# LOCAL MODULARITY, PILLAY'S ONE-BASEDNESS AND CF-PROPERTY IN $o$-MINIMAL STRUCTURES

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ABSTRACT. In $o$-minimal structures, weak one-basedness, weak local modulaity, strong linearity, generic linearity, linearity and CF-property are equivalent. We discuss the implicatitons among modularity, local modularity, Pillay's one-basedness and CF-property.

CONTENTS



## 1. INDEPENDENCE RELATION IN ROSY THEORIES AND DIMENSION IN $\mathrm{U}^{\mathfrak{p}}=1$ THEORIES

Let $\mathcal{M}$ be a sufficiently saturated model of an $L$-theory $T$. $e \in \mathcal{M}^{\mathrm{eq}}$ is an imaginary iff $e=\bar{a}/E$, where $E(\bar{x},\bar{y})$ is an $\emptyset$-definible equivalence relation with $\mathrm{lh}(\bar{a})=\mathrm{lh}(\bar{x})=\mathrm{lh}(\bar{y})$, where $\bar{a}\subset\mathcal{M}$ is a finite tuple. For $e\in\mathcal{M}^{\mathrm{eq}}$ and $A\subset\mathcal{M}^{\mathrm{eq}}$ we write $e\in\mathrm{acl}^{\mathrm{eq}}(A)$ if $|\{\sigma(e):\sigma\in\mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/A)\}|$ is finite. $\bar{a},\bar{b},\bar{c},\dots$ denote finite tuples of $\mathcal{M}^{\mathrm{eq}}$ and $A,B,C,D\dots$ donote small subset of $\mathcal{M}^{\mathrm{eq}}$.

The independence calculus  See [A].
A *symmetric* ternary relation $*\mathop{\smile\!\!\!\!\!\!|}_{*}*$ on $\mathcal{M}^{\mathrm{eq}}$ has the independence calculus if the following 8 conditions hold:

(1) Invariance: $A\mathop{\smile\!\!\!\!\!\!|}_{B}C$ and $ABC\equiv A'B'C'$ imply $A'\mathop{\smile\!\!\!\!\!\!|}_{B'}C'$
(2) Normality: $A\mathop{\smile\!\!\!\!\!\!|}_{B}C$ implies $A\mathop{\smile\!\!\!\!\!\!|}_{B}BC$.
(3) Monotonicity: $A\mathop{\smile\!\!\!\!\!\!|}_{B}C$ and $A_0\subseteq A$ imply $A_0\mathop{\smile\!\!\!\!\!\!|}_{B}C$
(4) *Transitivity* : If $B\subseteq C\subseteq D$, then $A\mathop{\smile\!\!\!\!\!\!|}_{B}D$ iff $A\mathop{\smile\!\!\!\!\!\!|}_{B}C$ and $A\mathop{\smile\!\!\!\!\!\!|}_{C}D$
(5) Extention: There exists $A'\equiv_B A$ such that $A'\mathop{\smile\!\!\!\!\!\!|}_{B}C$.

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(6) Finite character: If $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B C$ for any finite tuple $\bar{a} \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$.
(7) *Local character*: For any $B, A \subset \mathcal{M}$, there exist a cardinal $\kappa(B)$ and $A_0 \subseteq A$ such that $|A_0| \leq \kappa(B)$ and $B \mathop{\smile\!\!\!\!\!\!\mid}_{A_0} A$.
(8) Anti-reflexivity: $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{a}$ implies $\bar{a} \in \mathrm{acl}^{\mathrm{eq}}(A)$.

We say $\mathcal{M}^{\mathrm{eq}}$ is rosy if it has the independence calculus.

If $\mathop{\smile\!\!\!\!\!\!\mid}$ is the thorn independence, symmetry$\Leftrightarrow$transitivity$\Leftrightarrow$local character modulo the above other properties. In any $\mathrm{U}^{\mathfrak{p}} = 1$ (rosy of rank one) structure $\mathcal{M}$, (i.e. $a \in \mathrm{acl}^{\mathrm{eq}}(A)$ iff $\mathrm{U}^{\mathfrak{p}}(a/A) = 0$, where $a \in \mathcal{M}$. $b \notin \mathrm{acl}^{\mathrm{eq}}(A)$ iff $\mathrm{U}^{\mathfrak{p}}(b/A) = 1$, where $b \in \mathcal{M}$.) for any $a, b \in \mathcal{M}$ and $B \subset \mathcal{M}^{\mathrm{eq}}$, $a \in \mathrm{acl}^{\mathrm{eq}}(Bb) \setminus \mathrm{acl}^{\mathrm{eq}}(B)$ implies $b \in \mathrm{acl}^{\mathrm{eq}}(Ba)$, as $a \not\mathop{\smile\!\!\!\!\!\!\mid}_B b$ iff $b \not\mathop{\smile\!\!\!\!\!\!\mid}_B a$, we can define $\dim(\bar{a}/A)$ and $\dim(\bar{a}/Ae)$. For any $\bar{a} \subset \mathcal{M}$, $e = \bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$ and $A \subset \mathcal{M}^{\mathrm{eq}}$, we define $\dim(e/A) := \dim(\bar{a}/A) - \dim(\bar{a}/Ae)$.

**Fact 1.1.** *Any o-minimal structure is rosy of rank one (i.e.* $\mathrm{U}^{\mathfrak{p}} = 1$*)*

## 2. CF-property

For real tuple $\bar{a} \in \mathcal{M}^n$ we write $\bar{a} \in \mathrm{dcl}^{\mathrm{eq}}(A)$ if $\{\sigma(\bar{a}) : \sigma \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/A)\} = \{\bar{a}\}$.

We say that $(\mathcal{M}, <, \cdots)$ is o-minimal structure if any definable set $X \subseteq \mathcal{M}$ is a finite union of intervals and points. $\mathrm{acl}^{\mathrm{eq}}(*) = \mathrm{dcl}^{\mathrm{eq}}(*)$ in the real sort $\mathcal{M}^n$ $(n < \omega)$ by lexicographic order in o-minimal structures.

Let $\varphi(\bar{z})$ be an $\emptyset$-definable set and let $C(x, y, \bar{z})$ be an $\emptyset$-definable set such that

$$C(x, y, \bar{z}) \to \varphi(\bar{z}) \wedge y \in \mathrm{dcl}(\bar{z}, x) \wedge \dim(x, y/\bar{z}) \leq 1$$

Then we write $y = f_{\bar{z}}(x)$.

For any $a \in \mathcal{M}$, we define an $a$-definable equivalent relation on $U$: For $\bar{c}, \bar{c}' \models \varphi(\bar{z})$, $\bar{c} \sim_a \bar{c}'$ iff there exists $a' > a$ such that $f_{\bar{c}}(x) = f_{\bar{c}'}(x)$ on the interval $(a, a')$. Put $\hat{\hat{z}}_a = \bar{z}/\sim_a \in \mathcal{M}^{\mathrm{eq}}$.

**Definition 2.1.** We say that an o-minimal structure $(\mathcal{M}, <, \cdots)$ has **C**ollapse of **F**amilies of functions-property if for any $a \in \mathcal{M}$ and any $\bar{c} \models \varphi(\bar{z})$ we have

$$\dim(\hat{\hat{c}}_a/a) \leq 1.$$

We call $\hat{\hat{c}}_a$ Peterzil's germ of $\bar{c}$ at $a$.

An example of non CF-property: Multiplication violates CF-property.
In $(\mathbb{R}, +, \cdot, <)$, put $f_{(b_1, b_2)}(x) = b_1 x + b_2$. Then $(b_1, b_2) \sim_a (c_1, c_2)$ iff $(b_1, b_2) = (c_1, c_2)$. So $\hat{\hat{b}}_a = (b_1, b_2)/\sim_a$ is interdefinable with $(b_1, b_2)$ over $a$. So $\dim(\hat{\hat{b}}_a/a) = 2$.

## 3. An example of non-locally modular $o$-minimal structure with CF-property

Definition of partial endomorphisms:
Suppose that $(I, +, <)$ is a group-interval(roughly speaking it has a partial definable group operation on the interval $I$) in an $o$-minimal structure. We say that an $\emptyset$-definable partial unary function $f$ is a partial endomorphism if

(1) $dom(f) = I$ or $dom(f) = (-c, c)$ for some $c \in I$.
(2) If $a, b, a + b \in dom(f)$, then $f(a + b) = f(a) + f(b)$.

**Fact 3.1.** [LPe]*: Suppose that* $(I, +, <)$ *is a group-interval in a locally modular o-minimal structure* $\mathcal{M}$ *and let* $f$ *be a partial endomorphism in* $I$*. Then there exists a total endomorphism with* $dom(g) = \mathcal{M}$ *such that* $f = g|dom(f)$*.*

In $\mathbb{R}$, put $\pi|(-1,1)(x) = \pi \cdot x$ for each $x \in (-1,1)$.
We can define $\pi \cdot y = n \cdot \pi|(-1,1)\left(\dfrac{y}{n}\right)$ for each $y \in (-n,n)$, where $n \in \mathbb{N}$.
We can not extend $\pi|(-1,1)$ to non-standard part, so $\mathrm{Th}(\mathbb{R},+,0,1,<,\pi|(-1,1))$ is not locally modular by Fact 3.1. As $\mathrm{Th}(\mathbb{R},+,0,1,<,\pi|(-1,1))$ does not interpret any field-interval, it has CF-property by trichotomy theorem [PeS].

## 4. Pillay's one-basedness for strong types of real tuples in $o$-minimal structures

For any real type $p$ over $C = \mathrm{acl}^{\mathrm{eq}}(C)$, we can find $\bar{a}\bar{b} \models p$ such that $\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) = |\bar{a}|$ and $\bar{b} = f(\bar{a},\bar{c})$, where $f$ is an $\emptyset$-definble function and $\bar{c} \subseteq C$.
We say that $\bar{a}$ is a generic tuple for $p$.

The germs of definable functions:
We define an $\bar{a}$-definable equvalence relation $E_{f,\bar{a}}$ as follows. $E_{f,\bar{a}}(\bar{c},\bar{c}') \Leftrightarrow$ there exists an open neighborhood $U$ of $\bar{a}$ such that $f(\bar{x},\bar{c}), f(\bar{x},\bar{c}')$ are defined on $U$ and $f(\bar{x},\bar{c})|U = f(\bar{x},\bar{c}')|U$ OR neither of $f(\bar{x},\bar{c}), f(\bar{x},\bar{c}')$ is defined on an open neighborhood of $\bar{a}$.

$$\bar{c}_{E_{f,\bar{a}}} := \bar{c}/E_{f,\bar{a}} \text{ is the germ of definable function } f \text{ around } \bar{a}.$$

We call $\bar{c}_{E_{f,\bar{a}}}$ Pillay's germ of $\bar{c}$ at $\bar{a}$ (with respect to $f$).

Weak canonical bases for strong type $\mathrm{stp}(\bar{a}\bar{b}/C) := \mathrm{tp}(\bar{a}\bar{b}/\mathrm{acl}^{\mathrm{eq}}(C))$:
An algebraically closed set $D$ is said to be weak canonical base of $\mathrm{stp}(\bar{a}\bar{b}/C)$ if $D = \mathrm{acl}^{\mathrm{eq}}(D)$ is the smallest subset of $\mathrm{acl}^{\mathrm{eq}}(C)$ with $\bar{a}\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_D C$, and we write $\mathrm{wcb}(\mathrm{stp}(\bar{a}\bar{b}/C))$ for the $D$.

If weak canonical bases exist, the germs of definable functions will be interdefinable with weak canonical bases over generic tuples.

A strong type without weak canonical base:
We work in $\mathrm{Th}(\mathbb{R},+,0,1,<,\pi|(-1,1))$. Take $a,b,c > \mathbb{R}$ with $|a-b| < 1, |c-\pi\cdot b| < 1$ and $a,b,c$ are independent in a saturated model of $\mathrm{Th}(\mathbb{R},+,0,1,<,\pi(*))$, where $\pi(*)$ is totally defined. Put

$$d := \pi(a-b) + c.$$

Then $\mathrm{wcb}(\mathrm{stp}(a,d/b,c))$ does not exist. (If it existed, then $\pi(a) \in \mathrm{dcl}(a)$ would follow in $\mathrm{Th}(\mathbb{R},+,0,1,<,\pi|(-1,1))$. Let $f(x,y,z) = \pi\cdot(x-y)+z$.
Claim: $(b,c)_{E_{f,a}} \in \mathrm{dcl}(a,d)$.
Let $\sigma \in \mathrm{Aut}(\mathcal{M}/a,d)$. Then we have
$f(a,b,c) = \pi(a-b)+c = d = \sigma(d) = \pi(a-\sigma(b)) + \sigma(c) = f(a,\sigma(b),\sigma(c))$,

$$E_{f,a}((b,c),(\sigma(b),\sigma(c)))$$

Later we will say that $\mathrm{stp}(a,d/b,c)$ is Pillay's one-based.

**Fact 4.1.** [Pi] *Suppose that* $\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) = |\bar{a}|$ *and* $\bar{b} = f(\bar{a},\bar{c})$, *where* $f$ *is an* $\emptyset$*-definble function and* $\bar{c} \subseteq C = \mathrm{acl}^{\mathrm{eq}}(C)$.

(1) *If* $\mathrm{wcb}(\mathrm{stp}(\bar{a}\bar{b}/C))$ *exists, there exists* $\bar{d} \subseteq \mathrm{wcb}(\mathrm{stp}(\bar{a}\bar{b}/C))$ *such that* $\mathrm{dcl}^{\mathrm{eq}}(\bar{d}, \bar{a}) = \mathrm{dcl}^{\mathrm{eq}}(\bar{c}_{E_{f,\bar{a}}}, \bar{a})$.
(2) *If there exists* $\bar{d} \subseteq C$ *such that* $\mathrm{acl}^{\mathrm{eq}}(\bar{d}, \bar{a}) = \mathrm{acl}^{\mathrm{eq}}(\bar{c}_{E_{f,\bar{a}}}, \bar{a})$*, then* $\mathrm{acl}^{\mathrm{eq}}(\bar{d}) = \mathrm{wcb}(\mathrm{stp}(\bar{a}\bar{b}/C))$*, the weak canonical base of* $\mathrm{stp}(\bar{a}\bar{b}/C)$ *exists.*

Pillay's germ $\bar{c}_{E_{f,\bar{a}}}$ is an almost weak canonical base over a generic tuple $\bar{a}$.

**Definition 4.2.** Suppose that $\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) = |\bar{a}|$ and $\bar{b} = f(\bar{a}, \bar{c})$, where $f$ is an $\emptyset$-definble function and $\bar{c} \subseteq C = \mathrm{acl}^{\mathrm{eq}}(C)$.
We say that $\mathrm{stp}(\bar{a}, \bar{b}/C)$ is Pillay's one-based if

$$\bar{c}_{E_{f,\bar{a}}} \in \mathrm{acl}^{\mathrm{eq}}(\bar{a}, f(\bar{a}, \bar{c})) = \mathrm{acl}^{\mathrm{eq}}(\bar{a}, \bar{b}).$$

If weak canonical bases exist, Pillay's one-basedness implies usual one-basedness: We have $\mathrm{wcb}(\bar{a}, \bar{b}/\mathrm{acl}^{\mathrm{eq}}(C)) \subseteq \mathrm{acl}^{\mathrm{eq}}(\bar{a}, \bar{b})$, so $\dim(\bar{a}, \bar{b}/C) = \dim(\bar{a}, \bar{b}/C \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}, \bar{b})) \geq \dim(\bar{a}, \bar{b}/C)$.

**Question 4.3.** (1) *In o-minimal structures, modularity implies Pillay's one-basedness for any strong type of real tuples. Does local modularity imply Pillay's one-basedness for any strong type of real tuples in o-minimal structures? It is known that local modularity implies CF-property* [Pe].
(2) *In o-minimal theories, is there non Pillay's one-based strong type without its weak canonical base? There exists Pillay's one-based strong type without its weak canonical base as we mentioned above in this section.*
(3) *Local modularity implies one-basedness under assuming the existence of weak canonical bases for any strong type in rosy theories* [Y]. *Is there locally modular rosy theory with a strong type without its weak canonical base?*

## 5. Pillay's one-basedness for any strong type of real tuple in *o*-minimal case implies CF-property

Peterzil's germ and Pillay's one are almost same by the following fact.

**Fact 5.1.** [LPe]*: Assume dense linear ordering.* $\hat{\bar{c}}_a = \hat{\bar{c}}'_a \Leftrightarrow \bar{c}_{E_{f,a}} = \bar{c}'_{E_{f,a}}$*. In particular, we see* $\mathrm{dcl}^{\mathrm{eq}}(\hat{\bar{c}}_a, a) = \mathrm{dcl}^{\mathrm{eq}}(\bar{c}_{E_{f,a}}, a)$ *and* $\dim(\hat{\bar{c}}_a/a) = \dim(\bar{c}_{E_{f,a}}/a)$.

*Proof.* ($\Leftarrow$): Clear. ($\Rightarrow$): $\hat{\bar{c}}_a = \hat{\bar{c}}'_a$ iff $f(x, \bar{c}) = f(x, \bar{c}')$ on $(a, a')$ for some $a' > a$. Take $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(a) \models \mathrm{tp}(a/\bar{c}, \bar{c}') \cup \{a < x < a'\}$. (If $a \not\perp \bar{c}, \bar{c}'$, then $f(x, \bar{c}) = f(x, \bar{c}') = d \in \mathrm{dcl}^{\mathrm{eq}}(\bar{c}, \bar{c}')$ is constant around $a$: Let $M_0$ be a prime model over $\bar{c}, \bar{c}'$. Let $a_+ \models \{x : a < x < e \text{ if } a < e \in M_0\}$. We have $a_+ \notin M_0$. By $f(a_+, \bar{c}) = f(a_+, \bar{c}')$ and dense linear ordering, there exists a closed interval $J = [e_1, e_2]$ containing $a_+$ such that $e_i \in M_0$ and $f(x, \bar{c}) = f(x, \bar{c}')$ on $J$. Note that $e_1 \leq a$. Otherwise $a < e_1$, then $a_+ < e_1$ follows, a contradiction.) So we may assume $a \perp \bar{c}, \bar{c}'$. Take a thorn non-forking extension of $\mathrm{tp}(a/\bar{c}, \bar{c}')$ over $\bar{c}, \bar{c}', a, a'$ satisfying $a < x < a'$. As $f(x, \bar{c}) = f(x, \bar{c}')$ on the interval $I = (a, a')$ containing $\sigma(a)$, we have $\bar{c}_{E_{f,\sigma(a)}} = \bar{c}'_{E_{f,\sigma(a)}}$. By considering $\sigma^{-1}$ which fixes $\bar{c}$ and $\bar{c}'$, we see $\bar{c}_{E_{f,a}} = \bar{c}'_{E_{f,a}}$. □

**Remark 5.2.** For any generic tuple $\bar{a}$, we have $f_{\bar{c}}(\bar{a}) \in \mathrm{dcl}(\bar{c}_{E_{f,\bar{a}}}, \bar{a})$.

Suppose that $\sigma \in \mathrm{Aut}(\mathcal{M}/\bar{a})$ such that $\bar{c}_{E_{f,\bar{a}}} = \sigma(c)_{E_{f,\bar{a}}}$. Then we have $f(x, \bar{c}) = f(x, \sigma(\bar{c}))$ on an open neighborhood containing $\bar{a}$. So we see $f_{\bar{c}}(\bar{a}) = f_{\sigma(\bar{c})}(\bar{a}) =$

$\sigma(f_{\bar{c}}(\bar{a}))$ as desired.

**Proposition 5.3.** *Pillay's one-basedness implies CF-property.*

By the above remark and Pillay's one-basedness, $f_{\bar{c}}(a)$ is interalgebraic with $\bar{c}_{E_{f,a}}$ over $a$. Fact 5.1 and $\dim(f_{\bar{c}}(a)/a) \leq 1$ imply $\dim(\hat{\bar{c}}_a/a) = \dim(\bar{c}_{E_{f,a}}/a) = \dim(f_{\bar{c}}(a)/a) \leq 1$. □

## 6. Does CF-property imply Pillay's one-basedness in $o$-minimal structures?

A consideration for unary generic tuple $a$:
We have $\dim(\bar{c}_{E_{f,a}}/a) \leq 1$ by CF-property and Fact 5.1 and $f_{\bar{c}}(a) \in \mathrm{dcl}(\bar{c}_{E_{f,a}}, a)$ by Remark 5.2.
Assume that $\dim(f_{\bar{c}}(a)/a) = 1$.
We have $\dim(a, f_{\bar{c}}(a), \bar{c}_{E_{f,a}}) = \dim(f_{\bar{c}}(a)/\bar{c}_{E_{f,a}}, a) + \dim(\bar{c}_{E_{f,a}}/a) + \dim(a) \leq 2$. On the other hand we have $\dim(a, f_{\bar{c}}(a), \bar{c}_{E_{f,a}}) = \dim(\bar{c}_{E_{f,a}}/a, f_{\bar{c}}(a)) + \dim(f_{\bar{c}}(a)/a) + \dim(a) = \dim(\bar{c}_{E_{f,a}}/a, f_{\bar{c}}(a)) + 2$.
Therefore we have $\bar{c}_{E_{f,a}} \in \mathrm{acl}^{\mathrm{eq}}(a, f_{\bar{c}}(a))$.
A consideration for n-ary generic tuple $\bar{a}$:
We have $\dim(\bar{a}, f_{\bar{c}}(\bar{a}), \bar{c}_{E_{f,\bar{a}}}) = \dim(f_{\bar{c}}(\bar{a})/\bar{c}_{E_{f,\bar{a}}}, \bar{a}) + \dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) + \dim(\bar{a}) = \dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) + \dim(\bar{a})$. On the other hand, we have $\dim(\bar{a}, f_{\bar{c}}(\bar{a}), \bar{c}_{E_{f,\bar{a}}}) = \dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}, f_{\bar{c}}(\bar{a})) + \dim(f_{\bar{c}}(\bar{a})/\bar{a}) + \dim(\bar{a})$.

**Remark 6.1.** The following are equivalent.

1. $\dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}, f_{\bar{c}}(\bar{a})) = 0$
2. $\dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) = \dim(f_{\bar{c}}(\bar{a})/\bar{a})$.
3. $\dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) \leq \dim(f_{\bar{c}}(\bar{a})/\bar{a})$.

*Proof.* Use Remark 5.2 $f_{\bar{c}}(\bar{a}) \in \mathrm{dcl}(\bar{a}, \bar{c}_{E_{f,\bar{a}}})$. □.

**Question 6.2.** *Is there an o-minimal theory with CF-property with a non Pillay's one-based strong type? CF-property with elimination of imaginaries and the existence of weak canonical bases for any strong type imply modularity in o-minimal theories* [Y].

## 7. Appendix : Weakly local modularity implies strong linearity in geometric structures

The following proof appears at Propostion 2.17 in [BV], but it is complicated. We give a modified proof.
Suppose that $(\mathcal{M}, \downarrow)$ is weakly local modular and $\hat{\mathfrak{C}} := \{\hat{C}(x, y, \hat{a}) : \hat{a} \models \hat{\varphi}(\hat{z})\}$ is 2-dimensional almost normal interpretable family of plane curve.
So $\dim(\hat{\varphi}(\hat{z})) \geq 2$. We seek a contradiction.
Suppose that $\mathcal{M} \models \hat{\varphi}(\hat{a})$ and $\dim(\hat{a}) \geq 2$. Put $k = \dim(\bar{a})$ and $\bar{a} = a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n$, where $a_{\leq k}$ is acl-independent and $a_j \in \mathrm{acl}(a_{\leq k})$ for $j = k+1, \ldots, n$. Take $c, d$ such that $\hat{C}(c, d, \hat{a}), c \downarrow \bar{a}, \dim(cd) = 2$. By wealy local modularity, there exists $B$ such that $B \downarrow \bar{a}, cd$ and $\bar{a} \downarrow_{\mathrm{acl}(\bar{a},B)\cap\mathrm{acl}(cdB)} cd$. Put $X = \mathrm{acl}(\bar{a}, B)$ and $Y = \mathrm{acl}(cdB)$.
Then we have $X \downarrow_{X\cap Y} Y$.

Claim 1 : $\dim(X \cap Y/B) = 1$

$\dim(XY/B) = \dim(Y/XB) + \dim(X/B) = \dim(cd/\bar{a}, B) + \dim(X/B)$ (as $cd \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{\bar{a}} B$)$=$ $\dim(cd/\bar{a}) + \dim(X/B) = 1 + k$. Since $k+1 = \dim(XY/B) = \dim(X/Y) + \dim(Y/B) = \dim(X/X \cap Y) + 2$, we have $\dim(X/X \cap Y) = k - 1 < k = \dim(X/B)$. So Claim 1 follows.

Take $t$ such that $\mathrm{acl}(tB) = X \cap Y$. We have $\bar{a} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{tB} cd$. As $\bar{a} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{tB} cd$ and $d \in \mathrm{acl}(c\bar{a}, B)$, $d \in \mathrm{acl}(ctB)$ follows. On the other hand, we have $\dim(\bar{a}/tB) = \dim(X/X \cap Y) < \dim(X/B) = k$, we may assume $a_k \in \mathrm{acl}(a_{\le k-1} tB)$. Take $\bar{u}, \bar{v} \subset B$ such that $d \in \mathrm{acl}(ct\bar{u})$ and $a_k \in \mathrm{acl}(a_{\le k-1} t\bar{v})$. Put $\bar{w} := \bar{u}\bar{v} \subset B$. As $\bar{w} \mathop{\smile\hskip-0.9em\vert\hskip0.5em} cd\bar{a}$, we have $a_k \notin \mathrm{acl}(a_{\le k-1}\bar{w})$ and $d \notin \mathrm{acl}(c, \bar{w})$. So we have $a_k \in \mathrm{acl}(a_{\le k-1} t\bar{w}) \setminus \mathrm{acl}(a_{\le k-1}\bar{w})$ and $d \in \mathrm{acl}(ct\bar{w}) \setminus \mathrm{acl}(c\bar{w})$. By exchange property of algebraic closure for geometric structures, $t \in \mathrm{acl}(a_{\le k}\bar{w})$ and $t \in \mathrm{acl}(cd\bar{w})$.

Take $\bar{b} \models \mathrm{tp}(\bar{a}/\mathrm{acl}(cd\bar{w}))$ such that $\bar{a} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{cd\bar{w}} \bar{b}$. Note that $\mathrm{acl}(cd\bar{w}) = \mathrm{acl}(cdt\bar{w}) = \mathrm{acl}(ct\bar{w})$. Take $\bar{b} \models \mathrm{tp}(\bar{a}/\mathrm{acl}(cd\bar{w}))$ such that $\bar{a} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{cd\bar{w}} \bar{b}$. Not that $\mathrm{acl}(cd\bar{w}) = \mathrm{acl}(cdt\bar{w}) = \mathrm{acl}(ct\bar{w})$.

Claim 2: $c \notin \mathrm{acl}(\bar{a}, \bar{b}, t\bar{w})$

As $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em} \bar{a}$ and $\bar{a}, cd \mathop{\smile\hskip-0.9em\vert\hskip0.5em} B$, we have $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em} \bar{a}, B$. As $t, \bar{w} \in \mathrm{acl}(\bar{a}, B)$, $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em} \bar{a}, t\bar{w}$ follows. By automorphism fixing $\mathrm{acl}(cd\bar{w})$ which sends $\bar{a}$ to $\bar{b}$, we have $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em} \bar{b}, t\bar{w}$. So $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{t\bar{w}} \bar{b}$ follows. Since $\bar{a} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{ct\bar{w}} \bar{b}$, we have $\bar{a}, c \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{t\bar{w}} \bar{b}$. So $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{\bar{a}, t, \bar{w}} \bar{b}$ and $c \notin \mathrm{acl}(\bar{a}, t\bar{w})$, Claim 2 follows. Take $\bar{b} \models \mathrm{tp}(\bar{a}/\mathrm{acl}(cd\bar{w}))$ such that $\bar{a} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{cd\bar{w}} \bar{b}$. Not that $\mathrm{acl}(cd\bar{w}) = \mathrm{acl}(cdt\bar{w}) = \mathrm{acl}(ct\bar{w})$.

Claim 3: $\hat{\bar{a}} \notin \mathrm{acl}(cd\bar{w})$

Note that $c \mathop{\smile\hskip-0.9em\vert\hskip0.5em} \bar{a}$ and $d \in \mathrm{acl}(c\hat{\bar{a}})$. As $B \mathop{\smile\hskip-0.9em\vert\hskip0.5em} cd\bar{a}$ and $\bar{w} \subset B$, we have $\bar{w} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{cd} \bar{a}$. So $\dim(\hat{\bar{a}}/cd\bar{w}) = \dim(\hat{\bar{a}}/cd) = \dim(cd\hat{\bar{a}}) - \dim(cd) = \dim(c\hat{\bar{a}}) - 2 = \dim(c) + \dim(\hat{\bar{a}}) - 2 = \dim(\hat{\bar{a}}) - 1 \ge 1$, because we assume $\dim(\hat{\bar{a}}) \ge 2$.

Claim 4: $\hat{\bar{a}} \notin \mathrm{acl}(\hat{\bar{b}})$: By Claim 3 and $\hat{\bar{a}} \mathop{\smile\hskip-0.9em\vert\hskip0.5em}_{cd\bar{w}} \hat{\bar{b}}$.

By Claim 2, $c \notin \mathrm{acl}(\bar{a}, \bar{b})$. As $\bar{b} \models \mathrm{tp}(\bar{a}/\mathrm{acl}(cd\bar{w}))$, we have $\hat{C}(c, d, \hat{\bar{a}}) \wedge \hat{C}(c, d, \hat{\bar{b}})$. $\{\hat{\bar{a}} \models \varphi(\hat{\bar{z}}) : \hat{C}(x, y, \hat{\bar{a}}) \wedge \hat{C}(x, y, \hat{\bar{b}})$ is infinite$\}$ is infinite by Claim 4. Here, we use elimination of $\exists^\infty$ for geometric structures. $\hat{\mathfrak{C}} := \{\hat{C}(x, y, \hat{\bar{a}}) : \hat{\bar{a}} \models \hat{\varphi}(\hat{\bar{z}})\}$ is not almost normal, a contradiction. $\Box$

I want a direct proof of the converse implication i.e. strong linearity implies weakly local modularity. Suppose non-weakly local modulatity. Construct a 2-dimensional almost normal interpretable family of plane curve.

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