DIAMETER OF A TYPE-DEFINABLE LASCAR STRONG TYPE OVER A HYPERIMAGINARY

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Abstract. Using the so called open anaylsis, Ludomir Newelski proved that type-definable Lascar strong types over real parameters have finite diameters in 2003. Later in 2008, Rodrigo Pelaez proved the same result in a more direct way, using the notion of c-free and weakly c-free extensions over a complete type, which were also introduced by Newelski. Recall that a hyperimaginary is an equivalence class of an $\emptyset$-type-definable equivalence relation. We will extend the result of Pelaez to the context of hyperimaginaries, i.e. type-definable Lascar strong types over a hyperimaginary always have finite diameters.

Fix a complete theory T with a language $\mathcal{L}$ and a monster model $\mathcal{M}$ of T. Let E be an $\emptyset$-type definable equivalence relation, a a real (possibly infinite) tuple, and $\boldsymbol{e} = a/E$ a hyperimaginary. Assume that there is a partial type $\pi(x)$ over ab such that $b_0 \models \pi(x)$ iff $b_0 \equiv^{\mathrm{L}}_{\boldsymbol{e}} b$, i.e. the Lascar strong type of b over $\boldsymbol{e}$ is type-definable. We will show that there is a natural number $n_b < \omega$ such that for all b_0 having the same Lascar strong type of b over $\boldsymbol{e}$, the diameter(or Lascar distance) between b and b_0 over $\boldsymbol{e}$ is at most n_b, i.e. $d_{\boldsymbol{e}}(b, b_0) \leq n_b$.

The same result over an empty(or a set of real) parameter(s) is established in [4]. Many concepts and arguments will be from [4] but some critical modifications and new concepts will also be introduced to handle the situations over a hyperimaginary. We begin with the space of complete types over a hyperimaginary $\boldsymbol{e}$.

Definition 1. (1) Say a partial type $p(x)$ over a is a complete type over $\boldsymbol{e}$ if there is c such that for any $\varphi(x) \in \mathcal{L}(a)$, (if $f(c) \models \varphi(x)$ for all $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$, then $\varphi \in p$). Denote this $p(x)$ by $\mathrm{tp}_x(c/\boldsymbol{e})$, and call "the" complete type of c over $\boldsymbol{e}$ (with representation a). We may omit x and write $\mathrm{tp}(c/\boldsymbol{e})$ if the variable is clear from the context.

(2) $S_x(\boldsymbol{e}) = \{p(x) : p(x)$ is a complete type over $\boldsymbol{e}\}$ is the set of all complete types over a hyperimaginary $\boldsymbol{e}$ in the variable x. Likewise, we may omit x.

(3) For $\varphi(x) \in \mathcal{L}(a)$, denote $[\varphi(x)]_{\boldsymbol{e}} = \{p(x) \in S_x(\boldsymbol{e}) : \varphi(x) \in p(x)\}$. Define likewise for a partial type $\Phi(x)$ over a.

(4) For $\varphi(x) \in \mathcal{L}(a)$, denote $[\varphi(x)]^{con}_{\boldsymbol{e}} = \{p(x) \in S_x(\boldsymbol{e}) : p$ is consistent with $\varphi\}$. Define likewise for a partial type $\Phi(x)$ over a.

Remark 2. (1) The above definition of a complete type over a hyperimaginary can replace the old version for it : Recall that $\exists y(E(a, y) \wedge \mathrm{tp}_{xy}(ca))$ would be a complete type of c over $\boldsymbol{e}$ in the old fashion, which is a subset of $p(x)$, and their solution sets coincide.

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(2) We say $p(x) = \mathrm{tp}(c/\boldsymbol{e}) \in S(\boldsymbol{e})$ "the" complete type of c over $\boldsymbol{e}$ since it is unique : The parameters are restricted to a and $\mathrm{tp}(c/\boldsymbol{e})$ is the maximal set of formulas $\varphi(x)$ over a such that $\{f(c) : f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})\} \subseteq \varphi(\mathcal{M})$.

(3) $p(x) \in [\varphi(x)]_{\boldsymbol{e}}$ iff $p(x)$ is inconsistent with $\neg\varphi(x)$ iff $p(x) \notin [\neg\varphi(x)]^{con}_{\boldsymbol{e}}$, so $S_x(\boldsymbol{e}) \setminus [\varphi(x)]_{\boldsymbol{e}}$ is NOT the same as $[\neg\varphi(x)]_{\boldsymbol{e}}$ in general.

Lemma 3. (1) *$\{[\varphi(x)]_{\boldsymbol{e}} : \varphi(x) \in \mathcal{L}(a)\}$ forms a basis of $S_x(\boldsymbol{e})$.*
(2) *$S_x(\boldsymbol{e})$ is Hausdorff and compact.*

Proof. (1) $[x = x]_{\boldsymbol{e}} = S_x(\boldsymbol{e})$ and $[\varphi(x)]_{\boldsymbol{e}} \cap [\psi(x)]_{\boldsymbol{e}} = [\varphi(x) \wedge \psi(x)]_{\boldsymbol{e}}$.

(2) If $p \neq q \in S_x(\boldsymbol{e})$, then $p \wedge q$ is inconsistent, so by compactness, there is $\varphi(x) \in p$ and $\psi(x) \in q$ such that $\varphi(x) \wedge \psi(x) \equiv \emptyset$. Then $p \in [\varphi]_{\boldsymbol{e}}, q \in [\psi]_{\boldsymbol{e}}$ and $[\varphi]_{\boldsymbol{e}} \cap [\psi]_{\boldsymbol{e}} = \emptyset$, proving that $S_x(\boldsymbol{e})$ is Hausdorff.
To show compactness, suppose there is an open cover $\{[\varphi_i(x)]_{\boldsymbol{e}} : i \in I\}$ without any finite subcover. Pick any finite subset, say $\{\varphi_{i_0}, \cdots, \varphi_{i_k}\}$. Then by supposition, there is $p(x) \in S_x(\boldsymbol{e})$ such that $p(x) \notin [\varphi_{i_0}]_{\boldsymbol{e}} \cup \cdots \cup [\varphi_{i_k}]_{\boldsymbol{e}}$. Then by Remark 2(3), $p(x)$ is consistent with $\neg\varphi_{i_l}$ for each $l = 0, \cdots, k$. So by compactness, there is $q(x) \in S_x(\boldsymbol{e})$ consistent with $\neg\varphi_i$ for each $i \in I$(since $\exists x_{i \in I}(\bigwedge_{i \in I} \neg\varphi_i(x_i) \wedge \bigwedge_{i,j \in I} x_i \equiv_{\boldsymbol{e}} x_j)$ is finitely consistent). Then by Remark 2(3) again, this $q(x) \notin \bigcup_{i \in I} [\varphi_i]_{\boldsymbol{e}} = S_x(\boldsymbol{e})$, which is impossible.

□

Let $\pi(x)$ be a partial type over ab such that $b_0 \models \pi$ iff $b \equiv^{\mathrm{L}}_{\boldsymbol{e}} b_0$.

Definition 4. (1) $U \subseteq \mathcal{M}$ (not necessarily small) is called c-free over π if there is $n < \omega$ such that $f_0, \cdots f_{n-1} \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$ such that $\pi(\mathcal{M}) \subseteq \bigcup_{i<n} f_i(U)$ and $f_i(\pi(\mathcal{M})) = \pi(\mathcal{M})$ for every $i < n$.

(2) $\varphi(x) \in \mathcal{L}(\mathcal{M})$ is c-free over π if $\varphi(\mathcal{M})$ is c-free over π.

(3) A (partial) type $q(x)$ is c-free over π if for any $\varphi(x)$ such that $q(x) \vdash \varphi(x)$, $\varphi(x)$ is c-free over π.

Definition 5. $U \subseteq \mathcal{M}$ is weakly c-free over π if there is $V \subseteq \mathcal{M}$ such that V is not c-free over π but UV is c-free over π. Define for $\varphi(x) \in \mathcal{L}(\mathcal{M})$ and a partial type $q(x)$ to be weakly c-free over π in the same manner as above.

Remark 6. If $U \subseteq V$ and U is (weakly) c-free over π, then V is (weakly) c-free over π.

Definition 7. Define $F_x : S_x(a) \to S_x(\boldsymbol{e})$, $F_x(\mathrm{tp}_x(c/a)) = \mathrm{tp}_x(c/\boldsymbol{e})$.

Remark 8. F_x is possibly NOT continuous since $F_x^{-1}([\varphi(x)]_{\boldsymbol{e}})$ can be a proper subset of $[\varphi(x)] \subseteq S_x(a)$: Even if $p(x) \notin [\varphi(x)]_{\boldsymbol{e}}$, it may happen that $F_x^{-1}(p) \cap [\varphi(x)] \neq \emptyset$ since there can be $c \vDash \varphi(x)$ such that there is some $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$, $f(c) \nvDash \varphi(x)$.

Lemma 9. *For any (partial) type $\Phi(x)$ over a, denote $[\Phi(x)] = \{p(x) \in S(a) : \Phi(x) \subseteq p(x)\}$. Then $F_x([\Phi(x)]) = \{q \in S(\boldsymbol{e}) : q$ is consistent with $\Phi\}$ and F_x is a closed map.*

Proof. Let $\Phi(x)$ be a partial type over a. Then

$$\begin{aligned} F_x([\Phi(x)]) &= F_x(\{p \in S(a) : p(\mathcal{M}) \subseteq \Phi(\mathcal{M})\} \\ &= F_x(\{p \in S(a) : \forall(\exists)c \vDash p,\, c \vDash \Phi\} \\ &= \{F_x(p) \in S(\boldsymbol{e}) : \forall(\exists)c \vDash p,\, c \vDash \Phi\} \\ &= \{q \in S(\boldsymbol{e}) : \exists c \vDash q \text{ such that } c \vDash \Phi\} \\ &= \{q \in S(\boldsymbol{e}) : q \text{ is consistent with } \Phi\}. \end{aligned}$$

By compactness, $q \in S(\boldsymbol{e})$ is consistent with Φ iff q is consistent with every finite subset Φ_0 of Φ. Thus $F_x([\Phi(x)]) = \bigcap_{\substack{\Phi_0 \subseteq \Phi \\ \Phi_0 \,:\, \text{finite}}} [\bigwedge \Phi_0]^{con}_{\boldsymbol{e}}$ where $\bigwedge \Phi_0$ is the conjunction of all formulas in Φ_0, which is a single formula in $\mathcal{L}(a)$. Note that by Remark 2(3), each $[\bigwedge \Phi_0]^{con}_{\boldsymbol{e}}$ is a basic closed set in $S(\boldsymbol{e})$, thus $F_x([\Phi(x)]) = \bigcap_{\substack{\Phi_0 \subseteq \Phi \\ \Phi_0 \,:\, \text{finite}}} [\bigwedge \Phi_0]^{con}_{\boldsymbol{e}}$ is closed. □

Definition 10. (1) $P = \{q(x,y) \in S_{xy}(\boldsymbol{e}) : q(x,y) \cup \pi(x) \cup \pi(y) \text{ is consistent}\}$.
(2) For $c \models \pi$, $P_{w,c} = \{q(x,y) \in P : q(c,y) \text{ is weakly c-free over } \pi\}$.

So far, no assumption on $\pi(x)$ is used, i.e. $\pi(x)$ would be any partial type having some parameters. Now we start exploiting the fact that $\pi(x)$ type-defines the Lascar strong type of b over $\boldsymbol{e}$.

Lemma 11. (1) *For any* $p(x,y) = \text{tp}(b_1b_2/\boldsymbol{e}) \in P$, $b_1 \equiv^{\text{L}}_{\boldsymbol{e}} b_2$.
(2) $P = [\pi(x) \cup \pi(y)]^{con}_{\boldsymbol{e}}$ *is closed in* $S_{xy}(\boldsymbol{e})$.
(3) *Define* $\Delta_{w,c}(x,y) = \{\varphi(x,y) \in \mathcal{L}(a) : \varphi(c,y) \text{ is not weakly c-free over } \pi\}$. *Then* $P_{w,c} = P \cap \bigcap_{\varphi \in \Delta_{w,c}} [\neg\varphi(x,y)]^{con}_{\boldsymbol{e}}$. *In particular,* $P_{w,c}$ *is closed in* $S_{xy}(\boldsymbol{e})$.
(4) *There is a partial type* $\Phi_{w,c}(x,y)$ *over* a *such that* $p \in P_{w,c}$ *iff for all(some)* $d \models p$, $d \models \Phi_{w,c}$. *Thus* $P_{w,c} = [\Phi_{w,c}(x,y)]_{\boldsymbol{e}} = [\Phi_{w,c}(x,y)]^{con}_{\boldsymbol{e}}$.

Proof. (1) Say $b^*_1 b^*_2 \models \text{tp}(b_1b_2/\boldsymbol{e}) \wedge \pi(x) \wedge \pi(y)$. Then $b^*_1 \equiv^{\text{L}}_{\boldsymbol{e}} b \equiv^{\text{L}}_{\boldsymbol{e}} b^*_2$ and $f(b^*_1 b^*_2) = b_1b_2$ for some $f \in \text{Aut}_{\boldsymbol{e}}(\mathcal{M})$, hence $b_1 = f(b^*_1) \equiv^{\text{L}}_{\boldsymbol{e}} f(b^*_2) = b_2$.
(2) By Lemma 9, $P = [\pi(x) \cup \pi(y)]^{con}_{\boldsymbol{e}} = F_{xy}([\pi(x) \cup \pi(y)])$ is closed.
(3) Suppose there is $p(x,y) \in \bigcap_{\varphi \in \Delta_{w,c}} [\neg\varphi(x,y)]^{con}_{\boldsymbol{e}}$ but $p(x,y) \notin P_{w,c}$. Then $p(c,y)$ is not weakly c-free over π, so there is $\psi(y) \in \mathcal{L}(\mathcal{M})$ such that $\psi(y)$ is not weakly c-free over π and $p(c,y) \vdash \psi(y)$. By compactness and Remark 6, there is $\varphi(c,y) \in p(c,y)$ not weakly c-free over π. But then $\varphi(x,y) \in \Delta_{w,c}(x,y)$, so $\neg\varphi(x,y)$ must be consistent with $p(x,y)$, a contradiction since $\varphi(x,y) \in p(x,y)$. Conversely, suppose there is $p(x,y) \in P_{w,c}$ but $p(x,y) \notin \bigcap_{\varphi \in \Delta_{w,c}} [\neg\varphi(x,y)]^{con}_{\boldsymbol{e}}$. Then there is $\varphi(x,y) \in \Delta_{w,c}(x,y)$ such that $\neg\varphi(x,y)$ is inconsistent with $p(x,y)$, hence $p(x,y) \in [\varphi(x,y)]_{\boldsymbol{e}}$ by Remark 2(3). But then $\varphi(c,y)$ is not weakly c-free over π and $p(c,y) \vdash \varphi(c,y)$, a contradiction to $p(x,y) \in P_{w,c}$. Now $P_{w,c}$ is clearly closed by Remark 2(3).
(4) $\text{tp}(b_1b_2/\boldsymbol{e}) \in P_{w,c}$ iff
$\text{tp}(b_1b_2/\boldsymbol{e})$ is consistent with $\neg\psi(x,y)$ for any $\psi(x,y)$ such that $\psi(c,y)$ is not weakly c-free over π (by (3)) iff
$b_1b_2 \models \bigwedge_{\psi \in \Delta_{w,c}} \exists zw(xy \equiv_{\boldsymbol{e}} zw \wedge \neg\psi(z,w))$, say $\Phi_{w,c}(x,y)$. Notice that this $\Phi_{w,c}(x,y)$ is $\boldsymbol{e}$-invariant, so $\text{tp}(b_1b_2/\boldsymbol{e})$ is consistent with $\Phi_{w,c}(x,y)$ iff $\Phi_{w,c}(x,y) \subseteq \text{tp}(b_1b_2/\boldsymbol{e})$.

□

Proposition 12. *Let* $\emptyset \neq S \subseteq P_{w,c}(x,y)$ *where* $S = [\Psi(x,y)]_{\boldsymbol{e}}$ *for some (partial) type* Ψ *such that* $[\Psi(x,y)]_{\boldsymbol{e}} = [\Psi(x,y)]^{con}_{\boldsymbol{e}}$. *Then there are* $c_1, \cdots, c_k \models \pi$ *such that for any* $b_0 \models \pi$, *there is* $d \models \pi$ *such that*
(1) $\text{tp}(b_0d/\boldsymbol{e}) \in S$.
(2) $\text{tp}(c_id/\boldsymbol{e}) \in S$ *for some* $1 \leq i \leq k$.

Proof. We start with a claim.

Claim 1. $\Psi(c,y)$ *is weakly c-free over* π.

Proof of Claim 1. Suppose not, so there is $\delta'(c,y)$ such that $\delta'(c,y)$ is not weakly c-free over π and $\Psi(c,y) \vdash \delta'(c,y)$. Then by compactness and Remark 6, there is $\delta(c,y) \in$

$\bigwedge \Psi(c,y) = \{\psi_1 \wedge \cdots \wedge \psi_n : \psi_l \in \Psi(c,y), l \in \omega\}$, which is not weakly c-free over π. By Lemma 11(3), every element of $P_{w,c}$ must be consistent with $\neg\delta(x,y)$, i.e. $P_{w,c} \subseteq [\neg\delta(x,y)]^{con}_{\boldsymbol{e}}$. But we have $[\Psi(x,y)]^{(con)}_{\boldsymbol{e}} = S \subseteq P_{w,c}$ and $[\Psi(x,y)]^{(con)}_{\boldsymbol{e}} \cap [\neg\delta(x,y)]^{con}_{\boldsymbol{e}} = \emptyset$ (If not empty, say $p(x,y) \in S(\boldsymbol{e})$ witness it, then $p(\mathcal{M}) \subseteq \Psi(\mathcal{M})$ and $p(\mathcal{M}) \cap \neg\delta(\mathcal{M}) \neq \emptyset$, a contradiction), so every $p(x,y) \in P_{w,c}$ is not an element of S, a contradiction to $S \neq \emptyset$. □

Using Claim 1, say $V \subseteq \mathcal{M}$ is not c-free over π but $\pi(\mathcal{M}) \subseteq \bigcup_{i=1}^k g_i(\Psi(c,\mathcal{M}) \cup V)$ where $g_i \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$ and $g_i(\pi(\mathcal{M})) = \pi(\mathcal{M})$ for each $1 \leq i \leq k$.

Claim 2. *For each $b_0 \vDash \pi$, there is $d \in \Psi(b_0,\mathcal{M})$ such that $d \in \bigcup_{i=1}^k \Psi(g_i(c,\mathcal{M}))$.*

Proof of Claim 2. Suppose not, so that there is $b_0 \vDash \pi$ such that for every $d \in \Psi(b_0,\mathcal{M})$, $d \notin \bigcup_{i=1}^k \Psi(g_i(c,\mathcal{M}))$. Notice that $d \vDash \pi$: $\mathrm{tp}(b_0 d/\boldsymbol{e}) \in P$ and $b_0 \vDash \pi$, so there is $b_0^* d^* \vDash \mathrm{tp}(b_0 d/\boldsymbol{e}) \wedge \pi(x) \wedge \pi(y)$. Then there is $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$ such that $f(b_0^* d^*) = b_0 d$, and this f also fixes $b/\equiv^{\mathrm{L}}_{\boldsymbol{e}}$ since $b \equiv^{\mathrm{L}}_{\boldsymbol{e}} b_0 \equiv^{\mathrm{L}}_{\boldsymbol{e}} b_0^*$. Thus $d = f(d^*) \vDash \pi$. So for all $d \in \Psi(b_0,\mathcal{M})$, $d \in \pi(\mathcal{M}) \subseteq \bigcup_{i=1}^k g_i(\Psi(c,\mathcal{M}) \cup V)$, hence $d \in \bigcup_{i=1}^k g_i(\Psi(c,\mathcal{M}) \cup V) \setminus \bigcup_{i=1}^k \Psi(g_i(c,\mathcal{M})) \subseteq \bigcup_{i=1}^k g_i((\Psi(c,\mathcal{M}) \cup V) \setminus \Psi(c,\mathcal{M}))$. Since b_0 and c realizes π, there is $g_{k+1} \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$ such that $g_{k+1}(c) = b_0$ and $g_{k+1}(\pi(\mathcal{M})) = \pi(\mathcal{M})$. Also, since $g_{k+1}(\Psi(c,\mathcal{M}) \cup V) = (g_{k+1}(\Psi(c,\mathcal{M}) \cup V) \setminus \Psi(b_0,\mathcal{M})) \cup \Psi(b_0,\mathcal{M})$ and $\Psi(b_0,\mathcal{M}) \subseteq \bigcup_{i=1}^k g_i((\Psi(c,\mathcal{M}) \cup V) \setminus \Psi(c,\mathcal{M}))$, $g_{k+1}(\Psi(c,\mathcal{M}) \cup V) \subseteq \bigcup_{i=1}^{k+1} (g_i(\Psi(c,\mathcal{M}) \cup V) \setminus \Psi(c,\mathcal{M}))$. Since $g_{k+1}(\Psi(c,\mathcal{M}) \cup V)$ is c-free over π, $\bigcup_{i=1}^{k+1} (g_i(\Psi(c,\mathcal{M}) \cup V) \setminus \Psi(c,\mathcal{M}))$ is also c-free over π, hence $(\Psi(c,\mathcal{M}) \cup V) \setminus \Psi(c,\mathcal{M}) \subseteq V$ is c-free over π, a contradiction. □

Now by Claim 2, $g_1(c), \cdots, g_k(c)$ satisfies : For any $b \vDash \pi$, there is $d \in \pi$ such that $\mathrm{tp}(bd/\boldsymbol{e}) \in S (= [\Psi(x,y)]^{(con)}_{\boldsymbol{e}})$ and $\mathrm{tp}(c_i d/\boldsymbol{e}) \in S$ for some $1 \leq i \leq k$, completing the proof.

□

Let $\Gamma(x,y)$ be a partial type over a such that $b_0 b_1 \vDash \Gamma(x,y)$ iff there is an $\boldsymbol{e}$-indiscernible sequence beginning with b_0, b_1.

Definition 13. Say the diameter(or Lascar distance) between b_1 and b_2 is less than equal to n where $1 < n < \omega$, if

$$b_1 b_2 \vDash \exists y_1 \cdots y_{n-1} (\Gamma(x,y_1) \wedge \Gamma(y_1,y_2) \wedge \cdots \wedge \Gamma(y_{n-1},y))$$

and denote it by $d_{\boldsymbol{e}}(b_1,b_1) \leq n$. If $b_1 b_2 \vDash \Gamma(x,y)$, then say $d_{\boldsymbol{e}}(b_1,b_2) \leq 1$.

Theorem 14. *There is $n_b \in \omega$ such that if $b_0 \vDash \pi(x)$, then $d_{\boldsymbol{e}}(b,b_0) \leq n_b$.*

Proof. By Lemma 11(1), $P = \bigcup_{1 \leq n \in \omega} (P \cap [d_{\boldsymbol{e}}(x,y) \leq n]_{\boldsymbol{e}})$, thus

$$\begin{aligned} P_{w,c} = P \cap P_{w,c} &= \bigcup_{1 \leq n \in \omega} (P \cap [d_{\boldsymbol{e}}(x,y) \leq n]_{\boldsymbol{e}} \cap P_{w,c}) \\ &= \bigcup_{1 \leq n \in \omega} (P_{w,c} \cap [d_{\boldsymbol{e}}(x,y) \leq n]_{\boldsymbol{e}}. \end{aligned}$$

Note that $[d_{\boldsymbol{e}}(x,y) \leq n]_{\boldsymbol{e}} = [d_{\boldsymbol{e}}(x,y) \leq n]^{con}_{\boldsymbol{e}}$ for any $1 \leq n < \omega$: '$d_{\boldsymbol{e}}(x,y) \leq n$' is $\boldsymbol{e}$-invariant, so its solution set is a union of $\mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$-orbits. Thus by Lemma 11(4),

$$\begin{aligned} P_{w,c} \cap [d_{\boldsymbol{e}}(x,y) \leq 1]^{(con)}_{\boldsymbol{e}} &= [\Phi_{w,c}(x,y)]^{(con)}_{\boldsymbol{e}} \cap [d_{\boldsymbol{e}}(x,y) \leq 1]^{(con)}_{\boldsymbol{e}} \\ &= [\Phi_{w,c}(x,y) \wedge (d_{\boldsymbol{e}}(x,y) \leq 1)]^{(con)}_{\boldsymbol{e}}. \end{aligned}$$

Now Proposition 12 can be applied with $S = [\Phi_{w,c}(x,y) \wedge (d_{\boldsymbol{e}}(x,y) \leq 1)]_{\boldsymbol{e}}^{(con)}$, so that there are $c_1, \cdots c_k \models \pi$ such that for any $b_0 \models \pi$, there are $d, d_0 \models \pi$ such that

(1) $\mathrm{tp}(bd/\boldsymbol{e}), \mathrm{tp}(b_0 d_0/\boldsymbol{e}) \in S$

(2) $\mathrm{tp}(c_{k_*} d/\boldsymbol{e}), \mathrm{tp}(c_{k_0} d_0/\boldsymbol{e}) \in S$ for some $1 \leq k_*, k_0 \leq k$.

Let $M = \min\{n \in \omega : \{\mathrm{tp}(c_i c_j/\boldsymbol{e}) : 1 \leq i, j \leq k\} \subseteq [d_{\boldsymbol{e}}(x,y) \leq n]_{\boldsymbol{e}}\}$. Then

$$\begin{aligned} d_{\boldsymbol{e}}(b, b_0) &\leq d_{\boldsymbol{e}}(b, d) + d_{\boldsymbol{e}}(d, c_{k_*}) + d_{\boldsymbol{e}}(c_{k_*}, c_{k_0}) + d_{\boldsymbol{e}}(c_{k_0}, d_0) + d_{\boldsymbol{e}}(d_0, b_0) \\ &\leq 1 + 1 + M + 1 + 1 = 4 + M. \end{aligned}$$

□

Corollary 15. *Type-definable Lascar strong type over a hyperimaginary has finite diameter.*

References

[1] D. Lascar and A. Pillay, 'Hyperimaginaries and automorphism groups', *J. of Symbolic Logic* 66 (2001) 127-143.
[2] B. Kim, 'Simplicity Theory', Oxford University Press (2014).
[3] M. Ziegler, 'Introduction to the Lascar group', in *Tits buildings and the model theory of groups*, London Math. Lecture Note Series 291, Cambridge Univ. Press (2002) 279-298.
[4] Rodrigo Pelaez Pelaez, 'About The Lascar Group', Dissertation, University of Barcelona (2008).
[5] Ludomir Newelski, 'The diameter of a Lascar strong type', *Fundamenta Mathematicae* 176(2):157-170 (2003).

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