Dividing and Forking in Random Structures
– Easy Case –

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Dividing and forking are important concepts in model theory, both of which are related to abstract independence. Although these two concepts are not equivalent in general, in simple theories, they are equivalent. We demonstrate that these concepts differ in random hypergraphs omitting a certain kind of sub-hypergraphs. This work is an extension of the study that I presented at the 2022 meeting of the MSJ. More comprehensive results are examined as a jointwork with Kikyo, but in this paper, only the results obtained solely by the author will be presented.

Definition 1. Let $2 \leq m < l < \omega$, and let R be an m-ary predicate symbol. We study m-hypergraphs G represented by R. We simply refer to an m-element set $X \subset G$ with $R(X)$ as an R-edge or an edge. For a finite G, the number of edges in G is denoted by $\mathrm{e}(G)$, i.e., $\mathrm{e}(G) = |\{X \in [G]^m : G \models R(X)\}|$. Similarly $\mathrm{ne}(G)$ represents the number of m-element subsets of G that do not form edges, i.e., $\mathrm{ne}(G) = |\{X \in [G]^m : G \models \neg R(X)\}|$. For $s \in \omega$, let $H^m_{l,s}$ be the class of all finite hypergraphs G such that

$$X \in [G]^l \implies \mathrm{ne}(X) > s.$$

Notice that $G \in H^m_{l,s}$ means that G omits a subset $X \subset G$ of size l that satisfies $\mathrm{e}(X) \geq \binom{l}{m} - s$. If $l = m+1$ and $s = m-2$, then $\binom{l}{m} - s = 3$. Hence, the condition $G \in H^m_{m+1,m-2}$ is equivalent to the following:

$$X \in [G]^{m+1} \implies \mathrm{e}(X) < 3. \qquad (*)$$

$H^2_{3,0}$ is the class of all triangle free graphs. tetrahedron and tetrahedron omitting one face. $H^3_{4,1}$ is the class of all 3-hypergraphs that omit tetrahedrons and tetrahedrons with one omitted face. For an infinite set X, we say that X belongs to the class $H^m_{l,s}$ if every finite substructure of X is a member of this class.

Proposition 2. *1. For $s < \binom{l-2}{m-2}$, $H^m_{l,s}$ has the free-AP.*

2. For $s = \binom{l-2}{m-2}$, $H^m_{l,s}$ does not have the free-AP.

3. For $s > \binom{l-2}{m-2}$, $H^m_{l,s}$ does not have the AP.

Proof. We prove only 3. Assuming AP leads to a contradiction. By the definition of $H^m_{l,s}$, a complete hypergraph B of size $l-1$ belongs to $H^m_{l,s}$. Let $A \subset B$ be an $(l-2)$-element substructure, and consider $C = A \sqcup \{c\}$, a hypergraph with $ne(C) = 1$. In other words, C is a hypergraph obtained by eliminating one edge from a complete hypergraph of size $l-1$. Let $D \in H^m_{l,s}$ be an amalgam of B and C over A. Then, $D = B \sqcup \{c\}$ as a set. Moreover,

$$e(D) \geq e(B) + e(C) - e(A) = \binom{l}{m} - \left(\binom{l-2}{m-2} + 1\right) \geq \binom{l}{m} - s.$$

This is a contradiction. □

Let $F^m_{l,s}$ be the Fraisse limit of the class $H^m_{l,s}$.

Theorem 3. *For $3 \leq m < l$ and $s < \binom{l-3}{m-3}$, the theory of $F^m_{l,s}$ has SU-rank one.*

Proof. We work in a sufficiently saturated extension $\mathcal{M} \succ F^m_{l,s}$. Let A and $a \notin A$ be given. We show that $SU(a/A) = 1$. Let $p_A(x) = \mathrm{tp}(a/A)$ and let $I = \{A_i\}_{i\in\omega}$ be an indiscernible sequence with $A_0 = A$. It is sufficient to show that the following set is consistent:

$$\Gamma(x) = \bigcup_{i\in\omega} p_{A_i}(x).$$

Here $p_{A_i}(x)$ denotes the type obtained from $p(x)$ by replacing A with A_i. Letting $I^* := \bigcup I = \bigcup_{i\in\omega} A_i$, we consider the hypergraph on the nodes $I^* \cup \{x\}$ such that all edges are those explicitly represented in Γ. Namely, the relation R on $I^* \cup \{x\}$ is

$$(R \text{ on } I^*) \cup \{\{x\} \cup B : B \in [I^*]^{m-1},\ R(x, B) \in \Gamma(x)\}.$$

To see the consistency of Γ, it is sufficient to see that the hypergraph $I^* \cup \{x\}$ does not have a subset D of size l such that $\mathrm{ne}(D) \leq s$. So, for a contradiction, let $D \in [I^* \cup \{x\}]^l$ satisfy $\mathrm{ne}(D) \leq s$. D must contain x, since any finite subset of I^* belongs to $H^m_{l,s}$. Also, there are $i \neq j$ such that both $D \setminus A_i$ and $D \setminus A_j$ are non-empty, since otherwise D is isomorphic to a substructure of $A \cup \{a\}$. Choose $b \in D \setminus A_i$ and $c \in D \setminus A_j$. For any set X in $[D \setminus \{x, b, c\}]^{m-3}$, the m-element set $X \cup \{x, b, c\}$ does not form an edge, due to the definition of $\Gamma(x)$. Since $D \setminus \{x, b, c\}$ has the cardinality $l-3$, there are $\binom{l-3}{m-3}$-many such sets X. This means that $\mathrm{ne}(D) > s$. A contradiction. $\square$

Definition 4. Let A and B be subsets of a hypergraph. $ne(A/B)$ denotes the cardinality of the set $\{X \in [A \cup B]^m : B \subset X,\ A \cup B \models \neg R(X)\}$.

Clearly, $ne(A/B)$ is upper-bounded by $\binom{|A \setminus B|}{m-|B|}$. When using this notation, we usually assume A and B are disjoint.

Theorem 5. *For $s = m - 2$, in the theory of $F^m_{m+1,s}$, forking and dividing are different.*

Proof. Since we have already treated the case $m = 2$, we assume $m \geq 3$. We use the characterization (*) of $H^m_{m+1,m-2}$, and work in a large elementary extension $\mathcal{M}$ of $F^m_{m+1,m-2}$. Let $A = a_1 a_2 \dots a_{m-2}, b, c$ be an m-tuple with $\neg R(A)$. Let $\varphi(x, A)$ be the formula

$$R(x, a_1, \dots, a_{m-2}, b) \wedge R(x, a_1, \dots, a_{m-2}, c) \wedge \bigwedge_X \neg R(x, X, b, c),$$

where X ranges over all $(m-3)$-element subsets of $\{a_1, \dots, a_{m-2}\}$. (If $m = 3$, X does not appear in the definition.) First, observe that $\varphi(x, A)$ is consistent, meaning it has a solution in $\mathcal{M}$. This follows from the fact that the hypergraph $A \cup \{x\}$, whose edges are those explicitly represented in $\varphi(x, A)$, belongs to $K^m_{m+1,m-2}$.

Claim A. *There is an indiscernible sequence $(b_i, c_i)_{i \in \omega}$ over $a_1, \dots, a_{m-2}$ with $b_0, c_0 = b, c$ such that $R(a_1, \dots, a_{m-2}, b_i, c_j)$ iff $i < j$ for all $i, j \in \omega$.*

Prepare variables x_i and y_i for $i \in \omega$. We create a hypergraph with the set $H = \{a_1, \dots, a_{l-3}\} \cup \{x_i, y_i : i \in \omega\}$, where the R-edges of the hypergraph are defined by the set

$$\{\{a_1, \dots, a_{m-2}, x_i, y_j\} : i < j < \omega\}. \tag{**}$$

What we have to show is that H belongs to $H^{m,m+1}_{m-2}$. For showing this, let D be an $(m+1)$-element subset of the hypergraph H. We show that $\mathrm{e}(D) < 3$. If $\{a_1, \ldots, a_{m-2}\}$ is not a subset of D, then D has no edges. So we can assume $D = \{a_1, \ldots, a_{m-2}\} \cup X$, where $X \subset \{x_i, y_i : i \in \omega\}$ is a three elment set. There are several possibilities for the shape of X. But, in any case, we have $\mathrm{e}(D) \leq 2$, by (**). Thus, we can find a copy (over $a_1, \ldots, a_{m-2}$) of H in $\mathcal{M}$. By a simple compactness argument, this copy can be selected to form an indiscernible sequence over $\{a_1, \ldots, a_{m-2}\}$. (End of Proof of Claim A)

Claim B. *$\varphi(x, A)$ divides over $a_1, \ldots, a_{m-2}$.*

Select an indiscernible sequence $I := (b_i c_i)_{i \in \omega}$, as demonstrated to exist in Claim A. For a contradiction, suppose that $\{\varphi(x, a_1, \ldots, a_{m-2}, b_i, c_i) : i \in \omega\}$ is realized by $d \in \mathcal{M}$. Let D be the $(m+1)$-element set $\{a_1, \ldots, a_{m-2}, b_0, c_1, d\}$. The following m-element subsets are edges in D:

$$\{a_1, \ldots, a_{m-2}, b_0, d\},\ \{a_1, \ldots, a_{m-2}, c_1, d\} \text{ and } \{a_1, \ldots, a_{m-2}, b_0, c_1\}.$$

So, we must have $\mathrm{e}(D) \geq 3$. However, this is impossible, since $D \in H^m_{m+1,m-2}$. (End of Proof of Claim B)

Claim C. *Let $I = (b_i, c_i)_{i \in \omega}$ be an indiscernible sequence over $a_1, \ldots, a_{m-2}$ starting with $b_0, c_0 = b, c$. If there is no edge of the form $\{a_1, \ldots, a_{m-2}, b_i, c_j\}$, where $i \neq j$, then the set $\Gamma(x) := \{\varphi(x, a_1, \ldots, a_{m-2}, b_i, c_i) : i \in \omega\}$ is consistent.*

Let I^* denote the set $\{b_i : i \in \omega\} \cup \{c_i : i \in \omega\}$. We consider the hypergraph $H := \{a_1, \ldots, a_{m-2}\} \cup I^* \cup \{x\}$ whose edges are those explicitly represented in Γ. For showing that $\Gamma(x)$ has a solution in $\mathcal{M}$, it is sufficient to show that the above hypergraph H is in $H^m_{m+1,m-2}$. Let $D \subset H$ be any $(m+1)$-element set. According to the characterization (*), what we need to show is that $\mathrm{e}(D) < 3$. If $x \notin D$, then D is a subset of $\{a_1, \ldots, a_{m-2}\} \cup I^*$, so we are done. Thus, in what follows, we assume $x \in D$. There are two cases to consider. First we treat the case $\{a_1, \ldots, a_{m-2}\} \subset D$. In this case, D has one of the following form:

1. $D = \{a_1, \ldots, a_{m-2}\} \cup \{b_i, c_j\} \cup \{x\}$,
2. $D = \{a_1, \ldots, a_{m-2}\} \cup \{b_i, b_j\} \cup \{x\}$ $(i \neq j)$,
3. $D = \{a_1, \ldots, a_{m-2}\} \cup \{c_i, c_j\} \cup \{x\}$ $(i \neq j)$.

In the case of 1, if $i = j$, then D is isomorphic to the hypergraph $\{x\} \cup A$ defined by the formula $\varphi(x, A)$, so $\mathrm{e}(D) < 3$. Also, if $i \neq j$, then by our assumption, the only edges in D are $\{a_1, \ldots, a_{m-2}\} \cup \{b_i\} \cup \{x\}$ and $\{a_1, \ldots, a_{m-2}\} \cup \{c_j\} \cup \{x\}$, and we are done.

In the case of 2, by the definition of our hypergraph H, there is no edge that contains all of x, b_i, b_j. So, the possible edges are $\{a_1, \ldots, a_{m-2}\} \cup \{b_i\} \cup \{x\}$, $\{a_1, \ldots, a_{m-2}\} \cup \{b_j\} \cup \{x\}$, and $\{a_1, \ldots, a_{m-2}\} \cup \{b_i, b_j\}$. However, the last set cannot be an edge, because if it were, the $(m+1)$-element set $\{a_1, \ldots, a_{m-2}\} \cup \{b_0, b_1, b_2\}$ would have three edges due to the indiscernibility. The case 3 is treated in the same way.

Now, we treat the remaining case, i.e., some a_t $(1 \leq t \leq m-2)$ does not belong to D. Then, $|D \cap \{a_1, \ldots, a_{m-2}\}| \leq m-3$. Since other cases treated similarly, we assume $D \cap \{a_1, \ldots, a_{m-2}\} = \{a_1, \ldots, a_k\}$, where $k \leq m-3$. Let $B \subset D$ be an m-element set. We prove that, if B contains x, then B is not an edge. So, let B have the form $B = \{a_1, \ldots, a_k\} \cup C \cup \{x\}$, where $C \subset I^*$. Let $\mathrm{wd}(C) := \min\{|X| : X \subset \omega,\ C \subset \bigcup_{i \in X} \{b_i, c_i\}\}$. (wd stands for width.) If $\mathrm{wd}(C) \geq 2$, then B is not an edge, due to the definition of H. If $\mathrm{wd}(C) = 1$, then $k = m-3$, and B is isomorphic to $\{a_1, \ldots, a_{m-3}\} \cup \{b, c\} \cup \{x\}$. So, due to the definition of $\varphi(x)$, B is not an edge. Hence, the only possible edge is $D \setminus \{x\}$. Namely, $\mathrm{e}(D) \leq 1$. (End of Proof of Claim C)

Claim D. *Let $A^* = a_1, \ldots, a_{m-2}, b_1, \ldots, b_{n^*}$ be an R-free tuple, where n^* is sufficiently large. Let $\psi(x, A^*)$ be the formula*

$$\bigvee_{1 \leq i < j \leq n^*} \varphi(x, a_1, \ldots, a_{m-2}, b_i, b_j).$$

Then, $\psi(x, A^*)$ does not divide over $a_1, \ldots, a_{m-2}$.

Suppose, for the sake of contradiction, that $\psi(x, A^*)$ divides over $a_1, \ldots, a_{m-2}$. Choose an indiscernible sequence $I = (b_{i,1}, \ldots, b_{i,n^*})_{i \in \omega}$ witnessing the dividing, where $b_{0,1}, \ldots, b_{0,n^*} = b_1, \ldots, b_{n^*}$. Since ψ is a disjunction of the formulas $\varphi(x, a_1, \ldots, a_{m-2}, b_i, b_j)$ $(1 \leq i < j \leq n^*)$, the set $\Gamma_{i,j}(x) = \{\varphi(x, a_1, \ldots, a_{m-2}, b_{n,i}, b_{n,j}) : n \in \omega\}$ must be inconsistent for all $i < j \leq n^*$.

Hence, by Claim C, for all $i < j \leq n^*$, (at least) one of the following is true:

(i) $R(a_1, \ldots, a_{m-2}, b_{n,i}, b_{n',j})$ for all $n < n' < \omega$;

(ii) $R(a_1, \ldots, a_{m-2}, b_{n,j}, b_{n',i})$ for all $n < n' < \omega$.

So, using Ramsey's theorem, by symmetry of the argument, we can find $i < j < k \leq n^*$ such that the situation (i) holds true for each of (i, j), (j, k), and (i, k). Here we used the assumption that n^* is sufficiently large. Then, by the indiscernibility, the $(m+1)$-element set $X := \{a_1, \ldots, a_{m-2}\} \cup \{b_{0,i}, b_{1,j}, b_{2,k}\}$ satisfies $\mathrm{e}(X) \geq 3$. This implies that $X \notin H^m_{m+1,m-2}$, which leads to a contradiction. $\square$

References

[1] Personal communication with Kikyo.

[2] Katrin Tent, and Martin Ziegler. A Course in Model Theory. Cambridge, Cambridge University Press, 2012.