Groups in generic structures

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Abstract

We show that a normal generic structure with a rational coefficient does not have any infinite definable group.

1 Generic structures

In this short note, a graph means a simple (hyper-)graph: A structure $M = (M, R)$ with a relation R is a (hyper-)graph if R satisfies

- $\models \forall x_1 ... x_n [R(x_1 ... x_n) \to \bigwedge_{i \neq j} x_i \neq x_j]$;
- $\models \forall \bar{x} [R(\bar{x}) \to R(\sigma \bar{x})]$ for any permutation σ.

Then M denotes a set of vertices in (M, R), and R^M a set of (hyper-)edges in (M, R).

Let $A, B, C, \cdots$ denote (hyper-)graphs. A predimension $\delta(A)$ of a finite (hyper-)graph A is defined by

$$\delta(A) = |A| - \alpha |R^A|,$$

where the coefficient α satisfies $0 < \alpha \leq 1$.

For finite A, B, we write $\delta(A/B) = \delta(A \cup B) - \delta(A)$.

For finite A, B with $A \subset B$, A is said to be closed in B (written $A \leq B$), if $\delta(X/A) \geq 0$ for any $X \subset B - A$.

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It can be checked that if $A \leq B$ then $A \cap X \leq B \cap X$ for any $X \subset B$. For possibly infinite A, B, $A \leq B$ is defined by $A \cap X \leq B \cap X$ for any finite $X \subset B$.

For $A \subset B$, it can be seen that there is the smallest C with $A \subset C \leq B$. This C is called the closure of A in B, and denoted by $\mathrm{cl}_B(A)$.

Let $\mathbf{K}$ be a class of finite (hyper-)graphs A with $\emptyset \leq A$. Then a countable structure M is said to be $(\mathbf{K}, \leq)$-generic, if it satisfies the following:

- if A is a finite subset of M then $A \in \mathbf{K}$;
- if $A \subset M$ and $A \leq B \in \mathbf{K}$, then there is a $B' \leq M$ with $B' \cong_A B$;
- if A is a finite subset of M then $\mathrm{cl}_M(A)$ is finite.

The following fact can be found in [9].

Fact 1.1 Let M be a saturated generic structure. Then

1. $\mathrm{Th}(M)$ is stable.
2. If the coefficient is rational, then $\mathrm{Th}(M)$ is ω-stable.

Example 1.2 (rational coefficients)

1. Hrushovski's strongly minimal structures [4]
2. Baldwin's projective planes [1]
3. Holographic structures [6]

Example 1.3 (irrational coefficients)

1. Hrushovski's pseudoplanes [5]
2. Herwig's generic structures [3]
3. Sparse random graphs [2]

For finite $A, B \subset M$, let $\mathrm{d}_M(A) = \delta(\mathrm{cl}_M(A))$ and let $\mathrm{d}(B/A) := \mathrm{d}(BA) - \mathrm{d}(A)$. For possibly infinite C, let $d(B/C) = \inf\{d(B/C_0) : C_0 \subset_{\mathrm{fin}} C\}$.

For $A, B, C \subset M$, B and C are said to be d-independent over A, if it satisfies

- $\mathrm{cl}(BA) \cap \mathrm{cl}(CA) = \mathrm{cl}(A)$;
- $\mathrm{d}(B/C) = \mathrm{d}(B/A)$.

In many examples of generic structures, d-independence coincides with forking-independence.

Definition 1.4 We call a generic structure M normal, if it satisfies

1. M is saturated,
2. forking-independence coincides with d-independence,
3. $a \in \mathrm{acl}(A)$ implies $\mathrm{d}(a/A) = 0$ for any $aA \subset M$.

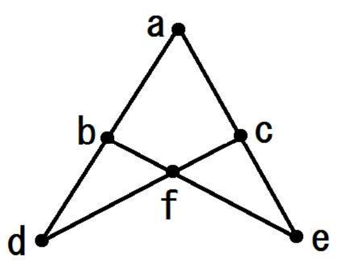
Example 1.2.1-3 and Example 1.3.1-2 are normal generic structures, but Example 1.3.3 is not normal because the structure is not saturated.

2 Proposition

The following fact can be found, for instance, in [8].

Fact 2.1 Let G be an ω-stable group and let p be a generic type of G. For any realizations $a, b, c \models p$ with $\{a, b, c\}$ independent, we can get a *group configuration* $C = \{a, b, c, d, e, f\}$, i.e.,

1. $a, b, c, d, e, f \models p$;
2. any two points are independent;
3. any point on a line is algebraic over the other two points on the line;
4. any three non-collinear points are independent.



Assumption 2.2 In what follows, we assume that

- M is a normal generic structure with a rational coefficient;
- G is an infinite group definable in $\mathrm{Th}(M)$.

We can take a generic type p of G, since $\mathrm{Th}(M)$ is ω-stable by Fact 1.1.

Let $\mathrm{d}(p)$ mean $\mathrm{d}(e)$ for a realization e of p.

Lemma 2.3 If $\mathrm{d}(p) > 0$, then we have a contradiction.

Proof. Let $d(p) = k > 0$. Take realizations $a, b, c \models p$ such that $\{a, b, c\}$ is independent. By Fact 2.1, we can take a group configuration C generated by $\{a, b, c\}$. Note that d-independence coincides with forking-independence since M is a normal generic structure. Then

$$\mathrm{d}(C) = \mathrm{d}(abc) = \mathrm{d}(a) + \mathrm{d}(b) + \mathrm{d}(c) = 3k.$$

On the other hand, let $L_1, ...L_4$ be the lines of C and $P_1, ..., P_6$ the points of C. By Assumption 2.2 and Definition 1.4.3, we have $\mathrm{d}(L_i) = 2k$ and $\mathrm{d}(P_j) = k$. Then

$$\mathrm{d}(C) = \sum_{1 \leq i \leq 4} \mathrm{d}(L_i) - \sum_{1 \leq j \leq 6} \mathrm{d}(P_j) = 4 \cdot 2k - 6 \cdot k = 2k.$$

This is a contradiction.

A type p is said to be trivial, if, for any $a, b, c \models p$, whenever $\{a, b, c\}$ is pairwise independent then it is independent.

Remark 2.4 If $\mathrm{d}(p) = 0$, then p is trivial.

Proof. Take realizations $a, b, c \models p$ such that $\{a, b, c\}$ is pairwise independent. To prove that $\{a, b, c\}$ is independent, it is enough to show that a and bc are d-independent. Since $\mathrm{d}(p) = 0$, we have

$$\mathrm{cl}(a) \cap \mathrm{cl}(bc) = \mathrm{cl}(\emptyset).$$

Since $0 \leq \mathrm{d}(a/bc) \leq \mathrm{d}(a) = 0$, we have

$$\mathrm{d}(a/bc) = d(a).$$

So a and bc are d-independent. Hence $\{a, b, c\}$ is independent.

Lemma 2.5 If $\mathrm{d}(p) = 0$, then we have a contradiction

Proof. Take realizations $a, b \models p$ such that $\{a, b\}$ is independent. Put $c = a \cdot b$, where $\cdot$ is the group operation. Since p is generic, $\{a, b, c\}$ is pairwise independent. So, by Remark 2.4, $\{a, b, c\}$ is independent. This is a contradiction.

Proposition 2.6 Let M be a normal generic structure with a rational coefficient. Then there is no infinite group definable in $\mathrm{Th}(M)$.

Proof. Let M be a normal generic structure with a rational coefficient. Suppose that there would be an infinite group G. By Fact 1.1, M is ω-stable, and so is G. Then we can take a generic type p of G. By Lemma 2.3 and 2.5, we have a contradiction. Hence there is no infinite group definable in $\mathrm{Th}(M)$.

Question 2.7 Is it possible to remove the condition "with a rational coefficient" from Proposition 2.6?

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