IMAGINARIES, STATIONARITY OF TYPES AND STABILITY

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ABSTRACT. Let T be a rosy theory having weak canonical bases with respect to a strict independence relation $\mathop{\smile\!\!\!\!\!\!|}$. Suppose that any type over algebraically closed sets in the real sort is $\mathop{\smile\!\!\!\!\!\!|}$-stationary. THEN T is stable, non-forking relation coincides with $\mathop{\smile\!\!\!\!\!\!|}$, and geometric elimination of imaginaries implies weak elimination of imaginaries.

1. NOTATIONS AND WELL-KNOWN FACTS ON STATIONARY TYPES

Let T be an L-theory, and $\mathcal{M}$ be a sufficiently saturated model of T. We work in $\mathcal{M}^{\mathrm{eq}} := \{a_E : a_E$ is the E-class of a, where a is a finite tuple of M and $E(x,y)$ is a ϕ-definable equivalence relation with $\mathrm{lh}(x) = \mathrm{lh}(y) = \mathrm{lh}(a)$ $\}$. $A, B, C, \cdots$ denote small subsets of $\mathcal{M}^{\mathrm{eq}}$. $a, a', \alpha, \cdots$ denote finite tuples in $\mathcal{M}$ (i.e. real finite tuples). We write $a \subset_\omega \mathcal{M}$. $A \equiv_B A'$ denotes $\mathrm{tp}(A/B) = \mathrm{tp}(A'/B)$ for any $A, A', B \subset \mathcal{M}^{\mathrm{eq}}$. For $i \in \mathcal{M}^{\mathrm{eq}}$, we write $i \in \mathrm{acl}^{\mathrm{eq}}(A)$ if the orbit of i by automorphisms fixing A pointwise is finite, and $i \in \mathrm{dcl}^{\mathrm{eq}}(A)$ if i is fixed by automorphisms fixing A pointwise. Adler [A] introduces a strict independence relation $* \mathop{\smile\!\!\!\!\!\!|}_* *$ for triplet of small subsets of $\mathcal{M}^{\mathrm{eq}}$ satisfying (1)-(9):

1. invariance: If $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $ABC \equiv A'B'C'$, then $A' \mathop{\smile\!\!\!\!\!\!|}_{B'} C'$.
2. monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_B C, A' \subseteq A$ and $C' \subseteq C$, then $A' \mathop{\smile\!\!\!\!\!\!|}_B C'$.
3. right base monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_B D$ and $B \subseteq C \subseteq D$, then $A \mathop{\smile\!\!\!\!\!\!|}_C D$.
4. left transitivity: If $B \subseteq C \subseteq D, D \mathop{\smile\!\!\!\!\!\!|}_C A$ and $C \mathop{\smile\!\!\!\!\!\!|}_B A$, then $D \mathop{\smile\!\!\!\!\!\!|}_B A$.
5. left normality: $A \mathop{\smile\!\!\!\!\!\!|}_B C$ implies $AB \mathop{\smile\!\!\!\!\!\!|}_B C$.
6. extension: If $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $C \subseteq D$, then there exists $A'(\equiv_{BC} A)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_B D$.
7. left finite character: If $a \mathop{\smile\!\!\!\!\!\!|}_B C$ for any $a \subset_\omega A$, then $A \mathop{\smile\!\!\!\!\!\!|}_B C$.
8. local character: For any A there exists a cardinal $\kappa(A)$ such that, for any B there exists $B_0 \subseteq B$ with $|B_0| < \kappa(A)$ and $A \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.
9. anti-reflexivity: $A \mathop{\smile\!\!\!\!\!\!|}_B A$ implies $A \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$.

T is said to be *rosy* if there exists a strict independence relation on $\mathcal{M}^{\mathrm{eq}}$. Theorem 1.14 in [A] shows that (1)-(8) imply symmetry: $A \mathop{\smile\!\!\!\!\!\!|}_B C \Leftrightarrow C \mathop{\smile\!\!\!\!\!\!|}_B A$.

Fact 1.1. Let T be rosy with a strict independence relation $\mathop{\smile\!\!\!\!\!\!|}$. Then we have $A \mathop{\smile\!\!\!\!\!\!|}_B \mathrm{acl}^{\mathrm{eq}}(B)$ for any $A, B \subset \mathcal{M}^{\mathrm{eq}}$.

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Proof. By local character and right base monotonicity, we have $A \mathop{\smile\!\!\!\!\!\!\,|}_B B$. By extension, there exists $A' \equiv_B A$ such that $A' \mathop{\smile\!\!\!\!\!\!\,|}_B \mathrm{acl}^{\mathrm{eq}}(B)$. By invariance, we have $A \mathop{\smile\!\!\!\!\!\!\,|}_B \mathrm{acl}^{\mathrm{eq}}(B)$. □

An algebraically closed set $C \subset \mathcal{M}^{\mathrm{eq}}$ is said to be the $\mathop{\smile\!\!\!\!\!\!\,|}$-weak canonical base of $\mathrm{tp}(a/B)$ if C is the smallest algebraically closed subset of $\mathrm{acl}^{\mathrm{eq}}(B)$ with $a \mathop{\smile\!\!\!\!\!\!\,|}_C B$. $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\!\,|}}(a/B)$ denotes the weak $\mathop{\smile\!\!\!\!\!\!\,|}$-weak cononical base of $\mathrm{tp}(a/B)$. We say that a rosy T has $\mathop{\smile\!\!\!\!\!\!\,|}$-weak canonical bases if there exists $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\!\,|}}(a/B)$ for any $a \subset_\omega \mathcal{M}$ and $B \subset \mathcal{M}^{\mathrm{eq}}$. If T has $\mathop{\smile\!\!\!\!\!\!\,|}$-weak canonical bases, then $\mathop{\smile\!\!\!\!\!\!\,|} = \mathop{\smile\!\!\!\!\!\!\,|}^{\mathfrak{p}}$ by Theorem 3.3 in [A]. We say that T has geometric elimination of imaginaries (GEI) if there exists $\alpha \subset_\omega \mathcal{M}$ such that $a_E \in \mathrm{acl}^{\mathrm{eq}}(\alpha)$ and $\alpha \in \mathrm{acl}^{\mathrm{eq}}(a_E)$ for any $a \subset_\omega \mathcal{M}$ and any ϕ-definable equivalence relation $E(x, y)$ in T. We say that T has weak elimination of imaginaries (WEI) if T has GEI with $a_E \in \mathrm{dcl}^{\mathrm{eq}}(\alpha)$ in the above definition of GEI.

Definition 1.2. Let T be rosy with a strict independence relation $\mathop{\smile\!\!\!\!\!\!\,|}$, and $a \subset_\omega \mathcal{M}, A \subset \mathcal{M}^{\mathrm{eq}}$. We say $\mathrm{tp}(a/A)$ is $\mathop{\smile\!\!\!\!\!\!\,|}$-stationary if for any $a' \subset_\omega \mathcal{M}, B = \mathrm{acl}^{\mathrm{eq}}(B) \subset \mathcal{M}^{\mathrm{eq}}$ such that $A \subset B$, $a \equiv_A a', a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_A B$, we have $a \equiv_B a'$.

The following is also well-known, which appeared in foot note on pp.20 in [Y2].

Fact 1.3. *Let T be any rosy theory with a strict independence relation $\mathop{\smile\!\!\!\!\!\!\,|}$.*

(1) *If $\mathrm{tp}(a/A)$ is $\mathop{\smile\!\!\!\!\!\!\,|}$-stationary and $i \in \mathrm{dcl}^{\mathrm{eq}}(a)$, then $\mathrm{tp}(i/A)$ is also $\mathop{\smile\!\!\!\!\!\!\,|}$-stationary.*
(2) *If $\mathrm{tp}(i/A)$ is $\mathop{\smile\!\!\!\!\!\!\,|}$-stationary, $i \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $i \in \mathrm{dcl}^{\mathrm{eq}}(B)$, then $i \in \mathrm{dcl}^{\mathrm{eq}}(A)$.*

Proof. For (1): Suppose that $i \equiv_A i', i \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $i' \mathop{\smile\!\!\!\!\!\!\,|}_A B$. We need to show $i \equiv_B i'$. By $i \in \mathrm{dcl}^{\mathrm{eq}}(a)$ and compactness, there exists a ϕ-definable function f such that $f(a) = i$. We may assume that $a \mathop{\smile\!\!\!\!\!\!\,|}_{Ai} B$ by extension, so as $i \mathop{\smile\!\!\!\!\!\!\,|}_A B$, we have $a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ by left transitivity, left normality and monotonicity. Take a' with $ia \equiv_A i'a'$. Again we may assume $a' \mathop{\smile\!\!\!\!\!\!\,|}_{Ai'} B$ by extension, so $a' \mathop{\smile\!\!\!\!\!\!\,|}_A B$ follows. As $a \equiv_A a', a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_A B$, by $\mathop{\smile\!\!\!\!\!\!\,|}$-stationarity of $\mathrm{tp}(a/A)$, we have $a \equiv_B a'$. Since $i = f(a), i' = f(a')$, we see $i \equiv_B i'$, as desired.
For(2): Note that $i \in \mathrm{acl}^{\mathrm{eq}}(A)$. Suppose that $i' \equiv_A i$. As $i' \in \mathrm{acl}^{\mathrm{eq}}(A)$ and Remark 1.1 and symmetry, $i' \mathop{\smile\!\!\!\!\!\!\,|}_A B$ follows. By $\mathop{\smile\!\!\!\!\!\!\,|}$-stationarity of $\mathrm{tp}(i/A)$, we see $i \equiv_B i'$, so $i = i'$ by $i \in \mathrm{dcl}^{\mathrm{eq}}(B)$. □

2. Stability

Remark 2.1. *If T has GEI, stationarity of types over algebraically closed sets in the real sort implies stationarity of types over algebraically closed sets in imaginary sorts.*

Proof. Take $a \subset_\omega M, A = \mathrm{acl}^{\mathrm{eq}}(A) \subset \mathcal{M}^{\mathrm{eq}}$ arbitrarily. Suppose that $A \subset B = \mathrm{acl}^{\mathrm{eq}}(B) \subset \mathcal{M}^{\mathrm{eq}}, a \equiv_A a', a \mathop{\smile\!\!\!\!\!\!\,|}_A B$ and $a' \mathop{\smile\!\!\!\!\!\!\,|}_A B$. We need to show $a \equiv_B a'$. By GEI, there exists $A_0 = \mathrm{acl}(A_0), B_0 = \mathrm{acl}(B_0) \subset \mathcal{M}$ such that $\mathrm{acl}^{\mathrm{eq}}(A_0) = A, \mathrm{acl}^{\mathrm{eq}}(B_0) = B$. As $A_0 \subseteq A$, we have $a \equiv_{A_0} a'$. By Remark 1.1, we have $a \mathop{\smile\!\!\!\!\!\!\,|}_{A_0} \mathrm{acl}^{\mathrm{eq}}(B_0), a' \mathop{\smile\!\!\!\!\!\!\,|}_{A_0} \mathrm{acl}^{\mathrm{eq}}(B_0)$. By stationarity over algebraically closed sets in the real sorts, we see $a \equiv_{\mathrm{acl}^{\mathrm{eq}}(B_0)=B} a'$. □

The following proposition is pointed out by Byunghan Kim at RIMS model theory workshop 2020.

Proposition 2.2. *Let T be rosy.*

(1) *Suppose that* $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ *is* $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}$*-stationary for any finite real tuple* $a \subset_\omega \mathcal{M}$ *and small* $A \subset \mathcal{M}^{\mathrm{eq}}$*. Then T is stable and thorn-forking coincides with forking.*

(2) *Suppose that* $\mathrm{tp}(a/\mathrm{acl}(A))$ *is* $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}$*-stationary for any finite real tuple* $a \subset_\omega \mathcal{M}$ *and small* $A \subset \mathcal{M}$*. Then T is stable and thorn-forking coincides with forking.*

(3) *Suppose that* $\mathrm{tp}(a/\mathrm{bbd}^{\mathrm{heq}}(A))$ *is* $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}$*-stationary for any finite real tuple* $a \subset_\omega \mathcal{M}$ *and small* $A \subset \mathcal{M}^{\mathrm{heq}}$*. (i.e. for any* $B = \mathrm{bbd}^{\mathrm{heq}}(B) \supset \mathrm{bbd}^{\mathrm{heq}}(A)$*, if* $a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_{\mathrm{bbd}^{\mathrm{heq}}(A)} B, a' \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_{\mathrm{bbd}^{\mathrm{heq}}(A)} B$ *and* $a \equiv_{\mathrm{bbd}^{\mathrm{heq}}(A)} a'$ *imply* $a \equiv_B a'$*.) Then T is stable and thorn-forking coincides with forking.*

Proof. (1): If $M \models T$, then we have $\mathrm{acl}^{\mathrm{eq}}(M) = \mathrm{dcl}^{\mathrm{eq}}(M)$. In particular $\mathrm{tp}(a/M) = \mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(M)) = \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(M))$ and $a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_M \mathrm{acl}^{\mathrm{eq}}(M)$ by Fact 1.1. As $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(M)) = \mathrm{tp}(a/M)$ is $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}$-stationary, T has the independence theorem over models with respect to thorn-forking. By Theorem 2.7 (2) and Fact 2.4 in [EO], for any finite real tuple a and any small $A \subset \mathcal{M}^{\mathrm{eq}}$ there exists $A_0 \subseteq A$ with $|A_0| \leq |T|$ and $a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_{A_0} A$.(Otherwise there would exist a chain of thorn-foking extension of length $|T|^+$, then $\mathfrak{p}_{\psi,\theta,k}$-rank drops infinitely many times for a fixed triplet (ψ, θ, k), so T would not be rosy.) By Kim-Pillay criterion:Theorem 3.3.1 in [K], thorn-forking coincides with forking. As any type of a finite real tuple over a model M is forking-stationary, by Corollary 2.3.17 in [W], it is definable over M. By Theorem 2.15 in [P], we see that T is stable.
(2): A similar proof of (1) works for (2).
(3): By Remark and Definiton 4.1.4 (2) in [K], note that $\mathrm{bbd}^{\mathrm{heq}}(M) = \mathrm{dcl}^{\mathrm{heq}}(M)$ for any model M of T. In particular $\mathrm{tp}(a/M) = \mathrm{tp}(a/\mathrm{dcl}^{\mathrm{heq}}(M)) = \mathrm{tp}(a/\mathrm{bbd}^{\mathrm{heq}}(M))$ and $a \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_M \mathrm{bbd}^{\mathrm{heq}}(M)$ by the proof of Fact 1.1. Now, a similar proof of (1) works for (3) in T^{heq}. □

3. GEI and WEI in rosy theories

The author's proof to the following proposition is improved by Anand Pillay as follows.

Proposition 3.1. *Let T be a rosy theory having weak canonical bases with respect to a strict independence relation* $\mathop{\smile\!\!\!\!\!\!|}$*. Suppose that any type over algebraically closed set in the real sort is* $\mathop{\smile\!\!\!\!\!\!|}$*-stationary. THEN GEI ⇔ WEI. (⇐ is clear.)*

Proof. Let $i = a_E$ and take b such that $b \equiv_i a$ and $b \mathop{\smile\!\!\!\!\!\!|}_i a$. By GEI, there exist $B \subset \mathcal{M}$ such that $\mathrm{wcb}_{\mathop{\smile\!\!\!\!\!\!|}}(b/ai) = \mathrm{acl}^{\mathrm{eq}}(B)$. Put $C := \mathrm{acl}(B)$.

Note that $C \subseteq \mathrm{acl}^{\mathrm{eq}}(B) \subseteq \mathrm{acl}^{\mathrm{eq}}(i)$, as $b \mathop{\smile\!\!\!\!\!\!|}_i ai$. We also have $b \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{dcl}^{\mathrm{eq}}(ai)$ by Fact 1.1, symmetry, left transitivity and monotonicity. As $\mathrm{tp}(b/C)$ is $\mathop{\smile\!\!\!\!\!\!|}$-stationary and $i \in \mathrm{dcl}^{\mathrm{eq}}(b) \cap \mathrm{dcl}^{\mathrm{eq}}(ai)$, $i \in \mathrm{dcl}^{\mathrm{eq}}(C)$ by Fact 1.3. So we have $C \subseteq \mathrm{acl}^{\mathrm{eq}}(i)$ and $i \in \mathrm{dcl}^{\mathrm{eq}}(C)$, as desired. □

Theorem 3.2. *Let T be a rosy theory having weak canonical bases with respect to a strict independence relation* $\mathop{\smile\!\!\!\!\!\!|}$*. Suppose that any type over algebraically closed* *set in the real sort is $\mathop{\smile\!\!\!\!\!\!\mid}$-stationary. THEN T is stable, non-forking coincides $\mathop{\smile\!\!\!\!\!\!\mid}$, and GEI implies WEI.*

Proof. By Therorem 3.3 in [A], we have $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{p}}$. By Proposition 2.2 (2), T is stable and non-foking coincides with $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{p}}$. By Proposition 3.1, GEI implies WEI. □

Corollary 3.3. *Suppose T has a strict independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ having the intersection property. (i.e. $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_B A$ imply $a \mathop{\smile\!\!\!\!\!\!\mid}_{A\cap B} AB$ for any $a \subset_\omega \mathcal{M}$ and any small subsets of $\mathcal{M}$ such that $A = \mathrm{acl}(A), B = \mathrm{acl}(B)$.) If any type over algebraically closed set in the real sort is $\mathop{\smile\!\!\!\!\!\!\mid}$-stationary, then T is stable with WEI and non-forking coincides with $\mathop{\smile\!\!\!\!\!\!\mid}$ in T.*

Proof. By Proposition 3.4 in [Y3] and Theorem 3.2. □

Remark 3.4. (1) Let T be stable. Suppose that $\mathrm{tp}(a/A) \vdash \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ for any $a, A = \mathrm{acl}(A) \subset \mathcal{M}$. As $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ is stationary by stability of T, we have GEI=WEI in T by Proposition 3.1. (Note that $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ implies $a \mathop{\smile\!\!\!\!\!\!\mid}_A \mathrm{acl}^{\mathrm{eq}}(B)$ by symmetry, Fact 1.1 and left transitivity.) Now we show the stationarity of $\mathrm{tp}(a/A)$ i.e. if $\mathrm{acl}(A) = A \subseteq B = \mathrm{acl}^{\mathrm{eq}}(B), a \equiv_A a', a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_A B$, then $a \equiv_B a'$: Note that $A \subseteq \mathrm{acl}^{\mathrm{eq}}(A) \subseteq B$. By $\mathrm{tp}(a/A) \vdash \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ and $a \equiv_A a'$, we have $a \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} a'$. By the above we have $a \mathop{\smile\!\!\!\!\!\!\mid}_A B, a' \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $a \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} a'$, by stationarity of strong types in stability theory, we have $a \equiv_B a'$.

(2) Let T be a rosy theory with CM-triviality in the real sort. Then T is CM-trivial with GEI and weak canonical bases. (See Theorem 4.2 and Theorem 2.4 in [Y3].) If any type over algebraically closed sets in the real sort is $\mathop{\smile\!\!\!\!\!\!\mid}$-statinary in T, then T is stable with WEI, non-forking coincides with $\mathop{\smile\!\!\!\!\!\!\mid}$ by Theorem 3.2.

(3) Any relational structure M such that $\mathrm{Th}(M)$ is amalgamation over closed sets ARE stable CM-trivial in the real sort and having non-forking stationarity of types over algebraically closed sets in the real sort (see [Y1] and [VY]). So, $\mathrm{Th}(M)$ has WEI by (2).

(4) D.M.Evans gives an example of $\aleph_0$-categorical CM-trivial SU-rank one graph which does not have GEI. See [E] and [Y3]. This example is NOT CM-trivial in the real sort.

(5) If T is a theory having free amalgamation ternary relation, then T has NSOP_4, elimination of hyperimaginaries and WEI. See Theorem 4.4 and Theorem 5.6 in [C].

(6) We said that a definable set $D \subset \mathcal{M}$ is stably embedded if for any $n < \omega$ and $\mathcal{M}$-definable subset X of $\mathcal{M}^n$, there exists D-definable subset Y of $\mathcal{M}^n$ such that $X \cap D^n = Y \cap D^n$. The following are equivalent by Lemma 2.3.2 in [CH].
   (a) $D \subset \mathcal{M}$ is stably embedded and D has WEI.
   (b) $\mathrm{tp}(i/\mathrm{acl}^{\mathrm{eq}}(i) \cap D) \vdash \mathrm{tp}(i/D)$ for any $i \in \mathcal{M}^{\mathrm{eq}}$.

(7) Let D be a strongly minimal set with infinite $D \cap \mathrm{acl}(\phi)$. Then D has WEI. See Lemma 1.6 in [P1].

(8) D.M.Hoffmann points out a simple theory CCMA(=Compact Complex Manifolds with an Automorphism) having GEI and finite coding in Theorem 4.3.6 and Lemma 4.3.7 in [Ho]. But CCMA does not having EI by

Corollary 3.6 in [BHM]. As WEI+finite coding=EI by Proposition 1.6 in [CF], CCMA does not have WEI.

(9) The N-dimensional free pseudospace is ω-stable and N-ample but not $(N+1)$-ample with WEI. See [T] and [BMZ].

(10) The theory of the free group is stable, n-ample for all $n < \omega$ and has GEI up to adding some reasonable sorts. See Theorem 4.6 in [S].

(11) $\mathrm{ACF}, \mathrm{DCF}, \mathrm{SCF}^*_{\mathrm{e}}(\mathrm{e} < \omega)$ eliminate imaginaries. $\mathrm{SCF}_{\mathrm{e}}$ for $e \in \omega \cup \{\infty\}$ in the language of fields does not eliminate imaginaries and has finite coding, so it does not have WEI. See [MMP].

Question 3.5. Does $\mathrm{SCF}_{\mathrm{e}}$ for $e \in \omega \cup \{\infty\}$ in the language of fields geometrically eliminate imaginaries? Find a stable theory (moreover, strongly minimal set) having GEI not having WEI.

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