Linear dynamics of weighted composition operators on function spaces over boundaries of homogeneous trees

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1 Introduction

The notion of hypercyclic operators has played the central role in the study of infinite dimensional linear dynamics. A bounded linear operator $T : X \to X$ on a separable Banach space X is said to be *hypercyclic* (or *topologically transitive*) if it has a dense orbit, that is, there exists a vector $x \in X$ (called a *hypercyclic vector*) such that $\{T^n(x) \mid n \geq 0\}$ is dense in X. If T is hypercyclic and moreover the set $\mathrm{Per}(T)$ of all periodic points of T is dense, then T is said to be *Devaney chaotic*. The existence of a linear chaotic operator is an *infinite dimensional* phenomenon: no linear maps on finite dimensional vector spaces can be hypercyclic. The monographs [7] and [18] are excellent sources of information.

The class of weighted backward shifts on $\ell^p(\mathbb{Z})$ or $\ell^p(\mathbb{N}), (1 \leq p < \infty)$ is the most well-studied class of operators and it serves as a testing ground for various aspects of operator theory. The hypercyclicity of such operators has been characterized in term of weights of the shifts (cf. [7] and [18]). Recently the results were generalized to shift operators on sequence spaces over directed trees ([1], [19], [20], [25] and [27]). Another class of linear operators whose dynamics is well-studied is that of weighted composition operators on various spaces of analytic functions and on L^p-spaces over

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measure spaces [2], [4], [6], [9], [16], [28], etc. Among these papers, the one by Pavone [28] is most relevant to us. It studies the composition operators on the L^p-spaces over the boundary of an infinite homogeneous tree.

A homogeneous infinite tree is a discrete analogue of a hyperbolic space ([12], [13]). Each such tree T admits a natural compactification whose boundary, denoted by Ω throughout, is a compact zero-dimensional metrizable space that admits a regular Borel probability measure μ, for which the space $L^p(\Omega) := L^p(\Omega, \mu)$ of all complex-valued L^p-functions on Ω ($1 \leq p < \infty$) is defined. Every hyperbolic graph automorophism $\varphi : T \to T$ induces a homeomorphism $\varphi : \Omega \to \Omega$ which defines the composition operator $C_\varphi : L^p(\Omega) \to L^p(\Omega); f \mapsto f \circ \varphi$. The main theorem of [28] states that C_φ is Devaney chaotic whenever $\varphi : T \to T$ is a hyperbolic graph automorphism. In an attempt to enlarge a class of "natural" examples of hypercyclic operators, it seems to be natural to consider the class of *weighted* composition operators induced by such automorphisms. In [22], [23] the author studied such operators and the present article reports some of the results in these papers.

Every hyperbolic automorphism $\varphi : T \to T$ of an infinite homogeneour tree T admits a unique geodesic, called the axis of φ, that defines two points α, β of the boundary Ω of T. The induced homeomorphism $\varphi : \Omega \to \Omega$ has these two points as the only fixed points, one of which is attracting and the other is repelling. Such a homeomorphism φ and a positive weight u on Ω define the weighted composition operator $W_{u,\varphi} : L^p(\Omega) \to L^p(\Omega)$ given by

$$(W_{u,\varphi} f)(\omega) = u(\omega) \cdot f(\varphi(\omega)), \ \omega \in \Omega. \tag{1}$$

The results of [22], [23] refine the main theorem of [28] and demonstrate that hypercyclicity and its variants of the operator $W_{u,\varphi}$ on $L^p(\Omega)$ have close connection to the values $u(\alpha), u(\beta)$. This seems to be a new phenomenon observed in the linear dynamics-context and similar phenomena are observed for backward shift operators over directed trees ([21], [24]). For $p = \infty$, the non-seprarability of $L^\infty(\Omega)$ leads us to study corresponding problems for $C_{\alpha,\beta}(\Omega)$, the Banach space of all continuous functions on Ω which vanish at the fixed points α and β, endowed with the supremum norm. The results for $C_{\alpha,\beta}(\Omega)$ are natural counterparts to those for L^p-spaces ($p < \infty$), in the sense that their statements are formally obtained by letting $p = \infty$ in the corresponding results for L^p-spaces.

Our proof relies on the standard hypercyclicity criterion and its variants and also a combinatorial analysis of hyperbolic automorphisms on T.

2 Preliminaries

Basic references on homogeneous trees are [11], [13] and [17]. For an integer $q \geq 2$, let T be an infinite homogeneous tree of valency $(q+1)$: each vertex of T is a vertex of exactly $(q+1)$ edges. Throughout, we specify a vertex o of T, referred to the *root* of T. Each pair u, v of vertices of T admits a unique *geodesic* $[u, v]$; a sequence $u = u_0, u_1, \ldots, u_n = v$ of mutually distinct vertices of T such that u_i and u_{i+1} are adjacent for each $i = 0, \ldots, n-1$. The number n is called the *length* of the geodesic $[u, v]$. The set $V(T)$ of all vertices of T admits a metric d given by $d(v, w) =$ the length of the geodesic $[v, w]$, for which the metric space $(V(T), d)$ is a Gromov-hyperbolic space ([13]).

Definition 2.1. *Let T be an infinite homogenous tree of valency $(q+1)$.*

(1) A geodesic ray ω emanating from a vertex v_0 is an infinite sequence $(v_0, v_1, \ldots, v_n, \ldots)$ of vertices of T such that for each $\ell > k \geq 0$, the subsequence $v_k, v_{k+1} \ldots, v_\ell$ is the geodesic $[v_k, v_\ell]$. In what follows we use the symbol

$$\omega = [v_0, v_1, \ldots, v_n, \ldots)$$

to indicate that the geodesic ray ω consists of the above vertices.

(2) Let $\omega = [o = u_0, u_1, \ldots, u_n, \ldots)$ be a geodesic ray emanating from o. For a vertex $v \in V(T)$, let $m = \max\{k \geq 0 \mid u_k \in [o, v]\}$. Then $[v, u_m] \cup [u_{m+1}, \ldots)$ is a geodesic ray emanating from v. It is convenient to write the geodesic ray $[v, u_m] \cup [u_{m+1}, \ldots)$ as $[v, \omega)$. In particular, ω is also denoted by $[o, \omega)$.

(3) Let Ω be the set of all geodesic rays emanationg from o. .

(3.1) For a vertex $v \in V(T)$, let $E(v)$ and $E_T(v)$ be the subsets of Ω and $V(T)$ respectively defined by

$$E(v) = \{\omega \in \Omega \mid v \in [o, \omega)\},$$
$$E_T(v) = \{w \in V(T) \mid v \in [o, w]\}.$$

(3.2) The set $\bar{T} = V(T) \cup \Omega$ is topologized as a compactification of the countable discrete space $V(T)$ so that $\{E(v) \cup E_T(v) \mid v \in [o, \alpha)\}$ forms a neighbor-

hood basis of a point $\alpha \in \Omega$. The space Ω is a compact zero-dimensional metrizable space, called the boundary of T. Each $E(v)$ is an open and closed subset of Ω.

(4) A regular Borel probability measure μ on Ω is defined by

$$\mu(E(v)) = \frac{1}{(q+1)q^{d(v,o)-1}}, \ v \in V(T). \tag{2}$$

(p) For $p \in [1, \infty)$, let $L^p(\Omega) := L^p(\Omega, \mu)$.

A geodesic line (α, β) between two distinct points α, β of Ω and the projection map $\pi_{\alpha,\beta} : \Omega \setminus \{\alpha, \beta\} \to (\alpha, \beta)$ are defined as follows.

Definition 2.2. *Let Ω be the boundary of an infinite homogeneous tree T of valencey $(q+1)$, and let α, β be two distinct points of Ω.*

(1) Let $m_{\alpha\beta}$ be the unique point defined by $[o, m_{\alpha\beta}] = [o, \alpha) \cap [o, \beta)$ and let $(\alpha, \beta) = [m_{\alpha\beta}, \alpha) \cup [m_{\alpha\beta}, \beta)$, called the geodesic line between α and β.
(2) For a geodesic line (α, β) between α and β, and for a vertex $v \in V(T) \setminus (\alpha, \beta)$, let $\pi_{\alpha\beta}(v)$ denotes the unique point given by $\{\pi_{\alpha\beta}(v)\} = (\alpha, \beta) \cap [v, \alpha) \cap [v, \beta)$.
(3) For a point $\omega \in \Omega \setminus \{\alpha, \beta\}$, let $\omega = [o, v_1, v_2, \ldots)$. The limit point $\lim_{i\to\infty} \pi_{\alpha\beta}(v_i)$ exists and is denoted by $\pi_{\alpha\beta}(\omega)$. It is the unique point satisfying $\{\pi_{\alpha\beta}(\omega)\} = (\alpha, \beta) \cap (\omega, \alpha) \cap (\omega, \beta)$.

Now we introduce the notion of hyperbolic automorphims on trees.

Definition 2.3. *A graph automorphism φ is said to be hyperbolic with displacement $d(d \geq 1)$ if there exists a geodesic line (α, β), called the axis of φ, such that*

$$\varphi((\alpha, \beta)) = (\alpha, \beta) \ \text{ and } \ d(\varphi(v), v) = d \ \text{ for each } v \in (\alpha, \beta).$$

Let $\varphi : T \to T$ be a hyperbolic automorphism with axis (α, β). For a geodesic ray $\omega = [o = v_0, v_1, v_2, \ldots) \in \Omega$, the sequence $[\varphi(v_0), \varphi(v_1), \varphi(v_2), \ldots)$ is a geodesic ray emanating from $\varphi(o)$. Let $m = \max\{k \geq 0 \mid \varphi(v_k) \in [o, \varphi(v_0)]\}$. Then $[o, \varphi(v_m)] \cup [\varphi(v_{m+1}), \varphi(v_{m+2}), \ldots)$ is a geodesic ray emanating from o. Let $\varphi_\Omega(\omega)$ be the point of Ω given by:

$$\varphi_\Omega(\omega) = [o, \varphi(v_m)] \cup [\varphi(v_{m+1}), \varphi(v_{m+2}), \ldots).$$

This defines a homeomorphism, simply denoted by $\varphi : \Omega \to \Omega$, that satisfies

$$\begin{aligned} &\varphi(\alpha) = \alpha,\ \varphi(\beta) = \beta, \\ &\pi_{\alpha\beta}(\varphi(\omega)) = \varphi(\pi_{\alpha\beta}(\omega)), \quad \omega \in \Omega \setminus \{\alpha, \beta\}. \end{aligned} \tag{3}$$

Assume further that the axis (α, β) of φ contains o and let $(\alpha, \beta) = \{v_i\}_{i\in\mathbb{Z}}$ with $v_0 = o$. Let d be the displacement of φ. Then either

$$\varphi(v_i) = v_{i+d} \text{ for each } i \in \mathbb{Z}, \quad \text{or} \tag{4}$$

$$\varphi(v_i) = v_{i-d} \text{ for each } i \in \mathbb{Z}. \tag{5}$$

The homeomorphism $\varphi : \Omega \to \Omega$ has α, β as the only fixed points. Assuming (4), we see that α is attracting and β is repelling: for each $a \in [o, \alpha)$ and for each $b \in [o, \beta)$, there exists an integer N such that, for each $n \geq N$, we have

$$\varphi^n(\Omega \setminus E(b)) \subset E(a), \quad \varphi^{-n}(\Omega \setminus E(a)) \subset E(b). \tag{6}$$

Also we have

$$\varphi(E(u)) = E(\varphi(u)), \quad u \in V(T) \setminus (\alpha, \beta). \tag{7}$$

Let $\varphi_*\mu$ be the push-forward measure defined by $\varphi_*\mu(A) = \mu(\varphi^{-1}(A))$. For a vertex v of T with $|\pi_{\alpha\beta}(v)| = \ell$, one can prove:

$$\frac{\varphi_*\mu(E(v))}{\mu(E(v))} = \frac{\mu(E(\varphi^{-1}(v)))}{\mu(E(v))} = \frac{q^{d(v,o)}}{q^{d(v,o)\pm d+2\epsilon\ell}} \leq q^d$$

for some $\epsilon \in \{0, 1\}$. Thus the measure $\varphi_*\mu$ is absolutely continuous with respect to μ and the Radon-Nikodym derivative satisfies

$$\frac{d\varphi_*\mu}{d\mu} \leq q^d \quad \mu - \text{a.e.}.$$

This implies the weighted composition operator $W_{\varphi,u}$ on $L^p(\Omega)$ or $C(\Omega)$, induced by a hyperbolic automorphism φ on T with displacement d whose axis contains o and a continuous positive weight $u : \Omega \to (0, \infty)$, is a bounded linear operator whose operator norm on $L^p(\Omega)$ (resp. on $C(\Omega)$) is bounded by $\|u\|_\infty q^{d/p}$ (resp. $\|u\|_\infty$) (cf. [28]).

Next we recall the hypercyclic criterion and its variants.

Definition 2.4. *Let X be a Banach space and let $T : X \to X$ be a bounded linear operator on X.*

(1) The operator T is said to be topologically mixing if for each pair U, V of non-empty open sets of X, there exists a positive integer N such that $T^{-n}(U) \cap V \neq \emptyset$ for each $n \geq N$. Every topologically mixing operator is hypercyclic.

(2) The operator $T : X \to X$ on X is said to be frequently hypercyclic if there exists a vector $x \in X$ such that, for each non-empty open subset V of X, we have

$$\liminf_{N \to \infty} \frac{\mathrm{card}(\mathbf{N}(x, V) \cap \{1, \ldots, N\})}{N} > 0$$

where $\mathbf{N}(x, V) = \{n \in \mathbb{N} \mid T^n(x) \in V\}$.

(3) A family $\{T_1, \ldots, T_M\}$ of bounded linear operators on X is said to be disjointly hypercyclic if there exits a vector $x \in X$ such that

$$\{(T_1^n(x), \ldots, T_M^n(x)) \mid n \in \mathbb{Z}_{\geq 0}\}$$

is dense in $\oplus_{i=1}^M X$.

Our proof of the (frequent) hypercyclicity and the Devaney chaoticity is based on the Hypercyclicity criterion and its variants:

Theorem 2.5. *Let $T : X \to X$ be a bounded linear operator on a Banach space X.*

(1) ([7, Theorem 1.6]) If there exist a dense set $\mathcal{D}$ of X, an increasing sequence $\{n_k\}$ of positive integers and a sequence of functions $\{S_k : \mathcal{D} \to X\}$ such that

(1.1) $\lim_{k\to\infty} \|T^{n_k}(x)\| = \lim_{k\to\infty} \|S_k(y)\| = 0$ for each $x, y \in \mathcal{D}$,

(1.2) $T^{n_k} \circ S_k = \mathrm{id}_{\mathcal{D}}$,

then T is hypercyclic. If moreover, the whole sequence $\mathbb{N}$ can be chosen as the above sequence $\{n_k\}$, then T is topologically mixing.

(2) ([7, Thereom 6.10, Theorem 6.18]) If there exist a dense set $\mathcal{D}$ of X and a map $S : \mathcal{D} \to \mathcal{D}$ such that

(2.1) $\sum_n T^n(x)$ and $\sum_n S^n(x)$ are unconditionally convergent for each $x \in \mathcal{D}$,

(2.2) $T \circ S = \mathrm{id}_{\mathcal{D}}$,

then T is frequently hypercyclic and Devaney chaotic.

(3) ([10, Proposition 2.6]). Let $\{T_i \mid i = 1, \ldots, M\}$ be a family of bounded linear operators on X. Assume that there exist a sequence $\{n_k\}$ of positive integers, dense subsets $\mathcal{D}_0, \ldots, \mathcal{D}_M$ of X and a sequence of maps $S_{i,k} : \mathcal{D}_i \to X$ such that

(3.1) $\lim_{k\to\infty} \|T_i^{n_k}(x_0)\| = 0$ *for each* $x_0 \in \mathcal{D}_0$ *and for each* $i = 1, \dots, M$*,*

(3.2) $\lim_{k\to\infty} \|S_{i,k}(x_i)\| = 0$ *for each* $x_i \in \mathcal{D}_i$ *and for each* $i = 1, \dots, M$*,*

(3.3) $\lim_{k\to\infty} \|T_i^{n_k} S_{j,k}(x_i)\| = 0$ *for each* $x_i \in \mathcal{D}_i$ *and for each* $i, j \in \{1, \dots, M\}$ *with* $j \neq i$*, and*

(3.4) $\lim_{k\to\infty} \|T_i^{n_k} S_{i,k}(x_i) - x_i\| = 0$ *for each* $x_i \in \mathcal{D}_i$ *and for each* $i = 1, \dots, M$*.*

Then $\{T_i \mid i = 1, \dots, M\}$ *is disjointly hypercyclic.*

3 Results

Throughout T denotes an infinite homogeneous tree of valency $(q+1)$, $q \geq 2$, with the root o, and Ω denotes its boundary endowed with the topology and the measure described in the previous section. Also we assume that $p \in [1, \infty)$. The results of [22], [23] are motivated by the next theorem.

Theorem 3.1. *[28] Let* $\varphi : T \to T$ *be a hyperbolic automorphism of an infinite homogeneous tree* T *of valency* $(q+1)$*. The composition operator* $C_\varphi : L^p(\Omega) \to L^p(\Omega)$ *is hypercyclic and Devaney chaotic.*

It should be mentioned here that the above theorem is also a consequence of [14, Theorem M]. Our first result is stated as follows.

Theorem 3.2. *[22] Let* $\varphi : T \to T$ *be a hyperbolic automorphism on* T *with displacement* d *whose axis* (α, β) *contains* o *and assume that* α *(resp.* β*) is the attracting (resp. the repelling) fixed point of the induced homeomorphism* $\varphi : \Omega \to \Omega$*. Also let* u *be a continuous positive weight on* Ω *and let* $W = W_{u,\varphi} : L^p(\Omega) \to L^p(\Omega)$ *be the weighted composition operator (1) induced by* φ *and* u*.*

(1) If W *is hypercyclic, then we have* $u(\alpha) \geq q^{-d/p}$ *and* $u(\beta) \leq q^{d/p}$*.*

(2) Assume that $u(\alpha) > q^{-d/p}$ *and* $u(\beta) < q^{d/p}$*. Then we have the following.*

(2.1) The operator W *is topologically mixing, Devaney chaotic and frequently hypercyclic.*

(2.2) There exists a topological embedding $e : [0,1]^{\mathbb{Z}} \to L^p(\Omega)$ *such that* $W \circ e = e \circ \sigma^d$*, where* $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$ *is the shift homeomorphism defined by*

$$\sigma((x_n)_{n\in\mathbb{Z}})_i = x_{i+1}, \quad (x_n) \in [0,1]^{\mathbb{Z}}.$$

In particular the topological entropy (see [29]) of W *is* ∞*.*

The statement (2.2) above is inspired by [3, Theorem 10.2]. We observe that the operator $W := q^{-1/p}C_\varphi$ induced by a hyperbolic automorphism with displacemet 1 has the operator norm 1 and hence is not hypercyclic. This shows that the hypothesis " $u(\alpha) > q^{-d/p}, u(\beta) < q^{d/p}$" in (2) of the above theorem may not be weakend to " $u(\alpha) \geq q^{-d/p}, u(\beta) \leq q^{d/p}$."

It is shown in [22] that (i) the disjoint hypercyclicity of a family of operators "generically" leads to a situation, in which the attracting points of the associated automorphisms are mutually distinct and (ii) the existence of a common hypercyclic vector of a family of composition operators induced by hyperbolic automorphisms implies that the set of the associated attracting fixed points has empty interior. The above (i) leads to the set up of the next result.

Theorem 3.3. *[22] Let $\varphi_1, \ldots, \varphi_M$ be hyperbolic automorphisms on T with displacements $d_1 \ldots, d_M$ respectively such that the attracting fixed points $\alpha_1, \ldots, \alpha_M$ are mutually distinct. For $i = 1, \ldots, M$, let (α_i, β_i) be the axis of φ_i and assume that $o \in \cap_{i=1}^M (\alpha_i, \beta_i)$. Also let $u^1, \ldots, u^M$ be continuous positive weights such that*

$$u^i(\alpha_i) > q^{-d_i/p} \text{ and } u^i(\beta_i) < q^{d_i/p} \quad \text{or each } i = 1, \ldots, M.$$

Let $W_i = W_{u^i,\varphi_i} : L^p(\Omega) \to L^p(\Omega)$. Then the family $\{W_i : L^p(\Omega) \to L^p(\Omega) \mid i = 1, \ldots M\}$ is disjointly hypercyclic.

A similar (but more technical) result motivated by (ii) is proved in [23].

For $p = \infty$, the situation is slightly different. The space $L^\infty(\Omega)$ fails to be separable and any operator on $L^\infty(\Omega)$ cannot be hypercyclic, which leads us to study the space $C(\Omega)$ instead. Since the composition operator $C_\varphi : C(\Omega) \to C(\Omega)$ has norm one for each homeomorphism φ on Ω, any hypercyclic weighted composition operator on a closed subspace of $C(\Omega)$ must have nontrivial weight. Recall that for points α, β of Ω, $C_{\alpha,\beta}(\Omega)$ denotes the space of continuous functions on Ω that vanish on $\{\alpha, \beta\}$. All of the statement (2) of the next theorem are obtained formally from the corresponding statement of Theorem 3.2 by letting $p = \infty$.

Theorem 3.4. *[22] Let $\varphi : T \to T$ be a hyperbolic automorphism whose axis (α, β) contains o. Assume that α (resp. β) is the attracting (resp. the repelling) fixed point of $\varphi : \Omega \to \Omega$. Let u be a continuous positive weight on Ω and let $W = W_{u,\varphi}$ be the weighted composition operator (1) induced by u and φ.*

(1) The operator $W : C(\Omega) \to C(\Omega)$ is not hypercyclic.

(2) For the operator $W : C_{\alpha,\beta}(\Omega) \to C_{\alpha,\beta}(\Omega)$, we have the following:

(2.1) If W is hypercyclic, then we have $u(\alpha) \geq 1$ and $u(\beta) \leq 1$.

(2.2) Assume that $u(\alpha) > 1$ and $u(\beta) < 1$. Then we have the following.

(2.2.1) The operator W is topologically mixing, Devaney chaotic and frequently hypercyclic.

(2.2.2) There exists a topological embedding $e : [0,1]^{\mathbb{Z}} \to C_{\alpha,\beta}(\Omega)$ such that $W \circ e = e \circ \sigma^d$, where $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$ is the shift homeomorphism of Theorem 3.2, and W has the infinite topological entropy.

The composition opertor $C_\varphi : C_{\alpha,\beta}(\Omega) \to C_{\alpha,\beta}(\Omega)$ induced by a hyperbolic automorphism φ which has $\alpha, \beta \in \Omega$ as fixed points has the operator norm 1and thus is not hypercyclic. Hence the hypothesis "$u(\alpha) > 1 > u(\beta)$" in (2.2) cannot be weakend to "$u(\alpha) \geq 1 \geq u(\beta)$."

A result that corresponds to Theorem 3.3 for $C(\Omega)$ is also proved in [22].

4 Remark on invariant measure

The notion of frequent hypercyclicity is motivated by ergodic theory and it is natural to ask as to whether a frequently hypercyclic bounded linear opeartor $T : X \to X$ on a separable Banach space X admits an invariant (finite) measure ν on X so that the measure dynamics (T, X, ν) is ergodic. The next theorem provides us with a sufficient condition for the existence of such a measure. See also [5] for a stronger result.

Theorem 4.1. *[26] Let $T : X \to X$ be a bounded linear operator on a separable Banach space X. Assume that there exist a dense set $\mathcal{D}$ of X and a sequence of maps $\{S_n : \mathcal{D} \to X \mid n \geq 1\}$ such that*

(1) $\sum_n T^n(x)$ and $\sum_n S_n(x)$ are unconditionally convergent for each $x \in \mathcal{D}$,

(2) $T^n \circ S_n = \mathrm{id}_{\mathcal{D}}$ and $T^m \circ S_n = S_{n-m}$ for eachpair of integers n, m with $n > m$.

Then there exists a T-invariant Borel probability measure ν on X such that

(3) $\nu(U) > 0$ for each nonempty open subset U of X, and

(4) the dynamics (T, X, ν) is strongly mixing

The conclusion of the above theorem implies, via the Birkoff ergodic theorem [29], that T is frequently hypercyclic. The proofs of Theorem 3.1 (2.1) and Theorem 3.3 (2.2.1) imply the next corollary.

Corollary 4.2. *Let $\varphi : T \to T$ be a hyperbolic automorphism on T with displacement d whose axis (α, β) contains o and assume that α (resp. β) is the attracting (resp. the repelling) fixed point of the induced homeomorphism $\varphi : \Omega \to \Omega$. Also let u be a continuous positive weight on Ω and let W be the weighted composition operator on $L^p(\Omega)$ (resp. $C_{\alpha,\beta}(\Omega)$) induced by φ and u. If $u(\alpha) > q^{-d/p}$ and $u(\beta) < q^{d/p}$ (resp. $u(\alpha) > 1, u(\beta) < 1$), then $L^p(\Omega)$ (resp. $C_{\alpha,\beta}(\Omega)$) admits a W-invariant Borel probability measure ν such that $(W, L^p(\Omega), \nu)$ (resp. $(W, C_{\alpha,\beta}(\Omega), \nu)$) is strongly mixing.*

The study of measure theoretic dynamics of the operator W above is a subject of future study.

References

[1] E. Abakumov and A. Abbar, *Several dynamical properites of shifts acting on directed trees*, Canad. J. Math. 2025, pp. 1-30.

[2] M.R. Azimi and M.R. Jabbarzadeh, *Hypercyclicity of weighted composition operators on L^p-spaces,* Mediterr J. Math. 19 (2022), no.164, 16pp.

[3] S. Bartroll, F. Martinez-Giménez, M. Murill-Arcila and A. Peris, *Cantor sets, Berunoulli shifts and linear dynamics*, in Descrptive topology and functional analysis, Springer Proc. Math. Stat. 80 (2014), 195-207.

[4] F. Bayart, *Common hypercyclic vectors for composition operators*, J. Op. Theory 52 (2004), 353-370.

[5] F. Bayart and É. Matheron, *Mixing operators and small subsets of the circle,* J. reine angew. Math. 715 (2016), 75-123.

[6] F. Bayart, U.B. Darji and B. Pires, *Topological transitivity and mixing of composition operators,* J. Math. Anal. Appl. 465 (2018), 125-139.

[7] F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Math. 179 (2009), Cambridge.

[8] L. Bernal-González, *Disjoint hypercyclic operators,* Stud. Math. 182 (2007), 113-131.

[9] N.C. Bernardes, Jr., U.B. Darji and B. Pires, *Li-Yorke chaos for composition operators on L^p-spaces*, Monatsh. für Math. 191 (2020), 13-35.

[10] J. Bès and A. Peris, *Disjointness in hypercyclicity*, J. Math. Anal. Appl. 336 (2007), 297-315.

[11] P. Cartier, *Fonctions harmoniques sur un arbre,* Symp. Math. 9, Calcoclo Probab. Teor-Turbolenza 1971 (1972), 203-207.

[12] J.M. Cohen and F. Colonna, *Embeddings of trees in hyperbolic disk,* Complex variables 24 (1994), 311-335.

[13] M. Coornaert, T. Delzant and A. Papadopoulos, *Géométrie et théorie de groupes*, Lect. Notes in Math. 1441 (1990), Springer.

[14] E. D'Aniello, U.B. Darji and M. Maiuriello, *Shift-like operators on $L^p(X)$*, J. Math. Anal. Appl. 515 (2022), 126363.

[15] E. D'Aniello, U.B. Darji and M. Maiuriello, *Generalized hyperbolicity and shadowing in L^p-spaces*, J. Diff. Eq. 298 (2021), 68-94.

[16] U.B. Darji and B. Pires, *Chaos and frequent hypercyclicity of composition operators,* Proc. Edinb. Math. Soc., 64 (2021), 513-531.

[17] A. Figá-Talamanca and M.A. Picardello, *Harmonic analysis on free groups,* Lect. Pure and Appl. Math. 87 (1983), Marcel Dekker.

[18] K-G. Grosse-Erdmann and A. Peris Manguillot, Linear chaos, Springer University Text, Springer (2011).

[19] K-G. Grosse-Erdmann and D. Papathanasion, *Dynamics of weighted shifts on directed trees*, Indiana Univ. Math. J. 72 (1), (2023), 263-299.

[20] K-G. Grosse-Erdmann and D. Papathanasion, *Chaotic weighted shifts on directed trees*, preprint.

[21] A. Han and K. Kawamura, *Hypercyclicity of weighted backward shifts and branching growth of directed trees*, submitted.

[22] K. Kawamura, *Weighted composition operators on L^p-and C_0-function spaces over boundaries of homogeneous trees*, submitted.

[23] K. Kawamura, *Weighted composition operators on L^p-and C_0-function spaces over boundaries of homogeneous trees II*, to appear in Houston J. Math..

[24] K. Kawamura, *Hypercyclicity of weighted backward shifts on the little Hardy and the little Lipschitz spaces over directed trees*, submitted.

[25] R.A. Martinez-Avendaño, *Hypercyclicity of shifts on weighted L^p spaces of di-*

rected trees, J. Math. Analy. Appl. 446(1) (2017), 823-842.

[26] M. Murillo-Arcila and A. Peris, *Strong mixing measures for linear operators and frequent hypercyclicity,* J. Math. Anal. Appl. 398 (2013), 462-465.

[27] R.A. Martinez-Avendaño and E. Rivera-Guasco, *The forward and backward shift on the Lipschitz space of a tree*, Integr. Eq. Oper. Theory 92 (3) (2020). https://doi.org/10.1007/s00020-019-2558-7

[28] M. Pavone, *Chaotic composition operators on trees*, *Houston J. Math. 18 (1992), 47-56.*

[29] P. Walters, An introduction to Ergodic theory, G.T.M. 79, Springer (1982).

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