Classification of two-faced independences

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概 要

Two-faced independences are independence relations for pairs of noncommutative random variables, such as bifree independence, which models the relation between left and right regular representation of generators of the free group in the canonical tracial state. Around 2000, in works of Speicher, Ben Ghorbal & Schürmann, and Muraki, a complete classification of "single-faced" independences was obtained: the only independences in this case are tensor, free, Boolean, monotone and anti-monotone independence. I report on the current status of the classification program for two-faced independences.

1 Universal products and independences

Given a noncommutative probability space, i.e., a pair $(\mathcal{A}, \Phi)$ which consists of a unital $*$-algebra $\mathcal{A}$ and a state $\Phi\colon \mathcal{A} \to \mathbb{C}$, we call the numbers $\Phi(a^k)$ the *moments* of a selfadjoint element $a = a^* \in \mathcal{A}$; under suitable technical assumptions, the commutative $*$-algebra generated by a can be identified with an algebra of functions on a classical probability space such that the restriction of Φ becomes the classical expectation. For $a_1, a_2 \in \mathcal{A}$, expressions of the form $\Phi(a_{i_1} a_{i_2} \dots a_{i_n})$ are called *mixed moments*. In many noncommutative situations, there is a specific structure which allows to compute all mixed moments from the moments of a_1 and a_2 and this insight led to studying the idea of "noncommutative independences". Central limit theorems for such independences hold under very general conditions (see [1]) and rich theories can be developed with surprising applications, for

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example in spectral analysis of graphs or random matrix theory, see [9, 16] or [12], respectively, and references therein.

Classical stochastic independence can be described by the slogan "joint distribution = product distribution".[1] A natural idea when looking for independences in noncommutative probability is therefore to redefine the terms "joint distribution" and "product distribution" in a general ($*$-)algebraic context and use the same slogan to get a corresponding *noncommutative independence*. Based on this idea, Ben Ghorbal and Schürmann axiomatized noncommutative independences in [2] and proved that they correspond exactly to Speicher's "universal products" in [17]. It turned out that the number of noncommutative independences in the sense of [2] is finite and small. Under the assumption of symmetry, Speicher showed that there are only three such independences, namely tensor, free and Boolean [17, Thm. 2]. Muraki extended Ben Ghorbal and Schürmann's framework to cover also non-symmetric independences and proved that, without symmetry, monotone and antimonotone independence are the only additional examples [13, 14]. About 10 years later, Muraki presented a variant of his result where he replaces a certain normalizability condition by a positivity condition, leading to a simpler and more conceptual proof [15]. Assuming neither normalizability nor positivity, all independences can be rescaled, and Boolean independence even allows for a continuous 2-parameter deformation [6]. We refer to results of the mentioned kind as *classification of independences*.

More independences can only be found by loosening the axioms of Ben Ghorbal and Schürmann. A crucial (and quite restrictive) axiom is the axiom of universality: independence should be preserved under certain morphisms. Ben Ghorbal and Schürmann work with the category of (complex associative) algebras. A newer development, started by Voiculescu's study of

[1] More precisely, random variables $X_k\colon \Omega \to E_k$ ($i \in I$ from some, for sake of simplicity finite, index set I) defined on a classical probability space $(\Omega, \Sigma, \mathbb{P})$ with values in measurable spaces $(E_i, \mathcal{E}_i)$ are independent if and only of the joint distribution $\mathbb{P}_{(X_i : i \in I)}$ equals the product distribution $\prod_{i\in I} \mathbb{P}_{X_i}$, where both probability measures are defined with respect to the product σ-algebra $\prod_{i\in I} \mathcal{E}_i$.

bifreeness [18], is to consider the category of *multi-faced* algebras instead, i.e., algebras B with a given free product decomposition $B = \bigsqcup_{\bullet\in\mathcal{F}} B^\bullet$ of subalgebras $B^\bullet \subset B$ called *faces*. An $\mathcal{F}$-faced algebra with an involution $*$ which restricts to involutions of the faces $B^\bullet$ is called an $\mathcal{F}$-faced $*$-algebra. In these notes, $\mathcal{F}$ is always the two-element set $\mathcal{F} = \{\circ, \bullet\}$ and we will write *two-faced* instead of $\mathcal{F}$-faced.

One subtlety deserves attention. The algebras considered so far are not necessarily unital. When we need a unit, we work with the unitization. This is mainly important in the context of positivity.

A two-faced homomorphism between two-faced algebras A, B is an algebra homomorphism $j\colon A \to B$ with $j(A^\bullet) \subseteq B^\bullet$ for $\bullet \in \mathcal{F}$. The free product $\bigsqcup B_i$ of two-faced algebras B_i is a two-faced algebra with faces $(\bigsqcup B_i)^\bullet = \bigsqcup B_i^\bullet \subseteq \bigsqcup B_i$.

Given any product operation $\odot$, which takes as input linear functionals φ_i on two-faced algebras B_i and produces as output a linear functional $\bigodot \varphi_i$ on $\bigsqcup B_i$, we say that two-faced homomorphisms $j_i\colon B_i \to \mathcal{A}$ are $\odot$-independent w.r.t. a linear functional $\Phi\colon \mathcal{A} \to \mathbb{C}$ if

$$\Phi \circ \bigsqcup j_i = \bigodot (\Phi \circ j_i).$$

The definition stems from the interpretation of the left hand side as an analogue of joint distribution and of the right hand side as an analogue of a product distribution in classical probability.

Properties of the independence are reflected by properties of the corresponding product operation. The following definition (a special case of *u.a.u.-products* as defined in [11], see [3, Def. 3.3 and Rem. 3.4]) has proved fruitful.

Definition 1.1. A *two-faced universal product* is a product operation which assigns to each pair (φ_1, φ_2) of linear functionals on two-faced algebras B_1, B_2 a linear functional $\varphi_1 \odot \varphi_2$ on $B_1 \sqcup B_2$ with the following properties.

unitality: $0 \odot \varphi = \varphi = \varphi \odot 0$ for all $\varphi\colon B \to \mathbb{C}$, B an $\mathcal{F}$-faced algebra; $0 \in \{0\}'$ the trivial functional

associativity: $(\varphi_1 \odot \varphi_2) \odot \varphi_3 = \varphi_1 \odot (\varphi_2 \odot \varphi_3)$ for all $\varphi_i \colon B_i \to \mathbb{C}$, B_i two-faced algebras

universality: $(\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2) = (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2)$ for all $\varphi_i \colon B_i \to \mathbb{C}$, $j_i \colon A_i \to B_i$ two-faced algebra homomorphisms

We define two more properties which an two-faced universal product may have or not.

symmetry: $\varphi_1 \odot \varphi_2 = \varphi_2 \odot \varphi_1$ for all $\varphi_i \colon B_i \to \mathbb{C}$, B_i two-faced algebras

positivity: $1 \oplus \varphi_1, 1 \oplus \varphi_2 \geq 0 \implies 1 \oplus (\varphi_1 \odot \varphi_2) \geq 0$ for all $\varphi_i \colon B_i \to \mathbb{C}$, B_i two-faced $*$-algebras; $1 \oplus \varphi$ denotes the unital extension of a linear functional φ on B to the unitization of B

Every single-faced universal product $\odot$ can trivially be regarded as a two-faced universal product by "ignoring" the two-faced structure. Also, every two-faced universal product defines two single-faced universal products by restricting to the two faces. These restrictions usually give the name of a two-faced product (e.g., the bifree product restricts to the free product on both faces), but be aware it is not at all clear a priori how many different two-faced universal products exist with the same restrictions (if any).

Several non-trivial examples of two-faced universal products besides bifreeness have been found, but in contrast with the single-faced setting, we do not have a complete classification, not even if we restrict to positive and symmetric two-faced universal products. These notes deal with partial results obtained so far in this direction.

2 Highest coefficients of positive and symmetric two-faced universal products

In this section we summarize the main results of [7], in which positive *symmetric* 2-faced universal products are *almost* classified by studying their associated moment-cumulant relations. The crucial concept is that of *highest coefficients*, which we now recall.

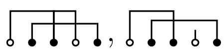
A set partition of $[n] := \{1, \ldots, n\}$ is a set $\pi = \{\beta_1, \ldots, \beta_k\}$ of nonempty pairwise disjoint subsets $\emptyset \neq \beta_i \subset [n]$ with $[n] = \bigcup \beta_i$. The elements of $[n]$ are called *legs* of the partition π.

A finite word $\mathbf{f} \in \{\circ, \bullet\}^n$ is called a *face structure* and a finite word $\mathbf{b} \in [k]^n$ for some $k \in \mathbb{N}$ is called a block structure.

We denote by

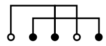
- $\mathcal{P}(\mathbf{f})$ the collection of all set partitions of $[n]$, for each face structure $\mathbf{f}$
- $\mathcal{P}$ the disjoint union of all $\mathcal{P}(\mathbf{f})$; formally, an element of $\mathcal{P}$ can be seen as a pair $(\pi, \mathbf{f})$ where π is a set partition of $[n]$ and $\mathbf{f} \in \mathcal{F}^n$ is a face structure.

Note that the set partitions in $\mathcal{P}(\mathbf{f})$ do not actually depend on the face structure, but we still need to distinguish between them. When we visualize partitions by their arc-diagrams in the usual way, this is done by coloring the nodes, for example



respectively correspond to

$$\{\{134\}, \{25\}\}, \{\{13\}, \{25\}, \{4\}\} \in \mathcal{P}(\circ\bullet\bullet\circ\bullet).$$

We identify a partition with its arc diagram, e.g.,  $=$ $\{\{134\}, \{25\}\} \in \mathcal{P}(\circ\bullet\bullet\circ\bullet)$.

With a pair $(\mathbf{f}, \mathbf{b})$ of a face structure $\mathbf{f}$ and a block structure $\mathbf{b}$ of the same length n, we associate a partition $\pi_{\mathbf{b}}^{\mathbf{f}} \in \mathcal{P}(\mathbf{f})$ with blocks $\beta_i := \{\ell \in [n] : \mathbf{b}(\ell) = i\}$. Conversely, given any $\pi = \{\beta_1, \ldots, \beta_k\} \in \mathcal{P}$ (with blocks labeled $1, \ldots, k$), we denote $\mathbf{f}_\pi$

its face structure, such that $\pi \in \mathcal{P}(\mathbf{f}_\pi)$ and $\mathbf{b}_\pi$ its block structure, i.e., the word such that $\mathbf{b}_\pi(\ell) = i$ whenever $\ell \in \beta_i$.

Definition 2.1. Let $\odot$ be a positive two-faced universal product. The *highest coefficients* of $\odot$ are the unique coefficients $\alpha(\pi), \pi \in \mathcal{P}$, such that for all $a_1 \dots a_n \in B_1 \sqcup \dots \sqcup B_k$, $a_\ell \in B_1^\circ \cup B_1^\bullet \cup \dots \cup B_k^\circ \cup B_k^\bullet$ with

- $\mathbf{f}_\pi \in \{\circ, \bullet\}^n$ encoding face membership,
- $\mathbf{b}_\pi \in [k]^n$ encoding block membership,

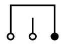
i.e., $a_\ell \in B_{\mathbf{b}_\pi(\ell)}^{\mathbf{f}_\pi(\ell)}$ for all $\ell \in [n]$, the expansion of the universal product according to the 'Central Coefficient Theorem' [11, Thm. 4.2] takes the form

$$\varphi_1 \odot \dots \odot \varphi_k(a_1 \dots a_n) = \alpha(\pi) \cdot \varphi_1 \left(\prod_{\mathbf{b}(\ell)=1}^{\rightarrow} a_\ell \right) \dots \varphi_k \left(\prod_{\mathbf{b}(\ell)=k}^{\rightarrow} a_\ell \right) + \text{nonlinear terms},$$

where linearity is meant with respect to $\varphi_1, \dots, \varphi_k$.

Example 2.2. If, in the notation of the previous definition,

$$\varphi_1 \odot \varphi_2(a_1 a_2 a_3) = \lambda \varphi_1(a_1 a_3) \varphi_2(a_2) + \mu \varphi_1(a_1) \varphi_1(a_3) \varphi_2(a_2)$$

with $\pi =$ , $\beta_1 = \{1, 3\}$, and $\beta_2 = \{2\}$ (i.e., $a_1 \in B_1^\circ, a_2 \in B_2^\circ, a_3 \in B_1^\bullet$), then $\alpha(\pi) = \lambda$.

It follows from Manzel and Schürmann's cumulant theory for universal products that the highest coefficients are enough to determine the universal product completely [11, Thm. 7.2]. The simplified idea is that (in the symmetric case!) independence is characterized by vanishing of mixed cumulants, and in the definition of cumulants, which are some sort of "linearization", only the highest coefficients of the universal product appear.

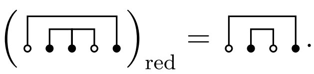
While a universal product is determined by its highest coefficients, it remained a sort of mystery what exactly a family of numbers has to fulfill in order be the highest coefficients of some universal product. The following results shed some light on this problem.

Theorem 2.3. *Let $\odot$ be a positive and symmetric two-faced universal product. Then its highest coefficients fulfill:*

(i) $\alpha(\,|\,) = 1$ *(the face is arbitrary).*

(ii) $\alpha(\,|\;|\,) = 1$ *(both faces are arbitrary).*

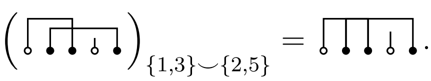
(iii) $\alpha_\pi = \alpha_{\mathrm{red}(\pi)}$, *where the* reduced *partition* $\mathrm{red}(\pi)$ *is obtained from π by merging neighboring legs of the same face in the same block into only one leg of that face in that block, e.g.,*



(iv) *Suppose $\pi \in \mathcal{P}(\mathbf{f})$ has blocks β_1, β_2 with neighboring legs in the same face, i.e. there exist $i \in \beta_1$, $j \in \beta_2$, $|i-j| = 1$, $\mathbf{f}(i) = \mathbf{f}(j)$. Then*

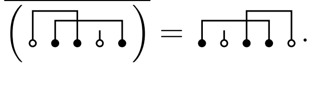
$$\alpha_\pi = \alpha_{\pi_{\beta_1 \smile \beta_2}} \cdot \alpha_{\{\beta_1, \beta_2\}}$$

where $\pi_{\beta_1 \smile \beta_2}$ is obtained from π by replacing the two separate blocks β_1, β_2 by their union $\beta_1 \cup \beta_2$, e.g.,



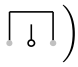
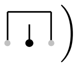
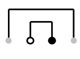
(v) $\alpha_\pi = \alpha_\sigma$ *whenever π and σ only differ in the faces of first or last leg. (We therefore will often not color the first and last leg at all.)*

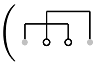
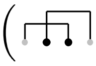
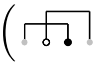
(vi) $\alpha_{\overline{\pi}} = \overline{\alpha_\pi}$, *where $\overline{\pi}$ denotes the mirror image of π, i.e., $\mathbf{b}_{\overline{\pi}}(\ell) = \mathbf{b}_\pi(n+1-\ell)$ and $\mathbf{f}_{\overline{\pi}}(\ell) = \mathbf{f}_\pi(n+1-\ell)$, e.g.,*



Conversely, every family of complex numbers $(\alpha_\pi)_{\pi \in \mathcal{P}}$ which fulfills these properties, is the family of highest coefficients of a (automatically unique) symmetric 2-faced universal product.

This theorem tells us in particular, how one can calculate highest coefficients of complicated partitions from highest coefficients of simpler partitions. Indeed, it turns out that, using the properties (i) to (vi), all highest coefficients are determined by the following six *basic coefficients* for *nestings* and *crossings* (the faces for the first and last leg are arbitrary):

$\nu_\circ := \alpha\Big($  $\Big)$, $\quad \nu_\bullet := \alpha\Big($  $\Big)$, $\quad \nu_{\circ\bullet} := \alpha\Big($  $\Big)$

$\xi_\circ := \alpha\Big($  $\Big)$, $\quad \xi_\bullet := \alpha\Big($  $\Big)$ $\quad \xi_{\circ\bullet} := \alpha\Big($  $\Big)$

Furthermore, it follows that these basic coefficients must fulfill the following compatibility conditions:

- $\nu_\circ, \nu_\bullet, \xi_\circ, \xi_\bullet \in \{0,1\}$, $\nu_{\circ\bullet}, \xi_{\circ\bullet} \in \{0\} \cup \mathbb{T}$
- $\nu_\circ = 0 \implies \xi_\circ = \nu_{\circ\bullet} = \xi_{\circ\bullet} = 0$
- $\nu_\bullet = 0 \implies \xi_\bullet = \nu_{\circ\bullet} = \xi_{\circ\bullet} = 0$
- $\nu_{\circ\bullet} \neq \xi_{\circ\bullet} \implies \xi_\circ = \xi_\bullet = 0$
- $\xi_\circ = 0$ or $\xi_\bullet = 0 \implies \nu_{\circ\bullet} = 0$ or $\xi_{\circ\bullet} = 0$

This information is actually enough to determine all families of coefficients which fulfill properties (i) to (vi). The result is described by Figure 1: the Hasse diagram for the set of symmetric universal products whose highest coefficients fulfill properties (i) to (vi) with respect to the pre-order $\odot_1 \leq \odot_2 \iff \forall_{\pi\in\mathcal{P}} |\alpha_1(\pi)| \leq |\alpha_2(\pi)|$. We can conclude:

Theorem 2.4. *Every positive and symmetric universal product is one of the products in Figure 1.*

It should be emphasized that this theorem does **not** state that all of the products in Figure 1 are actually positive!

3 Representing universal products on tensor or free product of Hilbert spaces

In this section, we describe results of [5] which prove positivity of all but four of the universal products in Figure 1, with the

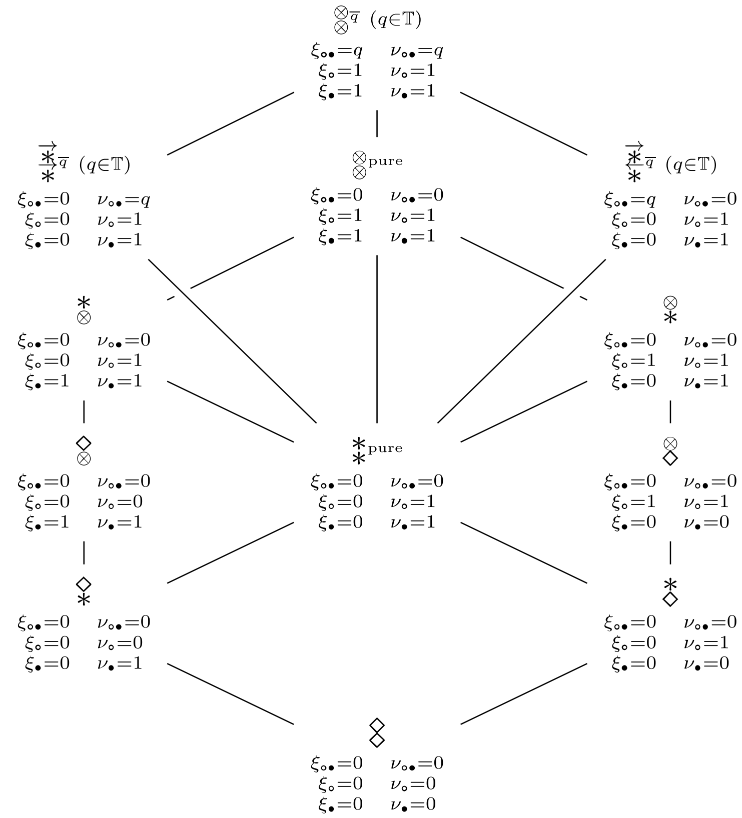


図 1: Hasse diagram of 2-faced symmetric universal products with properties (i) to (vi) for the pre-order $\odot_1 \leq \odot_2 \iff \forall_{\pi\in\mathcal{P}}|\alpha_1(\pi)| \leq \|\alpha_2(\pi)|$.

exception of the free-tensor product $\overset{*}{\otimes}$, the tensor-free product $\overset{\otimes}{*}$, the pure tensor product $\overset{\otimes}{\otimes}{}^{\text{pure}}$, and the pure free product $\overset{*}{*}{}^{\text{pure}}$. Positivity in these exceptional cases remains an open problem for now. In [5] also non-symmetric universal products are considered, but for these notes we focus on the symmetric case.

Let us recall the construction of the bifree product of linear functionals described in [18, Prop. 2.9 and Cor. 2.10 b]. For every linear functional $\varphi\colon A \to \mathbb{C}$ on an algebra (the role of unitality shall not be discussed here) one can find a vector space H with a decomposition $H = \mathbb{C}\Omega \oplus \hat{H}$ where Ω is a non-zero vector and $\hat{H}$ a complementary subspace such that $\varphi(a)\Omega = P_\Omega \pi(\cdot)\Omega$. If φ (or its unital extension $1 \oplus \varphi$) is a state, one can use the GNS representation of φ (or of $1 \oplus \varphi$) and write $\varphi = \langle \Omega, \pi(\cdot)\Omega\rangle$; in this case, π is a $*$-representation, $\hat{H}$ is the orthogonal complement of the unit vector Ω, and P_Ω is the orthogonal projection onto $\mathbb{C}\Omega$. We will use the pre-Hilbert space notation also in the pure linear algebra setting, i.e., $\langle\Omega,(\lambda\Omega\oplus\hat{h})\rangle := \lambda$ for $\lambda\in\mathbb{C}, \hat{h}\in\hat{H} =: \Omega^\perp$. The free product of vector spaces H_i with such a decomposition, $H_i = \mathbb{C}\Omega_i\oplus\hat{H}_i$, is defined as

$$H_1 * H_2 := \mathbb{C}\Omega \oplus \bigoplus_{\substack{n\geq 1\\ \varepsilon_1\neq\ldots\neq\varepsilon_n}} \hat{H}_{\varepsilon_1}\otimes\cdots\otimes\hat{H}_{\varepsilon_n}$$

where the direct sum runs over all finite alternating sequences in $\{1,2\}^*$. This space can be written in different ways as a tensor product with tensor factors H_i:

$$H_1 * H_2 = H_1\otimes H_{(2\ldots)} = H_2\otimes H_{(1\ldots)} = H_{(\ldots 2)}\otimes H_1 = H_{(\ldots 1)}\otimes H_2$$

with

$$H_{(2\ldots)} := \mathbb{C}\Omega_{(2\ldots)} \oplus \bigoplus_{\substack{n\geq 1\\ 2=\varepsilon_1\neq\ldots\neq\varepsilon_n}} \hat{H}_{\varepsilon_1}\otimes\cdots\otimes\hat{H}_{\varepsilon_n}$$

under the obvious identifications and with analogous definitions of $H_{(1\ldots)}$, $H_{(\ldots 2)}$, and $H_{(\ldots 1)}$. Now, if $\varphi_i = \langle\Omega_i,\pi_i(\cdot)\Omega_i\rangle$ are linear functionals on algebras A_i described by representations $\pi_i\colon A_i\to L(H_i)$ with representation spaces $H_i = \mathbb{C}\Omega_i\oplus\hat{H}_i$, then their free product is defined as

$$\varphi_1 * \varphi_2 := \langle\Omega, \pi_1 \overrightarrow{*} \pi_2(\cdot)\Omega\rangle$$

for the representation $\pi_1\overrightarrow{*}\pi_2\colon A_1\sqcup A_2\to L(H_1 * H_2)$ given by

$$\pi_1\overrightarrow{*}\pi_2(a) = \begin{cases}\pi_1(a)\otimes\mathrm{id}_{(2\ldots)}, & a\in A_1,\\ \pi_2(a)\otimes\mathrm{id}_{(1\ldots)}, & a\in A_2.\end{cases}$$

We say that the free product of linear functionals is *represented on the free product of representation spaces*. By symmetry of the construction, one also has

$$\varphi_1 * \varphi_2 := \langle\Omega, \pi_1 \overleftarrow{*} \pi_2(\cdot)\Omega\rangle$$

for the representation $\pi_1 \overleftarrow{*} \pi_2 \colon A_1 \sqcup A_2 \to L(H_1 * H_2)$ given by

$$\pi_1 \overleftarrow{*} \pi_2(a) = \begin{cases} \mathrm{id}_{(\dots 2)} \otimes \pi_1(a), & a \in A_1, \\ \mathrm{id}_{(\dots 1)} \otimes \pi_2(a), & a \in A_2. \end{cases}$$

So we have two different ways to represent the free product of linear functionals on the free product of representation spaces. Nothing keeps us from combining those products of representations into one product of representations of two-faced algebras, i.e., for representations $\pi_i \colon A_i \to L(H_i)$ of two-faced algebras A_i, we define

$$\pi_1 \overset{\rightarrow}{\underset{*}{*}} \pi_2(a) = \begin{cases} \pi_1 \overrightarrow{*} \pi_2(a) = \begin{cases} \pi_1(a) \otimes \mathrm{id}_{(2\dots)}, & a \in A_1^\circ, \\ \pi_2(a) \otimes \mathrm{id}_{(1\dots)}, & a \in A_2^\circ, \end{cases} \\ \pi_1 \overleftarrow{*} \pi_2(a) = \begin{cases} \mathrm{id}_{(\dots 2)} \otimes \pi_1(a), & a \in A_1^\bullet, \\ \mathrm{id}_{(\dots 1)} \otimes \pi_2(a), & a \in A_2^\bullet, \end{cases} \end{cases}$$

i.e., whether the operators act from the left or from the right depends on the face. This product of representations defines the bifree product of linear functionals

$$\varphi_1 \overset{\rightarrow}{\underset{*}{*}} \varphi_2 := \langle \Omega, \pi_1 \overset{\rightarrow}{\underset{*}{*}} \pi_2(\cdot)\Omega \rangle.$$

It is not hard too hard to show that this is a symmetric universal product and it is evidently positive, because if $1 \oplus \varphi_1$ and $1 \oplus \varphi_2$ are states on the unitalizations of $*$-algebras, then one can choose π_1, π_2 as their GNS representations, and it follows that $\pi_1 \overset{\rightarrow}{\underset{*}{*}} \pi_2$ is a $*$-representation.

The products associated with Liu's free-Boolean independence [10] or with bi-monotone independence of type II [8, 4] (a non-symmetric universal product) were obtained in a similar manner, which led to define in [5] *universal products of representations* in such a way that they give rise to universal products of linear functionals on multi-faced algebras as sketched in the example of bifreeness. In particular all universal products obtained that way are evidently positive.

We have to distinguish whether the representation space of the product representation $\pi_1 \odot \pi_2$ is the free product of representation spaces (as in the example of bifreeness) or the tensor product of representation spaces. In both cases, all universal products of linear functionals on two-faced algebras that can be obtained through the indicated construction are determined in [5]. A key observation is that it is possible to deform expressions of the form $\pi(a) \otimes \mathrm{id}$ in a way that is compatible with the construction of universal products of linear functionals from universal products of representations.

Observation 3.1. For $H = \mathbb{C}\Omega \oplus \hat{H}$, one can decompose a linear operator $T \in L(H)$ as a block operator matrix

$$T = \begin{pmatrix} \tau & (t')^* \\ t & \hat{T} \end{pmatrix} \in \begin{pmatrix} L(\mathbb{C}\Omega) & L(\hat{H}, \mathbb{C}\Omega) \\ L(\mathbb{C}\Omega, \hat{H}) & L(\hat{H}) \end{pmatrix} = L(H).$$

With respect to that decomposition, we define

$$T_\gamma := \begin{pmatrix} \tau & \overline{\gamma}(t')^* \\ \gamma t & \hat{T} \end{pmatrix}$$

for $\gamma \in \mathbb{T}$. Then $T \mapsto T_\gamma$ is a ($*$-)homomorphism and one can replace expressions $\pi(a) \otimes \mathrm{id}$ by deformed versions

$$\pi(a) \otimes P_\Omega + \pi(a)_\gamma \otimes P_{\Omega^\perp}$$

in the constructions of the free, bifree, or tensor product (exactly which complementary space of Ω is meant by $\Omega^\perp$ depends on the context).

If we construct a single-faced universal product, the product of linear functionals turns out to be independent of the choice of deformation parameters, but for two-faced independences, the choice of deformation parameters can have an influence. In principle, the deformation parameters can be chosen differently for the first and second factor (e.g., for $\pi_1 \otimes \mathrm{id}_{(2\ldots)}$ and $\pi_2 \otimes \mathrm{id}_{(1\ldots)}$ when deforming the bifree product), but as it turns out, for symmetric universal products it is sufficient that the deformation parameters only depend on the face, but not on whether one deals with the first or second factor. This yields us the following:

- The tensor product of linear functionals can be represented on the tensor product of representation spaces with

$$\pi_1 \otimes_\gamma \pi_2(a) := \begin{cases} \pi_1(a) \otimes P_{\Omega_2} + (\pi_1(a))_\gamma \otimes P_{\Omega_2^\perp}, & a \in A_1 \\ P_{\Omega_1} \otimes \pi_2(a) + P_{\Omega_1^\perp} \otimes (\pi_2(a))_\gamma, & a \in A_2 \end{cases}$$

- The free product of linear functionals can be represented on the free product of representation spaces with

$$\pi_1 \overrightarrow{*}_\gamma \pi_2(a) = \begin{cases} \pi_1(a) \otimes P_{\Omega_{(2\dots)}} + (\pi_1(a))_\gamma \otimes P_{\Omega_{(2\dots)}^\perp}, & a \in A_1, \\ \pi_2(a) \otimes P_{\Omega_{(1\dots)}} + (\pi_2(a))_\gamma \otimes P_{\Omega_{(1\dots)}^\perp}, & a \in A_2. \end{cases}$$

  or with

$$\pi_1 \overleftarrow{*}_\gamma \pi_2(a) = \begin{cases} P_{\Omega_{(\dots 2)}} \otimes \pi_1(a) + P_{\Omega_{(\dots 2)}^\perp} \otimes (\pi_1(a))_\gamma, & a \in A_1, \\ P_{\Omega_{(\dots 1)}} \otimes \pi_2(a) + P_{\Omega_{(\dots 1)}^\perp} \otimes (\pi_2(a))_\gamma, & a \in A_2. \end{cases}$$

- The Boolean product of linear functionals can be represented on the tensor product of representation spaces or on the free product of representation spaces with

$$\pi_1 \diamond \pi_2(a) := \begin{cases} \pi_1(a) \oplus 0_{H_1^\perp}, & a \in A_1 \\ \pi_2(a) \oplus 0_{H_2^\perp}, & a \in A_2 \end{cases}$$

  with respect to the canonical embeddings $H_i \hookrightarrow H_1 * H_2$ or $H_i \hookrightarrow H_1 \otimes H_2$, respectively (we use the same symbol $\diamond$ in both cases).

Theorem 3.2. *On the tensor or free product of representation spaces, one can represent two-faced symmetric universal products as indicated in Tables 1 and 2. In particular, except for possibly the free-tensor product* $\substack{*\\ \otimes}$*, the tensor-free product* $\substack{\otimes\\ *}$*, the pure tensor product* $\substack{\otimes\\ \otimes}^{pure}$*, and the pure free product* $\substack{*\\ *}^{pure}$*, all other universal products in Figure 1 are positive, and the four exceptional products can neither be represented on the tensor product nor on the free product of representation spaces.*

face 2 (•) / face 1 (∘)	tensor $\otimes_{\gamma_2}$	Boolean $\diamond$
tensor $\otimes_{\gamma_1}$	deformed tensor $\genfrac{}{}{0pt}{}{\otimes}{\otimes}_{\zeta}$	tensor-Boolean $\genfrac{}{}{0pt}{}{\otimes}{\diamond}$
Boolean $\diamond$	Boolean-tensor $\genfrac{}{}{0pt}{}{\diamond}{\otimes}$	Boolean (trivially 2-faced) $\genfrac{}{}{0pt}{}{\diamond}{\diamond}$

$(\zeta, \gamma_1, \gamma_2 \in \mathbb{T},\ \zeta = \gamma_1 \overline{\gamma_2})$

表 1: two-faced positive and symmetric universal products of linear functionals on the tensor product of representation spaces with given universal products of representations for the two faces

face 2 (•) / face 1 (∘)	left free $\overrightarrow{*}_{\gamma_2}$	right free $\overleftarrow{*}_{\gamma_2}$	Boolean $\diamond$
left free $\overrightarrow{*}_{\gamma_1}$	deformed free $\genfrac{}{}{0pt}{}{\overrightarrow{*}}{\overrightarrow{*}}_{\zeta}$	deformed bifree $\genfrac{}{}{0pt}{}{\overrightarrow{*}}{\overleftarrow{*}}_{\zeta}$	free-Boolean $\genfrac{}{}{0pt}{}{*}{\diamond}$
right free $\overleftarrow{*}_{\gamma_1}$	deformed bifree $\genfrac{}{}{0pt}{}{\overrightarrow{*}}{\overleftarrow{*}}_{\zeta}$	deformed free $\genfrac{}{}{0pt}{}{\overrightarrow{*}}{\overrightarrow{*}}_{\zeta}$	free-Boolean $\genfrac{}{}{0pt}{}{*}{\diamond}$
Boolean $\diamond$	Boolean-free $\genfrac{}{}{0pt}{}{\diamond}{*}$	Boolean-free $\genfrac{}{}{0pt}{}{\diamond}{*}$	Boolean (trivially 2-faced) $\genfrac{}{}{0pt}{}{\diamond}{\diamond}$

$(\zeta, \gamma_1, \gamma_2 \in \mathbb{T}, \zeta = \gamma_1 \overline{\gamma_2})$

表 2: two-faced positive and symmetric universal products of linear functionals on the free product of representation spaces with given universal products of representations for the two faces

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