SOME REMARKS ON TREE PROPERTIES

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1. Introduction

Since it was first introduced in [3], it is still unknown whether the class of SOP_1 theories and the class of SOP_2 theories are the same. Recently in [1], it has shown that SOP_1 and SOP_2 differ at the level of formulas. This means that there exists a formula witnessing SOP_1 but any conjunction of the formula does not witness SOP_2. In this note, we give a slightly stronger example. We construct a theory which has SOP_1, but every quantifier free formula does not witness SOP_2.

2. Preliminaries and notations

Notation 2.1. Let κ and λ be cardinals.

(i) By κ^λ we mean the set of all functions from λ to κ.
(ii) By $\kappa^{<\lambda}$ we mean $\bigcup_{\alpha<\lambda}\kappa^\alpha$ and call it a tree. If $\kappa = 2$, we call it a binary tree. If $\kappa \geq \omega$, then we call it an infinitary tree.
(iii) By $\emptyset$ or $\langle\rangle$, we mean the empty string in $\kappa^{<\lambda}$, which means the empty set (recall that every function can be regarded as a set of ordered pairs).

Let $\eta, \nu \in \kappa^{<\lambda}$.

(iv) By $\eta \trianglelefteq \nu$ we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say η and ν are comparable.
(v) By $\eta \perp \nu$ we mean that $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$. We say η and ν are incomparable if $\eta \perp \nu$.
(vi) By $\eta \wedge \nu$ we mean the maximal $\xi \in \kappa^{<\lambda}$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.
(vii) By $l(\eta)$ we mean the domain of η.
(viii) By $\eta <_{len} \nu$ we mean $l(\eta) < l(\nu)$.
(ix) By $\eta <_{lex} \nu$ we mean that $\eta \trianglelefteq \nu$, or $\eta \perp \nu$ and $\eta(l(\eta \wedge \nu)) < \nu(l(\eta \wedge \nu))$.
(x) By $\eta^\frown\nu$ we mean $\eta \cup \{(i + l(\eta), \nu(i)) : i < l(\nu)\}$.

Let $i, i_0, ..., i_{n-1} \in \kappa$.

(xi) By $\langle i_0...i_{n-1}\rangle$ we mean the function $\zeta \in \kappa^n$ such that $\zeta(k) = i_k$ for all $k \in n$. Sometimes we just write it $i_0...i_{n-1}$ if there is no confusion.
(xii) By $\langle i^{(n)}\rangle$ we mean the function $\zeta \in \kappa^n$ such that $\zeta(k) = i$ for all $k \in n$. Sometimes we just write it $i^{(n)}$ or i^n if there is no confusion. $\langle i^{(0)}\rangle$ is defined by $\langle\rangle$.

Definition 2.2. Let $\varphi(x, y)$ be an $\mathcal{L}$-formula.

(i) $\varphi(x, y)$ is said to be having the *tree property* (TP) if there exists a tree-indexed set $\langle a_\eta\rangle_{\eta\in\omega^{<\omega}}$ of parameters and $k \in \omega$ such that
- $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in \omega^\omega$ (path consistency),
- $\{\varphi(x, a_{\eta^\frown i})\}_{i\in\omega}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$.

(ii) $\varphi(x,y)$ is said to be having the *tree property of the first kind* (TP_1) if there is a tree-indexed set $\langle a_\eta\rangle_{\eta\in\omega^{<\omega}}$ of parameters such that
- $\{\varphi(x,a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta\in\omega^\omega$,
- $\{\varphi(x,a_\eta),\varphi(x,a_\nu)\}$ is inconsistent for all $\eta\perp\nu$.

(iii) $\varphi(x,y)$ is said to be having the *tree property of the second kind* (TP_2) if there is an array-indexed $(a_{i,j})_{i,j\in\omega}$ of parameters such that
- $\{\varphi(x,a_{n,\eta(n)})\}_{n\in\omega}$ is consistent for all $\eta\in\omega^\omega$,
- $\{\varphi(x,a_{i,j}),\varphi(x,a_{i,k}))\}$ is inconsistent for all $i,j,k\in\omega$ with $j\neq k$.

(iv) $\varphi(x,y)$ is said to be having the *1-strong order property* (SOP_1) if there is a binary-tree-indexed $\langle a_\eta\rangle_{\eta\in 2^{<\omega}}$ of parameters such that
- $\{\varphi(x,a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta\in {}^\omega 2$,
- $\{\varphi(x,a_{\eta^\frown 1}),\varphi(x,a_{\eta^\frown 0^\frown\nu})\}$ is inconsistent for all $\eta,\nu\in 2^{<\omega}$.

(v) $\varphi(x,y)$ is said to be having the *2-strong order property* (SOP_2) if there is a binary-tree-indexed $\langle a_\eta\rangle_{\eta\in 2^{<\omega}}$ of parameters such that
- $\{\varphi(x,a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta\in {}^\omega 2$,
- $\{\varphi(x,a_\eta),\varphi(x,a_\nu)\}$ is inconsistent for all $\eta\perp\nu$.

(vi) We say a theory has TP if there is a formula having TP with respect to its monster model of the theory. Sometimes we say that the theory is TP, and we call the theory an TP theory. We define TP_1 theory, TP_2 theory, SOP_1 theory, and SOP_2 theory in the same manner.

(vii) We say a theory is NTP if the theory is not TP, and we call the theory NTP theory. We define NTP_1 theory, NTP_2 theory, $NSOP_1$ theory, and $NSOP_2$ theory in the same manner.

Fact 2.3. (i) *A theory has TP if and only if it has TP_1 or TP_2.*
(ii) *If a theory has SOP_2, then the theory has SOP_1.*
(iii) *If a theory has SOP_1, then the theory has TP.*
(iv) *A theory has SOP_2 if and only if the theory has TP_1.*

Let $\mathcal{L}_{str}=\{\trianglelefteq,<_{lex},\wedge\}$ be languages where $\trianglelefteq$, $<_{lex}$ are binary relation symbols, and $\wedge$ is a binary function symbol. Then for cardinals κ and λ, a tree $\kappa^{<\lambda}$ can be regarded as an $\mathcal{L}_{str}$-structure whose interpretations of $\trianglelefteq,<_{lex},\wedge$ follow Notation 2.1.

Definition 2.4. Let $\overline{\eta}=(\eta_0,...,\eta_n)$ and $\overline{\nu}=(\nu_0,...,\nu_n)$ be finite tuples of $\kappa^{<\lambda}$.
(i) By $\mathrm{qftp}_{str}(\overline{\eta})$ we mean the set of quantifier free $\mathcal{L}_{str}$-formulas $\varphi(\overline{x})$ such that $\kappa^{<\lambda}\models\varphi(\overline{\eta})$.
(ii) By $\overline{\eta}\sim_{str}\overline{\nu}$ we mean $\mathrm{qftp}_{str}(\overline{\eta})=\mathrm{qftp}_{str}(\overline{\nu})$.

Let $\mathcal{L}$ be a language, T be a complete $\mathcal{L}$-theory, $\mathbb{M}$ be a monster model of T, and $(a_\eta)_{\eta\in\kappa^{<\lambda}}$, $(b_\eta)_{\eta\in\kappa^{<\lambda}}$ tree-indexed sets of parameters from $\mathbb{M}$. For $\overline{\eta}=(\eta_0,...,\eta_n)$, we write $\overline{a}_{\overline{\eta}}$ to denote $(a_{\eta_0},...,a_{\eta_n})$. For any finite set of $\mathcal{L}$-formulas Δ and a type Γ, we write Γ_Δ to denote $\{\varphi\in\Gamma:\varphi\in\Delta\}$. By $\overline{a}_{\overline{\eta}}\equiv_{\Delta,A}\overline{b}_{\overline{\nu}}$ we mean $\mathrm{tp}_\Delta(\overline{a}_{\overline{\eta}}/A)=\mathrm{tp}_\Delta(\overline{b}_{\overline{\nu}}/A)$.
(iii) We say $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ is strongly indiscernible (str-indiscernible) if $\mathrm{tp}(\overline{a}_{\overline{\eta}})=\mathrm{tp}(\overline{a}_{\overline{\nu}})$ for all $\mathrm{qftp}_{str}(\overline{\eta})=\mathrm{qftp}_{str}(\overline{\nu})$.
(iv) We say $(b_\eta)_{\eta\in\kappa^{<\lambda}}$ is strongly based (str-based) on $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ if for all $\overline{\eta}$ and a finite set of $\mathcal{L}$-formulas Δ, there is $\overline{\nu}$ such that $\overline{\eta}\sim_{str}\overline{\nu}$ and $\overline{b}_{\overline{\eta}}\equiv_\Delta\overline{a}_{\overline{\nu}}$.

Fact 2.5. *Let a tree-indexed set $(a_\eta)_{\eta\in\omega^{<\omega}}$ be given. Then there is a str-indiscernible $(b_\eta)_{\eta\in\omega^{<\omega}}$ which is str-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$.*

The proof can be found in [5], [4], and [8]. The above statement is called the modeling property of strong indiscernibility (str-modeling property).

Definition 2.6. Let κ and λ be cardinals.

(i) We say a subset X of $\kappa^{<\lambda}$ is an *antichain* if the elements of X are pairwise incomparable (*i.e.*, $\eta \perp \nu$ for all $\eta, \nu \in X$).

(ii) We say a formula $\varphi(x, y)$ has *antichain tree property* (ATP) if there exists a tree indexed set of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ such that
- for any antichain X in $2^{<\omega}$, the set $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent, and
- for any $\eta, \nu \in 2^{<\omega}$, if $\eta \trianglelefteq\!\!\!\!/\; \nu$, then $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

We say a theory has ATP if there exists a formula having ATP. If a theory does not have ATP, then we say the theory has NATP.

Remark 2.7. [1, Proposition 4.4, 4.6] If a formula witnesses ATP, then it witnesses TP_2 and SOP_1. In particular, if a theory has ATP, then it has TP_2 and SOP_1.

Remark 2.8. Thus if we want to show that SOP_1 and SOP_2 are not the same, then it is enough to show that there exists a theory which is ATP and NSOP_2.

Fact 2.9. [1, Section 6] *There exists a theory which has a witness of ATP, and any conjunction of the witness does not witness SOP_2.*

3. An ATP theory without quantifier free SOP_2 witness

We construct a structure of relational language whose theory has a quantifier free formula $\varphi(x, y)$ which forms an antichain tree, and every quantifier free formula in $\mathcal{L}$ does not witness SOP_2.

3.1. **Construction.**

Definition 3.1. Let $n \leq \omega$. An antichain $X \subseteq 2^{<n}$ is called a maximal antichain if there is no antichain $Y \subseteq 2^{<n}$ such that $X \subsetneq Y$.

We begin the construction with language $\mathcal{L} = \{R\}$ where R is a binary relation symbol. For each $n \in \omega$, let $\alpha_n \in \omega$ be the number of all maximal antichains in $2^{<n}$, and β_n be the set of all maximal antichains in $2^{<n}$. We can choose a bijection from α_n to β_n for each $n \in \omega$, say μ_n. For each $n \in \omega$, let A_n and B_n be finite sets such that $|A_n| = \alpha_n$ and $|B_n| = |2^{<n}|$. Their elements are denoted by

$$A_n = \{a^n_l : l < \alpha_n\}, \qquad B_n = \{b^n_\eta : \eta \in 2^{<n}\}.$$

Let C_n be the disjoint union of A_n and B_n for each $n \in \omega$. For each $n \in \omega$, let $\mathbb{C}_n$ be an $\mathcal{L}$-structure such that $\mathbb{C}_n = \langle C_n; R^{\mathbb{C}_n} \rangle$, where $R^{\mathbb{C}_n} = \{\langle a^n_l, b^n_\eta \rangle \in A_n \times B_n : \eta \in \mu_n(l)\}$.

For each $n \in \omega$, let ε_n be a map from $\alpha_n \cup 2^{<n}$ to $\alpha_{n+1} \cup 2^{<n+1}$ which maps $x \mapsto x$ for all $x \in \alpha_n \cup 2^{<n}$, and define $\varepsilon^*_n : C_n \to C_{n+1}$ by $a^n_l \mapsto a^{n+1}_{\varepsilon_n(l)}$ and $b^n_\eta \mapsto b^{n+1}_{\varepsilon_n(\eta)}$. Then ε^*_n is an embedding. Thus we can regard $\mathbb{C}_n$ as a substructure of $\mathbb{C}_{n+1}$ with respect to ε^*_n.

Let $\mathcal{P}_n$ be the power set of α_n, $\{\Gamma_i\}_{i<2^{\alpha_n}}$ an enumeration of $\mathcal{P}_n$, and p_n the cardinality of $\mathcal{P}_n$. For each $n \in \omega$, let δ_n be a map from $\alpha_n \cup 2^{<n}$ to $\alpha_{n+p_n} \cup 2^{<n+p_n}$

which maps $\eta \mapsto 0^{p_n}\frown\eta$ for all $\eta \in 2^{<n}$ and $l \mapsto \delta_n(l)$ for all $l \in \alpha_n$ where

$$\delta_n(l) = \mu_{n+p_n}^{-1}\begin{pmatrix} (0^{p_n}\frown\mu_n(l) \\ \cup\{0^{p_n-i-1}\frown\langle 1\rangle : l \notin \Gamma_i\} \\ \cup\{0^{p_n-i-1}\frown\langle 10\rangle : l \in \Gamma_i\} \\ \cup\{0^{p_n-i-1}\frown\langle 11\rangle : l \in \Gamma_i\} \end{pmatrix}.$$

Define $\delta_n^* : C_n \to C_{n+p_n}$ by $a_l^n \mapsto a_{\delta_n(l)}^{n+p_n}$ and $b_\eta^n \mapsto b_{\delta_n(\eta)}^{n+p_n}$. Then δ_n^* is an embedding. Thus we can regard $\mathbb{C}_n$ as a substructure of $\mathbb{C}_{n+p_n}$ with respect to δ_n^*.

Then we obtain a chain of $\mathcal{L}$-structures,

$$\mathbb{C}_1 \xhookrightarrow{\varepsilon_1^*} \mathbb{C}_2 \xhookrightarrow{\delta_2^*} \mathbb{C}_{2+p_2} \xhookrightarrow{\varepsilon_{2+p_2}^*} \cdots \mathbb{C}_n \xhookrightarrow{\varepsilon_n^*} \mathbb{C}_{n+1} \xhookrightarrow{\delta_{n+1}^*} \mathbb{C}_{n+1+p_{n+1}} \cdots .$$

Thus we choose $\mathbb{C}'_n$ recursively as follows

$$\begin{aligned} &\mathbb{C}'_1 = \mathbb{C}_1, \\ &\mathbb{C}'_2 = \mathbb{C}_2, \\ &\mathbb{C}'_{2i+1} = \mathbb{C}_{k+p_k} \text{ where } \mathbb{C}_k = \mathbb{C}'_{2i}, \\ &\mathbb{C}'_{2i+2} = \mathbb{C}_{k+1} \text{ where } \mathbb{C}_k = \mathbb{C}'_{2i+1}, \end{aligned}$$

and for each $i \in \omega$, let $A'_i = A_k$ and $B'_i = B_k$ where $\mathbb{C}'_i = \mathbb{C}_k$. Let $\mathbb{C} = \bigcup_{i\in\omega}\mathbb{C}'_i$, $A = \bigcup_{i\in\omega} A'_i$, $B = \bigcup_{i\in\omega} B'_i$, and C be the universe of $\mathbb{C}$.

3.2. **Verification.** In this section we show that $\mathrm{Th}(\mathbb{C})$ has ATP and there is no quantifier free formula in $\mathcal{L}$ which has SOP_2 modulo $\mathrm{Th}(\mathbb{C})$. First we check some properties of $\mathrm{Th}(\mathbb{C})$. Recall that $\{\nu \in 2^{<\omega} : \nu \trianglelefteq \eta\}$ is linearly ordered by $\trianglelefteq$ for each η. This fact will be used frequently to prove the following propositions and remarks.

Remark 3.2. It is clear that either $\mathbb{C} \models \exists x R(x,c)$ or $\mathbb{C} \models \exists y R(c,y)$, for all $c \in C$. In fact $\mathbb{C} \models \exists x R(x,c)$ if and only if $c \in B$, and $\mathbb{C} \models \exists y R(c,y)$ if and only if $c \in A$.

Proposition 3.3. *$R(x,y)$ forms an antichain tree in* $\mathrm{Th}(\mathbb{C})$.

Proof. By compactness, it is enough to show that for any $n \in \omega$, there exists $\langle c_\eta\rangle_{\eta\in 2^{<n}}$ such that $\{R(x,c_\eta) : \eta \in X\}$ is consistent if and only if X is an antichain in $2^{<n}$. Fix $n \in \omega$, and let $c_\eta = b_\eta^k$ for each $\eta \in 2^{<n}$, where $\mathbb{C}_k = \mathbb{C}'_n$. Note that $k \geq n$.

First we show that if $X \subseteq 2^{<n}$ is an antichain, then $\{R(x,c_\eta) : \eta \in X\}$ is consistent. Since $2^{<n} \subseteq 2^{<k}$, X can be regarded as an antichain in $2^{<k}$. Hence, there exists a maximal antichain $\bar{X} \subseteq 2^{<k}$ containing X. Let $l = \mu_k^{-1}(\bar{X})$. Then $\langle a_l^k, c_\eta\rangle \in R^{\mathbb{C}_k}$ for all $\eta \in X$ since $\langle a_l^k, c_\eta\rangle = \langle a_l^k, b_\eta^k\rangle$ and $\eta \in \bar{X} = \mu_k(\mu_k^{-1}(\bar{X})) = \mu_k(l)$ for all $\eta \in X$. Thus $\{R(x,c_\eta) : \eta \in X\}$ is consistent.

For the converse, suppose $X \subseteq 2^{<n}$ is not an antichain. Then there exist $\eta,\nu \in X$ such that $\eta \not\perp \nu$ in $2^{<n}$. Thus $\eta \not\perp \nu$ in $2^{<k}$. If $\{R(x,c_\eta) : \eta \in X\}$ is consistent, then $\{R(x,b_\eta^k), R(x,b_\nu^k)\}$ is consistent. Hence there exists $a \in A$ such that $\mathbb{C} \models R(a,b_\eta^k)\wedge R(a,b_\nu^k)$. By the construction of $\mathbb{C}$, there exists $k' \in \omega$ such that $a \in A_{k'}$ and we may assume $k = k'$. Hence there exists $l < \alpha_k$ such that $\mathbb{C} \models R(a_l^k, b_\eta^k) \wedge R(a_l^k, b_\nu^k)$. Thus $\eta,\nu \in \mu_k(l)$. There exists an antichain in $2^{<k}$ containing η and ν but it is not possible since $\eta \not\perp \nu$. □

Proposition 3.4. *There is no* $b_0, b_1, b_2, b_3 \in B$ *such that*

$$\begin{aligned}\mathbb{C} \models \quad & b_0 \neq b_1 \wedge b_2 \neq b_3 \\ & \wedge \exists x(R(x,b_0) \wedge R(x,b_1)) \wedge \exists x(R(x,b_2) \wedge R(x,b_3)) \\ & \wedge \neg\exists x(R(x,b_0) \wedge R(x,b_2)) \wedge \neg\exists x(R(x,b_0) \wedge R(x,b_3)) \\ & \wedge \neg\exists x(R(x,b_1) \wedge R(x,b_2)) \wedge \neg\exists x(R(x,b_1) \wedge R(x,b_3)).\end{aligned}$$

Proof. Suppose such b_0, b_1, b_2, b_3 exist. We may assume they are $b^n_{\eta_0}, b^n_{\eta_1}, b^n_{\eta_2}, b^n_{\eta_3}$ for some $n \in \omega$ and $\eta_0, \eta_1, \eta_2, \eta_3 \in 2^{<n}$. Since $\{R(x, b^n_{\eta_0}),\ R(x, b^n_{\eta_1})\}$ and $\{R(x, b^n_{\eta_2}), R(x, b^n_{\eta_3})\}$ are consistent, $b^n_{\eta_0} \perp b^n_{\eta_1}$ and $b^n_{\eta_2} \perp b^n_{\eta_3}$ in $2^{<n}$. Since $\{R(x, b^n_{\eta_0}), R(x, b^n_{\eta_2})\}$, $\{R(x, b^n_{\eta_0}),\ R(x, b^n_{\eta_3})\}$, $\{R(x, b^n_{\eta_1}),\ R(x, b^n_{\eta_2})\}$, $\{R(x, b^n_{\eta_1}),\ R(x, b^n_{\eta_3})\}$ are inconsistent, $b^n_{\eta_0} \not\perp b^n_{\eta_2}$, $b^n_{\eta_0} \not\perp b^n_{\eta_3}$, $b^n_{\eta_1} \not\perp b^n_{\eta_2}$, $b^n_{\eta_1} \not\perp b^n_{\eta_3}$. Thus $b^n_{\eta_2}, b^n_{\eta_3} \trianglelefteq b^n_{\eta_0} \wedge b^n_{\eta_1}$. Since $\{\nu \in 2^{<n} : \nu \trianglelefteq \nu' \wedge \nu''\}$ is linearly ordered by $\trianglelefteq$ for all $\nu' \neq \nu''$, we have $b^n_{\eta_2} \not\perp b^n_{\eta_3}$ and it is a contradiction. □

Proposition 3.5. *For all* $b_0, ..., b_n \in B$*, if* $\mathbb{C} \models \neg\exists x(R(x,b_0) \wedge \cdots \wedge R(x,b_n))$*, then there exist* $i < j \leq n$ *such that* $\mathbb{C} \models \neg\exists x(R(x,b_i) \wedge R(x,b_j))$*.*

Proof. Suppose $b_0, ..., b_n \in B$ and $\mathbb{C} \models \exists x(R(x,b_i) \wedge R(x,b_j))$ for all $i, j \leq n$. We may assume that there exists $k \in \omega$ such that for each $i < n$, there exists $\eta_i \in 2^{<k}$ such that $b_i = b^k_{\eta_i}$. As we observed in proof of Proposition 3.3, $\langle R(x,y), \langle b^k_\eta \rangle_{\eta \in 2^{<k}} \rangle$ forms an antichain tree with height k. Since $\{R(x, b^k_{\eta_i}), R(x, b^k_{\eta_j})\}$ is consistent for each $i, j < n$, the set of indices $\{\eta_0, ..., \eta_n\}$ is pairwisely incomparable. Thus $\{\eta_0, ..., \eta_n\}$ is an antichain in $2^{<k}$. Thus $\{(R(x, b^k_{\eta_0}), ..., R(x, b^k_{\eta_{n-1}})\}$ is consistent in $\mathbb{C}_k$, and hence it is consistent in $\mathbb{C}$. Therefore $\mathbb{C} \models \exists x(R(x,b_0) \wedge \cdots \wedge R(x,b_n))$. □

Proposition 3.6. *For all* $a_0, ..., a_n, a'_0, ..., a'_m \in A$*, if* $a_i \neq a'_j$ *for each* $i \leq n$*,* $j \leq m$*, then* $\mathbb{C} \models \exists y(R(a_0,y) \wedge \cdots \wedge R(a_n,y) \wedge \neg R(a'_0,y) \wedge \cdots \wedge \neg R(a'_m,y))$*.*

Proof. We may assume $a_0, ..., a_n, a'_0, ..., a'_m \in A_k$ for some $k \in \omega$, where $\mathbb{C}_k = \mathbb{C}'_{2d}$ for some $i \in \omega$. Hence there are $l_0, ..., l_n, l'_0, ..., l'_m < \alpha_k$ such that $a_0 = a^k_{l_0}, \ldots, a_n = a^k_{l_n}$ $a'_0 = a^k_{l'_0}$, ... , $a'_m = a^k_{l'_m}$ and $l_i \neq l'_j$ for each i and j. Let $\{\Gamma_i\}_{i<2^{\alpha_k}}$ be the enumeration of $\mathcal{P}_k$, the power set of α_k. Then there exists e such that $\Gamma_e = \{l'_0, ..., l'_m\}$. By the construction of δ_k, the maximal antichain $\mu_{k+p_k}(\delta_k(l'_j))$ in β_{k+p_k} does not contain $0^{(p_k-e-1)\frown}\langle 1 \rangle$ for all j. On the other hand, $\mu_{k+p_k}(\delta_k(l_i))$ contains $0^{p_k-e-1\frown}\langle 1 \rangle$ for all i since $l_i \notin \Gamma_e$. Thus we have

$$\mathbb{C} \models R(a^{k+p_k}_{\delta_k(l_i)}, b^{k+p_k}_{0^{p_k-e-1\frown}\langle 1 \rangle})$$

for all i, and

$$\mathbb{C} \models \neg R(a^{k+p_k}_{\delta_k(l'_j)}, b^{k+p_k}_{0^{p_k-e-1\frown}\langle 1 \rangle})$$

for all j. Since δ^*_k is an embedding from $\mathbb{C}'_{2d} = \mathbb{C}_k$ to $\mathbb{C}'_{2d+1} = \mathbb{C}_{k+p_k}$, we can say that there is $b \in B$ such that $\mathbb{C} \models R(a^k_{l_i}, b)$ for all i, and $\mathbb{C} \models \neg R(a^k_{l'_j}, b)$ for all j. This completes the proof. □

Proposition 3.7. *For all* $b_0, ..., b_n, b'_0, ..., b'_m \in B$*, if* $b_i \neq b'_j$ *for each* $i \leq n, j \leq m$*, and* $\mathbb{C} \models \exists x(R(x,b_i) \wedge R(x,b_j))$ *for all* i, j*, then* $\mathbb{C} \models \exists x(R(x,b_0) \wedge \cdots \wedge R(x,b_n) \wedge \neg R(x,b'_0) \wedge \cdots \wedge \neg R(x,b'_m))$*.*

Proof. As in Proposition 3.6, we may assume $b_0, ..., b_n, b'_0, ...b'_m \in B_k = B'_{2d+1}$ for some $k, d \in \omega$, hence there are $\eta_0, ..., \eta_n, \eta'_0, ..., \eta'_m \in 2^{<\kappa}$ such that $b_0 = b^k_{\eta_0}, ..., b_n = b^k_{\eta_n}, b'_0 = b^k_{\eta'_0}, ..., b'_0 = b^k_{\eta'_m}$ and $\eta_i \neq \eta'_j$ for each $i \leq n$ and $j \leq m$. Since $\mathbb{C} \models$

$\exists x(R(x, b_{\eta_i}) \wedge R(x, b_{\eta_j}))$ for each $i, j \leq n$, the set $\{\eta_0, ..., \eta_n\}$ forms an antichain in $2^{<k}$. By the construction we may assume $\mathbb{C}_k = \mathbb{C}'_{2d+1}$ is a substructure of $\mathbb{C}_{k+1} = \mathbb{C}'_{2d+2}$ with respect to ε^*_k. Thus $\eta_0, ..., \eta_n, \eta'_0, ..., \eta'_m$ can be regarded as elements of $2^{<k+1}$. Now we choose an antichain tree in $2^{<k+1}$ which contains $\eta_0, ..., \eta_n$ and does not contain $\eta'_0, ...\eta'_m$. Let Λ be a maximal antichain in $2^{<k+1}$ which is defined by

$$\{\nu \in 2^{<k+1} : \nu \in 2^k \text{ and } \eta_i \ntrianglelefteq \nu \text{ for all } i \leq n\} \cup \{\eta_0, ..., \eta_n\}.$$

Then $\eta_i \in \Lambda$ for all $i \leq n$ and $\eta'_j \notin \Lambda$ for all $j \leq m$. Let $l = \mu^-_{k+1}(\Lambda)$. Then $\mathbb{C} \models R(a^{k+1}_l, b^{k+1}_{\eta_i})$ for all $i \leq n$, and $\mathbb{C} \models \neg R(a^{k+1}_l, b^{k+1}_{\eta'_j})$ for all $j \leq m$. This completes the proof. □

Now we are ready to prove that $\mathrm{Th}(\mathbb{C})$ has no quantifier free formula which witnesses SOP_2. We begin the proof by showing that some conjunctions of basic formulas do not witness SOP_2.

Proposition 3.8. *The formula $\varphi(\bar{x}, \bar{x}', y) = \bigwedge_{i \leq n} R(x_i, y) \wedge \bigwedge_{j \leq m} \neg R(x'_j, y)$ does not witness SOP_2 modulo $Th(\mathbb{C})$ for all $n, m \in \omega$. That is, there is no $\langle \bar{c}^\eta, \bar{c}'^\eta \rangle_{\eta \in 2^{<\omega}} = \langle c^\eta_0, ..., c^\eta_n, c'^\eta_0, ..., c'^\eta_m \rangle_{\eta \in 2^{<\omega}}$ in a monster model $\mathbb{M}$ of $Th(\mathbb{C})$, such that*

(i) *$\{\varphi(\bar{c}^{\eta \lceil n}, \bar{c}'^{\eta \lceil n}, y) : n \in \omega\}$ is consistent for all $\eta \in {}^\omega 2$*
(ii) *$\{\varphi(\bar{c}^\eta, \bar{c}'^\eta, y), \varphi(\bar{c}^\nu, \bar{c}'^\nu, y)\}$ is inconsistent for all $\eta \perp \nu$ in $2^{<\omega}$.*

Proof. To get a contradiction we assume there exists $\langle \bar{c}^\eta, \bar{c}'^\eta \rangle_{\eta \in 2^{<\omega}}$ which witnesses SOP_2 with $\varphi(\bar{x}, \bar{x}', y)$. We may assume $\langle \bar{c}_\eta \rangle_{\eta \in 2^{<\omega}}$ is strongly indiscernible by the modeling property. Thus

$$\begin{aligned}\{&R(c^{\langle 0 \rangle}_0, y), ..., R(c^{\langle 0 \rangle}_n, y), \neg R(c'^{\langle 0 \rangle}_0, y), ..., \neg R(c'^{\langle 0 \rangle}_m, y),\\ &R(c^{\langle 1 \rangle}_0, y), ..., R(c^{\langle 1 \rangle}_n, y), \neg R(c'^{\langle 1 \rangle}_0, y), ..., \neg R(c'^{\langle 1 \rangle}_m, y)\}\end{aligned}$$

is inconsistent. By Proposition 3.6, there are $i \leq n$ and $j \leq m$ such that $c^{\langle 0 \rangle}_i = c'^{\langle 1 \rangle}_j$ or $c^{\langle 1 \rangle}_i = c'^{\langle 0 \rangle}_j$. We may assume $c^{\langle 0 \rangle}_i = c'^{\langle 1 \rangle}_j$. Then $c^{\langle 0 \rangle}_i = c'^{\langle 10 \rangle}_j$, $c^{\langle 0 \rangle}_i = c'^{\langle 11 \rangle}_j$ and $c^{\langle 10 \rangle}_i = c'^{\langle 11 \rangle}_j$ by strong indiscernibility. Thus $c^{\langle 10 \rangle}_i = c'^{\langle 11 \rangle}_j = c^{\langle 0 \rangle}_i = c'^{\langle 10 \rangle}_j$. But it is a contradiction because the set $\{R(c^{\langle 10 \rangle}_0, y), ..., R(c^{\langle 10 \rangle}_n, y), \neg R(c'^{\langle 10 \rangle}_0, y), ..., \neg R(c'^{\langle 10 \rangle}_m, y)\}$ is consistent. □

Proposition 3.9. *The formula $\varphi(x, \bar{y}, \bar{y}') = \bigwedge_{i \leq n} R(x, y_i) \wedge \bigwedge_{j \leq m} \neg R(x, y'_j)$ does not witness SOP_2 modulo $Th(\mathbb{C})$ for all $n, m \in \omega$. That is, there is no $\langle \bar{c}^\eta, \bar{c}'^\eta \rangle_{\eta \in 2^{<\omega}} = \langle c^\eta_0, ..., c^\eta_n, c'^\eta_0, ..., c'^\eta_m \rangle_{\eta \in 2^{<\omega}}$ in a monster model $\mathbb{M}$ of $Th(\mathbb{C})$, such that*

(i) *$\{\varphi(x, \bar{c}^{\eta \lceil n}, \bar{c}'^{\eta \lceil n}) : n \in \omega\}$ is consistent for all $\eta \in {}^\omega 2$*
(ii) *$\{\varphi(x, \bar{c}^\eta, \bar{c}'^\eta), \varphi(x, \bar{c}^\nu, \bar{c}'^\nu)\}$ is inconsistent for all $\eta \perp \nu$ in $2^{<\omega}$.*

Proof. As in Proposition 3.8, we assume there is a strongly indiscernible tree of parameters $\langle \bar{c}^\eta, \bar{c}'^\eta \rangle_{\eta \in 2^{<\omega}}$ which witnesses SOP_2 with $\varphi(x, \bar{y}, \bar{y}')$. Then

$$\begin{aligned}\{&R(x, c^{\langle 0 \rangle}_0), ..., R(x, c^{\langle 0 \rangle}_n), \neg R(x, c'^{\langle 0 \rangle}_0), ..., \neg R(x, c'^{\langle 0 \rangle}_m),\\ &R(x, c^{\langle 1 \rangle}_0), ..., R(x, c^{\langle 1 \rangle}_n), \neg R(x, c'^{\langle 1 \rangle}_0), ..., \neg R(x, c'^{\langle 1 \rangle}_m)\}\end{aligned}$$

is inconsistent. By Proposition 3.7, there are $i \leq n$, $j \leq m$ such that $c^{\langle 0 \rangle}_i = c'^{\langle 1 \rangle}_j \vee c^{\langle 1 \rangle}_i = c'^{\langle 0 \rangle}_j$, or there are $i, j \leq n$ such that $\{R(x, c^{\langle 0 \rangle}_i), R(x, c^{\langle 1 \rangle}_j)\}$ is inconsistent.

If it is in the first case, then we can find a contradiction as we observed in Proposition 3.8. Thus we assume $\{R(x, c_i^{\langle 0\rangle}), R(x, c_j^{\langle 1\rangle})\}$ is inconsistent for some $i, j \leq n$. By strong indiscernibility, $\{R(x, c_i^{\langle 0\rangle}), R(x, c_j^{\langle 11\rangle})\}$, $\{R(x, c_i^{\langle 00\rangle}), R(x, c_j^{\langle 1\rangle})\}$, and $\{R(x, c_i^{\langle 00\rangle}), R(x, c_j^{\langle 11\rangle})\}$ are inconsistent. On the other hand, $\{R(x, c_i^{\langle 0\rangle}), R(x, c_i^{\langle 00\rangle})\}$ and $\{R(x, c_j^{\langle 1\rangle}), R(x, c_j^{\langle 11\rangle})\}$ are consistent by the path consistency condition. But then $c_i^{\langle 0\rangle}, c_i^{\langle 00\rangle}, c_j^{\langle 1\rangle}, c_j^{\langle 11\rangle}$ violate Proposition 3.4. This completes the proof. □

To generalize the result of Propositions 3.9 and 3.8 to the case of quantifier free formulas, we need the following fact and propositions.

Fact 3.10. [2, Corollary 4.11] *Let T be an arbitrary theory. If T has SOP_2 then there is a formula in a single free variable witnessing SOP_2.*

Proposition 3.11. *Let T be a theory and suppose $\varphi(x, \bar{y}) \vee \psi(x, \bar{z})$ witnesses SOP_2 modulo T, where x is a free variable, and $\bar{y}$, $\bar{z}$ are parameter variables. Then $\varphi(x, \bar{y})$ witnesses SOP_2 or $\psi(x, \bar{z})$ witnesses SOP_2.*

Proof. Let $\langle \bar{b}^\eta, \bar{c}^\eta\rangle_{\eta \in 2^{<\omega}} \subseteq \mathbb{M}$ witnesses SOP_2 with $\varphi \vee \psi$, where $\mathbb{M}$ is a monster model of T. We may assume the tree of parameters is strongly indiscernible. Since $\{\varphi(x, \bar{b}^\eta) \vee \psi(x, \bar{c}^\eta), \varphi(x, \bar{b}^\nu) \vee \psi(x, \bar{c}^\nu)\}$ is inconsistent for all $\eta \perp \nu$, it is clear that $\{\varphi(x, \bar{b}^\eta), \varphi(x, \bar{b}^\nu)\}$ and $\{\psi(x, \bar{c}^\eta), \psi(x, \bar{c}^\nu)\}$ are inconsistent for all $\eta \perp \nu$. Since $\{\varphi(x, \bar{b}^{\langle 0^n\rangle}) \vee \psi(x, \bar{c}^{\langle 0^n\rangle}) : n \in \omega\}$ is consistent, there exists $a \in \mathbb{M}$ such that $\mathbb{M} \models \varphi(a, \bar{b}^{\langle 0^n\rangle}) \vee \psi(a, \bar{c}^{\langle 0^n\rangle})$ for all $n \in \omega$. By the pigeon hole principle, there exists an infinite subset I of ω, such that $\mathbb{M} \models \varphi(a, \bar{b}^{\langle 0^i\rangle})$ for all $i \in I$, or $\mathbb{M} \models \psi(a, \bar{c}^{\langle 0^i\rangle})$ for all $i \in I$. Without loss of generality, we may assume $\mathbb{M} \models \varphi(a, \bar{b}^{\langle 0^i\rangle})$ for all $i \in I$. By strong indiscernibility, $\{\varphi(x, \bar{b}^{\eta\lceil n}) : n \in \omega\}$ is consistent for all $\eta \in {}^{\omega}2$ because $\langle \eta\lceil n\rangle_{n\in\omega} \sim_{str} \langle 0^i\rangle_{i\in I}$. Thus $\varphi(x, \bar{y})$ witnesses SOP_2 with $\langle \bar{b}^\eta\rangle_{\eta\in 2^{<\omega}}$. □

Proposition 3.12. *In Th($\mathbb{C}$), there is no quantifier free formula witnessing SOP_2.*

Proof. Suppose φ is a quantifier free formula witnessing SOP_2 modulo $\mathrm{Th}(\mathbb{C})$. By Proposition 3.10, we may assume φ is of the form $\varphi(z, \bar{w})$ where z is a single free variable, $\bar{w}$ is a parameter variable. By using disjunctive normal form and Proposition 3.11, we may assume φ is a conjunction of basic formulas. Thus we may assume φ is of the form

$$\bigwedge R(z, w_i) \wedge \bigwedge \neg R(z, w'_j) \wedge \bigwedge R(w''_k, z) \wedge \bigwedge \neg R(w'''_l, z).$$

But by the construction of $\mathbb{C}$, we have either $\mathbb{C} \models \exists y R(c, y)$ or $\mathbb{C} \models \exists x R(x, c)$ for all $c \in \mathbb{C}$. Hence φ is of the form either

$$\bigwedge R(z, w_i) \wedge \bigwedge \neg R(z, w'_j)$$

or

$$\bigwedge R(w_i, z) \wedge \bigwedge \neg R(w'_j, z).$$

By Propositions 3.8 and 3.9 we know that both of the above formulas can not witness SOP_2 modulo $\mathrm{Th}(\mathbb{C})$. Thus we have a contradiction and this completes the proof. □

Corollary 3.13. *SOP_1 and SOP_2 are distinct at the level of formulas.*

Proof. By Proposition 3.12, there is no quantifier free formula witnessing SOP_2 in $Th(\mathbb{C})$. In particular, $\bigwedge_{i\leq n} R(x, y_i)$ does not witness SOP_2 modulo $Th(\mathbb{C})$ for all $n \in \omega$. By Proposition 3.3 and Proposition **??**, the formula $R(x, y)$ witnesses SOP_1. Hence SOP_1 and SOP_2 are distinct at the level of formulas. □

Remark 3.14. But $Th(\mathbb{C})$ has a witness of SOP_2. Define $\varphi(x, y)$ by

$$x \neq y$$
$$\wedge\neg\exists w(R(w, x) \wedge R(w, y))$$
$$\wedge\exists z(\exists w(R(w, x) \wedge R(w, z)) \wedge \neg\exists w(R(w, y) \wedge R(w, z))).$$

Then φ says "y is a predecessor of x in the set of parameters" (*i.e.*, $y \trianglelefteq x$). For each $\eta \in 2^{<\omega}$, let $b_\eta = b^n_\eta$ for some $n \in \omega$. By the constructions of $\mathbb{C}$, b_η is well-defined. Then $\{\varphi(x, b_\eta), \varphi(x, b_\nu)\}$ is inconsistent whenever $\eta \perp \nu$. By compactness $\{\varphi(x, b_{\eta\lceil n}) : n < \omega\}$ is consistent for each $\eta \in {}^\omega 2$. Thus $\langle\varphi(x, y), \langle b_\eta\rangle_{\eta\in 2^{<\omega}}\rangle$ witnesses SOP_2 modulo $Th(\mathbb{C})$.

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