be an irrational number. We call a fraction of integers p/q a *good approximation of* α *from below* if $\beta > p/q$ and whenever $\beta > p'/q'$ with $q' \leq q$ then $p/q \geq p'/q'$.

Fact 3.3 ([12, 14]) *Suppose* $1 > p/q > 0$ *where* p *and* q *are coprime positive integers. Then there is a tree* W *with the following properties: Let* L *be the set of all nodes of* W *and* F *the set of all leaves of* W.

(1) L *is a path in* W *with* p *vertices and* $p-1$ *edges.*

(2) $|F| = q - p + 1$. *Every leaf is adjacent to some vertex in* L.

(3) $\delta_{p/q}(W/F) = 0$.

(4) $B \cap F <_{p/q} B$ *for any proper substructure* B *of* W.

Note that W *is a biminimal intrinsic extension of* F. *We call* W *a* twig *for* p/q.

Lemma 3.4 *Let* α *be an irrational number with* $2/3 > \alpha > 1/2$. *Put* $\beta = 2\alpha - 1$. *Then* $1/3 > \beta > 0$. *Let* p/q *be a good approximation of* β *from below.*

(1) Let W *be the twig for* p/q *and* F *the set of leaves of* W. *Then* W *is a biminimal intrinsic extension of* F *with respect to* β.

(2) Let W' *be a subdivision of* W *such that there are exactly one subdividing point on each edge of* W. *Then* W' *is a biminimal intrinsic extension of* F *with respect to* α.

Proof. (1) Since $\beta > p/q$, we have $0 > p - q\beta$. Let B be a proper substructure of W with $B - F$ non-empty. Let $p' = |B - F|$ and $q' = e(B, F)$. Then $\delta_\beta(B/B \cap F) = p' - q'(p/q) > 0$ and $q' \le q$. So, $p'/q' > p/q$. Since $q' \le q$, we have $q' < q$ and thus $p'/q' > \beta$. Hence, $\delta_\beta(B/B \cap F) = p' - q'\beta > 0$.

(2) Note that any extension by a path is a strong extension. It is enough to show that $B \cap F <_\alpha B$ for any proper substructure B' of W' assuming that every leaf of B' belongs to F. In this case, B' is a subdivision of a substructure B of W. Therefore, $\delta_\alpha(B'/F) = \delta_\beta(B/F) > 0$. □

Lemma 3.5 *Let α be an irrational number with $2/3 > \alpha > 1/2$, $f = f_\alpha$ the Hrushovski's log-like function associated to α, p/q a good approximation of $\beta = 2\alpha - 1$ from below, and A a member of $\mathbf{K}_f$. If $|A| \le p$, then there are extensions $A \subseteq F \subseteq B$ such that $B \in \mathbf{K}_f$, $A <_\alpha F$, B is a biminimal intrinsic extension of F with respect to α, $\delta_\alpha(B/F) = p - q\beta = p + q - 2q\alpha$, $|B - F| = p + q$, and $|B| = 2q + 1$.*

Proof. Let W' be the subdivision of a twig for p/q from Lemma 3.4 (2).

Let $L = b_1 c_1 b_2 c_2 \cdots b_{p-1} c_{p-1} b_p$ be a new path with $2p - 1$ vertices. Let $V(A) = \{a_1, a_2, \ldots, a_k\}$. Connect each a_i to b_i with a new path of length 2. Let B_1 be the resulting graph. B_1 belongs to $\mathbf{K}_f$ by Lemma 2.6. By putting some paths of length 2 at each b_i with the other ends left as leaves, the resulting graph B belongs to $\mathbf{K}_f$ by Lemma 2.5. Also, we can make B so that $B//A$ is isomorphic to W'. Let F be the set of points in B which will be leaves in $B//A$. Then $A \subseteq F$ and F is a extension of A by independent points in F. □

Lemma 3.6 *Assume that $(\mathbf{K}_f, <)$ has the free amalgamation property. Let $B \in \mathbf{K}_f$ and suppose B is biminimal intrinsic extension of $F \subseteq B$. Let*

$$D = B_1 \otimes_F B_2 \otimes_F \cdots \otimes_F B_k$$

where each B_i is isomorphic to B over F. If D is normal to f then $D \in \mathbf{K}_f$.

Proof. Let C be a proper substructure of D. Let $C_i = C \cap B_i$.

Case $C \cap F$ is a proper substructure of F. In this case, $C \cap F < C_i$ for each i and $C \cap F$, and C_i belong to $\mathbf{K}_f$. Hence,

$$C = C_1 \otimes_{C \cap F} C_2 \otimes_{C \cap F} \cdots \otimes_{C \cap F} C_k$$

belong to $\mathbf{K}_f$ by the free amalgamation property.

Case $C \cap F = F$. In this case, $\delta_\alpha(C) > \delta_\alpha(D)$ and $|C| < |D|$. Therefore, C is normal to f because f is an increasing function. □

4 Model Completeness

Definition 4.1 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. A graph $A \in \mathbf{K}$ is *absolutely closed* in $\mathbf{K}$ if whenever $A \subseteq B \in \mathbf{K}$ then $A < B$.

Note that the notion of being absolutely closed in $\mathbf{K}$ is invariant under isomorphisms.

Fact 4.2 *[12] Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ and M a generic structure of $(\mathbf{K}, <)$. Assume that M is countably saturated. Suppose for any $A \in \mathbf{K}$ there is $C \in \mathbf{K}$ such that $A < C$ and C is absolutely closed in $\mathbf{K}$. Then the theory of M is model complete.*

Theorem 4.3 *Let α be an irrational number with $1 > \alpha > 0$, $f = f_\alpha$ the Hrushovski's log-like function associated to α, and M the generic structure of $(\mathbf{K}_f, <_\alpha)$. If f is unbounded, then the theory of M is model complete. Note that it is already known that the theory of M is not model complete if f is bounded [15].*

Proof. We assume $2/3 > \alpha > 1/2$. The proof will be similar for the case $n/(n+1) > \alpha > (n-1)/n$.

Let $2\alpha - 1 = [a_0, a_1, a_2, \ldots]$ be the simple continued fraction. Let p_n/q_n be the reduced fraction form of $[a_0, a_1, \ldots, a_n]$. Then we have $p_{n+1}/q_{n+1} < \alpha < p_n/q_n$ if n is odd. Also for any n, we have

$$\left| (2\alpha - 1) - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

Now, let A be a finite substructure of M. We want to show that there is some D with $A < D \subseteq M$

Since f is unbounded, there is an integer $n_A > 0$ such that $f(n_A) \geq \delta_\alpha(A)$.

Let

$$\varepsilon_A = \min\{\delta_\alpha(A) - \delta_\alpha(X) \mid A \subseteq X \subseteq M,\ X \text{ finite},\ \delta_\alpha(X/A) < 0\}.$$

If the set on the right hand side is empty then A is absolutely closed in M. So, we can assume that $\varepsilon_A > 0$.

Choose odd n such that $q_n > n_A$, $0 < 1/p_n < \varepsilon_A/2$, and $|A| < p_n$.

It is known that p_{n+1}/q_{n+1} is a good approximation of $2\alpha - 1$ from below.

Choose $B \in \mathbf{K}_f$ as in Lemma 3.5 with $p = p_{n+1}$ and $q = q_{n+1}$. Let F be as in Lemma 3.5 also.

We have $|B - F| = p_{n+1} + q_{n+1}$. Let

$$D_j = B_1 \otimes_F B_2 \otimes_F \cdots \otimes_F B_j$$

where each B_i is isomorphic to B over F. Since $\delta_\alpha(B_i/F) < 0$, $\delta_\alpha(D_j)$ decreases as j increases. Hence, $D_j \notin \mathbf{K}_f$ eventually. Let k be such that $D_k \in \mathbf{K}_f$ but $D_{k+1} \notin \mathbf{K}_f$. We have

$$\delta_\alpha(D_k) - \delta_\alpha(D_{k+1}) = |\delta_\alpha(B/F)| = |p_{n+1} - (2\alpha - 1)q_{n+1}| < \frac{1}{q_{n+2}} < \frac{\varepsilon_A}{2}.$$

Hence,

$$f(|D_k| + p_{n+1} + q_{n+1}) > \delta_\alpha(D_k) - \varepsilon_A/2.$$

We want to evaluate how the curve $y = f_\alpha(x)$ behaves. Choose ℓ with $x_{\ell-1} < |B| = 2q_{n+1} + 1 \leq x_\ell < e_\ell$.

We have $p_n/q_n > 2\alpha - 1$. Hence, $(p_n + q_n)/(2q_n) > \alpha$. Since $2q_n < e_\ell$, we have $(p_n + q_n)/(2q_n) \geq k_\ell/d_\ell > \alpha$. So,

$$\begin{aligned} f'(x_\ell) &= \frac{k_\ell - d_\ell\alpha}{k_\ell} = 1 - \frac{d_\ell}{k_\ell}\alpha \\ &\leq 1 - \frac{2q_n}{p_n + q_n}\alpha = \frac{1}{p_n + q_n}(p_n - q_n(2\alpha - 1)) \\ &< \frac{1}{(p_n + q_n)q_{n+1}}. \end{aligned}$$

Hence,

$$q_{n+1}f'(x_\ell) < \frac{1}{p_n + q_n} < \frac{\varepsilon_A}{4}.$$

Therefore, for any $x \geq x_\ell$,

$$(p_{n+1} + q_{n+1})f'(x) < 2q_{n+1}f'(x_\ell) < \frac{\varepsilon_A}{2}.$$

Hence,

$$\delta_\alpha(D_k) - f(|D_k|) < \varepsilon_A.$$

Now, choose a sequence of biminimal intrinsic extensions starting from D_k in $\mathbf{K}_f$. Since the δ_α-values of these extensions are decreasing, we get an absolutely closed structure D eventually. Since D belongs to $\mathbf{K}_f$, we have $\delta_\alpha(D_k) \geq \delta_\alpha(D) \geq f(|D_k|)$. Hence,

$$|\delta_\alpha(D/D_k)| < \varepsilon_A.$$

D is obtained by a sequence of minimal intrinsic extensions from a structure isomorphic to B. Hence, $A < D$. □

Similar argument works for f which was modified by F. Wagner [16]. Our proof depends heavily on the definition of f_α. We still don't know if the model completeness of the generic structure of $\mathbf{K}_f$ is true in more general setting.

Acknowledgments

The work is supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] J.T. Baldwin and K. Holland, Constructing ω-stable structures: model completeness, Ann. Pure Appl. Log. **125**, 159–172 (2004).

[2] J.T. Baldwin and S. Shelah, Randomness and semigenericity, Trans. Am. Math. Soc. **349**, 1359–1376 (1997).

[3] J.T. Baldwin and N. Shi, Stable generic structures, Ann. Pure Appl. Log. **79**, 1–35 (1996).

[4] R. Diestel, *Graph Theory*, Fourth Edition, Springer, New York (2010).

[5] D. Evans, Z. Ghadernezhad, and K. Tent, Simplicity of the automorphism groups of some Hrushovski constructions, Ann. Pure Appl. Logic **167**, 22–48 (2016).

[6] G.H. Hardy, and E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford University Press, Oxford (1979).

[7] E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, preprint (1988).

[8] E. Hrushovski, A new strongly minimal set, Ann. Pure Appl. Log. **62**, 147–166 (1993).

[9] K. Ikeda, H. Kikyo, Model complete generic structures, in *the Proceedings of the 13th Asian Logic Conference*, World Scientific, 114–123 (2015).

[10] H. Kikyo, Model complete generic graphs I, RIMS Kokyuroku **1938**, 15–25 (2015).

[11] H. Kikyo, Balanced Zero-Sum Sequences and Minimal Intrinsic Extensions, RIMS Kokyuroku **2079**, Balanced zero-sum sequences and minimal intrinsic extensions (2018).

[12] H. Kikyo, Model Completeness of Generic Graphs in Rational Cases, Archive for Mathematical Logic **57** (7-8), 769–794 (2018).

[13] H. Kikyo, Model completeness of the theory of Hrushovski's pseudoplane associated to 5/8, RIMS Kokyuroku **2084**, 39–47 (2018).

[14] H. Kikyo, On the automorphism group of a Hrushovski's pseudoplane associated to 5/8, RIMS Kokyuroku **2119**, 75–86 (2019).

[15] H. Kikyo, S. Okabe, On Hrushovski's pseudoplanes, Proceedings of the 14th and 15th Asian Logic Conferences, 175–194 (2019).

[16] F.O. Wagner, Relational structures and dimensions, in *Automorphisms of first-order structures*, Clarendon Press, Oxford, 153–181 (1994).

[17] F.O. Wagner, *Simple Theories*, Kluwer, Dordrecht (2000).

Graduate School of System Informatics
Kobe University
1-1 Rokkodai, Nada, Kobe 657-8501
JAPAN
`kikyo@kobe-u.ac.jp`

神戸大学大学院システム情報学研究科　桔梗　宏孝