A Combinatorial Moment Formula Associated with the (q, s)-Poisson Distribution

Nobuhiro ASAI（淺井暢宏）
Department of Mathematics,
Aichi University of Education,
Hirosawa 1, Igaya, Kariya 448-8542, Japan.
nasai[at]auecc.aichi-edu.ac.jp
and
Hiroaki YOSHIDA（吉田裕亮）
Department of Information Sciences,
Ochanomizu University,
Tokyo 112-8610, Japan.
yoshida[at]is.ocha.ac.jp

Abstract

This is a preliminary report of our recent work [5]. We shall examine two parameterized deformation, namely (q, s)-deformation of the classical Poisson random variable of parameter $\lambda > 0$ on the (q, s)-Fock space. We shall give the recurrence formula of the orthogonal polynomials associated with the (q, s)-Poisson distribution. Moreover, we shall announce a moment formula of the distribution from the view point of the set partition statistics. This formula gives us a very nice combinatorial interpretation to deformation parameters.

1 Deformed Poisson

From the probabilistic point of view, the Poisson distribution is one of important distributions much the same as Gaussian. Indeed, using Gaussian and Poisson distributions, one can construct arbitrary infinitely divisible distributions via Lévy-Khintchine representation. In noncommutative probabilistic framework on the Boson Fock space, Hudson-Parthasarathy [15] and Schürmann [18] showed that by adding an appropriate gauge operator, the Poisson random variable can be realized on the Boson Fock space. In [9], Bożejko-Speicher constructed the q-Fock space by deforming the inner product of free (full) Fock space with the positive definite function on the symmetric group. The corresponding q-creation and annihilation operators satisfy the q-commutation relation and the q-Gaussian operator (random variable) gives rise to the q-Gaussian distribution, which is the orthogonalizing probability measures for the Rogers' q-Hermite polynomials under the appropriate rescaling. The corresponding Poisson operator (random variables) on Boson and q-Fock spaces can be realized by adopting the number and the q-number operators as the gauge part, respectively. The q-Poisson distribution is the orthogonalizing probability measure for the q-Charlier polynomials of Saitoh-Yoshida type [19][20] (see also [1]).

For later discussions in Sections 2 and 3, we need following things. In Section 1.1, we shall quickly prepare two parameterized deformation, namely, (q, s)-Fock space, creation, annihilation, intermediate, and scalar operators, which follows the technique in the sense of [4][11]. In Section 1.2, we shall define the (q, s)-Poisson type operator (random variable) and the corresponding (q, s)-Poisson distribution of a parameter $\lambda > 0$. The (q, s)-Poisson is the orthogonalizing probability measure of the orthogonal polynomials in (1.6) regarded as a generalization of q-Charlier polynomials. It is because the following known examples are included: One is of Saitoh-Yoshida type if $s = 1$ and the other is of Al-Salam-Carlitz type if $s = q$ in [16] (see also [2][12]). It is easy to see that the classical Charlier polynomials [12] can be recovered if $s = 1$ and $q \to 1$. Moreover, one can obtain orthogonal polynomials of the free Poisson [23] if $s = 1, q = 0$ and of the Boolean Poisson [22] if $q = 0, s \to 0$.

1.1 Deformed Fock Space and Operator

Let $\mathscr{H}$ be a real Hilbert space equipped with the inner product $\langle \cdot | \cdot \rangle$, and Ω be a distinguished unit vector, the so-called vacuum vector. We denote by $\mathcal{F}_{\rm fin}(\mathscr{H})$ the set of all the finite linear combinations of the elementary vectors $\xi_1 \otimes \cdots \otimes \xi_n \in \mathscr{H}^{\otimes n}$ $(n = 0, 1, 2, \ldots)$, where $\mathscr{H}^{\otimes 0} = \mathbb{C}\Omega$ as convention.

The q-deformed Fock space (simply called q-Fock space) was introduced in [9], which gives an interpolation between the Boson (symmetric) and the Fermion (anti-symmetric) Fock spaces and, especially, the case $q = 0$ of which yields the canonical model in the free probability theory (See [23], for instance).

Let us now recall the minimum about the (q, s)-deformed Fock space obtained by the weighted q-deformed Fock space with the weight sequences $\tau_n = s^{n-1}$ $(n \geq 1)$ in [4][11].

For $-1 < q < 1$ and $0 < s \leq 1$, we introduce the new inner product $\big(\cdot \mid \cdot \big)_{(q,s)}$ on $\mathcal{F}_{\rm fin}(\mathscr{H})$ by

$$\big(\xi_1 \otimes \cdots \otimes \xi_n \,|\, \eta_1 \otimes \cdots \otimes \eta_m\big)_{(q,s)} = \delta_{m,n}\, s^{\frac{n(n-1)}{2}} \sum_{\sigma \in \mathfrak{S}_n} q^{i(\sigma)} \langle \xi_1 \,|\, \eta_{\sigma(1)} \rangle \cdots \langle \xi_n \,|\, \eta_{\sigma(n)} \rangle,$$

where $\mathfrak{S}_n$ is the n-th symmetric group of permutations and $i(\sigma)$ is the number of inversions of the permutation $\sigma \in \mathfrak{S}_n$ defined by

$$i(\sigma) = \#\big\{ (i, j) \,\big|\, 1 \leq i < j \leq n,\, \sigma(i) > \sigma(j) \big\}.$$

Since the positivity of the inner product $\big(\cdot \mid \cdot \big)_{(q,s)}$ is guaranteed (see [10] [11]), the following definitions are allowed:

Definition 1.1. *The* (q, s)*-deformed Fock space* is (simply called *the* (q, s)*-Fock space*) $\mathcal{F}_{(q,s)}(\mathscr{H})$ the completion of $\mathcal{F}_{\rm fin}(\mathscr{H})$ with respect to the inner product $\big(\cdot \mid \cdot \big)_{(q,s)}$.

Definition 1.2. Given the vector $\xi \in \mathscr{H}$, *the* (q, s)*-creation operator* $a^{\dagger}_{(q,s)}(\xi)$ is defined by the canonical left creation,

$$\begin{aligned} &a^{\dagger}_{(q,s)}(\xi)\, \Omega = \xi, \\ &a^{\dagger}_{(q,s)}(\xi)\, (\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n, \quad n \geq 1. \end{aligned} \tag{1.1}$$

The (q, s)*-annihilation operator* $a_{(q,s)}(\xi)$ is defined by the adjoint operator of $a^{\dagger}_{(q,s)}(\xi)$ with respect to the inner product $\big(\cdot \mid \cdot \big)_{(q,s)}$, that is, $a_{(q,s)}(\xi) = \Big(a^{\dagger}_{(q,s)}(\xi)\Big)^*$.

The action of the (q, s)-annihilation operator on the elementary vectors is a direct consequence of the above definition.

Proposition 1.3. *The* (q, s)*-annihilation operator* $a_{(q,s)}(\xi)$ *acts on the elementary vectors as follows:*

$$\begin{aligned} &a_{(q,s)}(\xi)\, \Omega = 0, \quad a_{(q,s)}(\xi)\, \xi_1 = \langle \xi | \xi_1 \rangle\, \Omega, \\ &a_{(q,s)}(\xi)\, (\xi_1 \otimes \cdots \otimes \xi_n) = s^{n-1} \sum_{k=1}^{n} q^{k-1} \langle \xi | \xi_k \rangle\, \xi_1 \otimes \cdots \otimes \overset{\vee}{\xi_k} \otimes \cdots \otimes \xi_n, \quad n \geq 2, \end{aligned} \tag{1.2}$$

where $\overset{\vee}{\xi_k}$ *means that* ξ_k *should be deleted from the tensor product.*

Moreover, let us recall other special operators on $\mathcal{F}_{(q,s)}(\mathcal{H})$.

Definition 1.4. (1) *The scalar operator* k_s is defined by

$$\begin{aligned} &k_s\, \Omega = \Omega, \\ &k_s\, (\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = s^n \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n, \quad n \geq 1. \end{aligned} \tag{1.3}$$

(2) *The intermediate operator* n_q is defined by

$$\begin{aligned} &n_q\, \Omega = 0, \\ &n_q\, (\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = [n]_q\, \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n, \quad n \geq 1 \end{aligned} \tag{1.4}$$

where *the* q*-integer*, $[n]_q$ for $n \in \mathbb{N} \cup \{0\}$, is given by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} \text{ with } [0]_q = 0.$$

We remark that the scalar operator k_s can be interpreted as a deformation of the identity operator $I = k_1$ (a contraction for $s \in (0,1)$) and the intermediate operator n_q has the eigenvalue $[n]_q$ for the eigenspace $\mathscr{H}^{\otimes n}$.

Proposition 1.5. *The (q,s)-creation and the (q,s)-annihilation operators satisfy the following deformed commutation relation,*

$$a_{(q,s)}(\xi)\, a^{\dagger}_{(q,s)}(\eta) - qs\, a^{\dagger}_{(q,s)}(\eta)\, a_{(q,s)}(\xi) = \langle \xi | \eta \rangle\, k_s, \qquad \xi, \eta \in \mathscr{H}.$$

Proof. One can get this from Proposition 3.2 in [4] with $\tau_n = s^{n-1}$ for $n \geq 1$. □

Let us consider the vacuum state φ for bounded operators on the (q,s)-Fock space $\mathcal{F}_{(q,s)}(\mathscr{H})$ as

$$\varphi(b) = (b\,\Omega \mid \Omega)_{(q,s)}, \qquad b \in \mathcal{B}\big(\mathcal{F}_{(q,s)}(\mathscr{H})\big),$$

which is called *the vacuum expectation of* b. One can employ $\big(\mathcal{B}\big(\mathcal{F}_{(q,s)}(\mathscr{H})\big), \varphi\big)$ as the noncommutative probability space, on which the model of the (q,s)-deformed Poisson type operator (random variable) will be discussed.

Remark 1.6. Let us denote the deformation in the sense of Blitvić [7] by $(q,t)_{Bl}$ in this paper to avoid confusions in notation with our (q,s)-deformation. One can see that the $(q/t,t)$-Fock space in our language is equivalent to the $(q,t)_{B\ell}$-Fock space. See [4].

1.2 Deformed Poisson Type Operator

From now on, let us treat the (q,s)-Fock space of one-mode case with the unit base vector ξ. The (q,s)-creation $a^{\dagger}_{(q,s)}(\xi)$ and the (q,s)-annihilation $a_{(q,s)}(\xi)$ operators are simply denoted by $a^{\dagger}_{(q,s)}$ and $a_{(q,s)}$, respectively. In case of one-mode, the operators $a^{\dagger}_{(q,s)}$, $a_{(q,s)}$, k_s, and n_q act on the elementary vectors immediately obtained from Definitions in Section 1.1 as follows.

Lemma 1.7. *For $s \in (0,1]$ and $q \in (-1,1)$,*

$$a^{\dagger}_{(q,s)}\, \xi^{\otimes m} = \xi^{\otimes(m+1)}, \;\; m \geq 0, \qquad a_{(q,s)}\, \xi^{\otimes m} = \begin{cases} s^{m-1}[m]_q\, \xi^{\otimes(m-1)}, & m \geq 1, \\ 0, & m = 0, \end{cases}$$

$$k_s\, \xi^{\otimes m} = s^m\, \xi^{\otimes m}, \;\; m \geq 0, \qquad n_q\, \xi^{\otimes m} = \begin{cases} [m]_q\, \xi^{\otimes m}, & m \geq 1, \\ 0, & m = 0. \end{cases}$$

From Proposition 1.5, one has immediately the commutation relation,

$$a_{(q,s)}\, a^{\dagger}_{(q,s)} - (qs) a^{\dagger}_{(q,s)} a_{(q,s)} = k_s.$$

Moreover, by direct commutations, one can get

Proposition 1.8. *For $q \in (-1,1)$ and $s \in (0,1]$, the following commutation relations hold:*

(1) *s-commutativity:*

$$\begin{cases} k_s a^{\dagger}_{(q,s)} = s\left(a^{\dagger}_{(q,s)} k_s\right), \\ s\left(k_s a_{(q,s)}\right) = a_{(q,s)} k_s. \end{cases}$$

(2) *Commutativity:*

$$k_s n_q = n_q k_s.$$

(3)

$$\begin{cases} a_{(q,s)} n_q - (qs) n_q a_{(q,s)} = \left((1-s) n_q + sI\right) a_{(q,s)}, \\ n_q a^{\dagger}_{(q,s)} - (qs) a^{\dagger}_{(q,s)} n_q = \left((1-s) n_q + sI\right) a^{\dagger}_{(q,s)}. \end{cases}$$

Definition 1.9. For $\lambda > 0$, let us define the bounded self-adjoint operator $\boldsymbol{p}_\lambda^{(q,s)}$ on the (q,s)-Fock space of one-mode by

$$\boldsymbol{p}_\lambda^{(q,s)} = n_q + \sqrt{\lambda}\left(a_{(q,s)}^\dagger + a_{(q,s)}\right) + \lambda k_s. \tag{1.5}$$

This is called *the (q,s)-Poisson type operator (random variable)* on a noncommutative probability space $\big(\mathcal{B}\big(\mathcal{F}_{(q,s)}(\mathscr{H})\big), \varphi\big)$. The probability distribution of $\boldsymbol{p}_\lambda^{(q,s)}$ with respect to the vacuum expectation is called *the (q,s)-Poisson distribution* of parameter λ and denoted by $\Pi_\lambda^{(q,s)}$ in this paper.

Theorem 1.10 ([5])**.** *The distribution $\Pi_\lambda^{(q,s)}$ is the orthogonalizing probability measure for the sequence of orthogonal polynomials $\{C_n^{(q,s)}(\lambda; x)\}$ determined by the following recurrence relation:*

$$\begin{aligned} &C_0^{(q,s)}(\lambda; x) = 1, \quad C_1^{(q,s)}(\lambda; x) = x - \lambda, \\ &C_{n+1}^{(q,s)}(\lambda; x) = (x - (\lambda s^n + [n]_q))\, C_n^{(q,s)}(\lambda; x) - \lambda s^{n-1}[n]_q\, C_{n-1}^{(q,s)}(\lambda; x), \quad n \geq 1. \end{aligned} \tag{1.6}$$

Remark 1.11. (1) One can consider the orthogonal polynomials given in (1.6) as a generalization of q-Charlier polynomials because the following known examples are included: One is of *Saitoh-Yoshida type* if $s = 1$ appeared in [19][20] (see also [1]) and the other is of *Al-Salam-Carlitz type* if $s = q$. See [2][12][16]. It is easy to see that the classical Charlier polynomials [12] can be recovered if $s = 1$ and $q \to 1$. Moreover, one can obtain orthogonal polynomials of the free Poisson [23] if $s = 1, q = 0$ and of the Boolean Poisson [22] if $q = 0, s \to 0$.
(2) In [5], we mention and compare the $(q,t)_{B\ell}$-Poisson operator of Ejsmont type in [14] and the recurrence formula for the orthogonal polynomials of the $(q,t)_{B\ell}$-Poisson distribution with $(q/t, t)$-Poisson in this paper. Since the scalar operator k_t is not considered in [14], the $(q,t)_{B\ell}$-Poisson operator of Ejsmont type is not regarded as a deformation of Al-Salam-Carlitz type [16] (see also [2][12]). Furthermore, (α, q)-Poisson operators of type B have been introduced by Asai-Yoshida [3] and Ejsmont [14], independently. Strictly speaking, for $\alpha \neq 0$, the definition of the conservation term in [3] is different from that in [14]. In addition, these two Poisson operators of type B do not contain the scalar operator k_s and hence are not of Al-Salam-Carlitz type. Therefore, the (q,s)-Poisson type operator and distribution treated in this paper are essentially different from those of type B in [3][14].

2 Set Partition and Statistics

In our moment formula, the set partitions will be employed as combinatorial objects. We shall quickly recall the definition of set partitions and introduce some partition statistics to state our moment formula in Theorem 3.3.

Definition 2.1. For the set $[n] = \{1, 2, \ldots, n\}$, a *partition of* $[n]$ is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of non-empty disjoint subsets of $[n]$ which are called *blocks* and whose union is $[n]$. For a block B, we denote by $|B|$ the size of the block B, that is, the number of the elements in the block B. A block B will be called *singleton* if $|B| = 1$.

Let $\mathcal{P}(n)$ denote the set of all partitions of $[n] = \{1, 2, \ldots, n\}$. Let $\pi \in \mathcal{P}(n)$ be a partition. A quadruple (a, b, c, d) of elements in $[n]$ is said to be crossing of π if the elements a, c are in some block of π and b, d are in another block of π. For elements $e, f \in [n]$, we say that f *follows* e in π if $e \leq f$, e and f belong to the same block of π, and there is no element of this block in the interval $[e, f]$.

Definition 2.2. We define a *restricted crossing* to be a crossing (a, b, c, d) such that c follows a and d follows b. The partition statistics $\mathrm{rc}(\pi)$, *the number of restricted crossings of* π, counts the restricted crossings in the partition π.

The restricted crossings have a natural interpretation in the graphic line representation of partitions as described, for example, in [6][21]. If the block B is not singleton (i.e. $|B| \geq 2$), then we write $B = \{b_1, b_2, \ldots, b_{|B|}\}$. That is, b_{j+1} follows b_j $(j = 1, 2, \ldots, |B| - 1)$, and put b_j's on the horizontal axis. We will join the points b_j and b_{j+1} by an arc above the horizontal axis. Then every restricted crossing appears by a pair of crossing arcs.

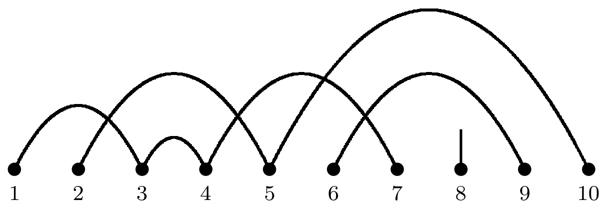
For our combinatorial formula, we shall introduce another partition statistics related to the last (maximum) elements of the blocks. For a block C of the partition $\pi \in \mathcal{P}(n)$ we consider the first (minimum) element f_C and the last (maximum) element ℓ_C in the block C. In case of singleton it means $f_C = \ell_C$. For an element $a \in [n]$, we say that the block C *covers* a if a does not belong to the block C but a is included in the interval $[f_C, \ell_C]$.

Definition 2.3. Let B be a block of a partition π. Then $\mathrm{dl}(B)$ counts the block that covers ℓ_B (the last element of B), which is called *the depth of the block B by the last element*. For a partition π, the statistics $\mathrm{td}(\pi)$ is defined by

$$\mathrm{td}(\pi) = \sum_{B\in\pi} \mathrm{dl}(B),$$

which we call *the total depth of the blocks by the last elements*.

Example 2.4. Let us consider a partition $\pi = \big\{\{1,3,4,7\},\{2,5,10\},\{6,9\},\{8\}\big\} \in \mathcal{P}(10)$ and put $B_1 = \{1,3,4,7\}$, $B_2 = \{2,5,10\}$, $B_3 = \{6,9\}$, and $B_4 = \{8\}$. Then one can see $\mathrm{rc}(\pi) = 4$ because the partition π has 4 restricted crossings, which can be represented as the pairs of crossing arcs $\big([1,3],[2,5]\big)$, $\big([2,5],[4,7]\big)$, $\big([4,7],[5,10]\big)$, and $\big([4,7],[6,9]\big)$ as illustrated below.



Moreover, the last element of the block B_1 is 7 covered by the blocks B_2 and B_3. Thus $\mathrm{dl}(B_1) = 2$. Since $\mathrm{dl}(B_2) = 0$, $\mathrm{dl}(B_3) = 1$, and B_4 is a singleton covered by B_2 and B_3, one can get $\mathrm{dl}(B_4) = 2$. Threfore, we have $\mathrm{td}(\pi) = 5$.

3 Combinatorial Moment Formula of the (q, s)-Poisson Distribution

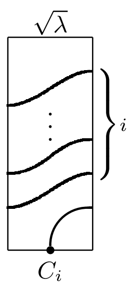
We are going to investigate the n-th moments of the (q,s)-Poisson distribution, $\Pi_\lambda^{(q,s)}$. Namely, we evaluate the vacuum expectation of the n-th power of the (q,s)-Poisson type operator (random variable),

$$\varphi\Big(\big(\boldsymbol{p}_\lambda^{(q,s)}\big)^n\Big) = \Big(\big(n_q + \sqrt{\lambda}a_{(q,s)} + \sqrt{\lambda}a^\dagger_{(q,s)} + \lambda k_s\big)^n\, \Omega \,\Big|\, \Omega\Big)_{(q,s)}.$$

In order to evaluate the vacuum expectation, we shall use the cards arrangement technique similar as in [13] for juggling patterns. For non-crossing cases, see [24][25]. We are now required to prepare four different cards to represent the restricted crossings. The cards and weights are explained in the subsequent sections.

3.1 Creation Cards

The creation card C_i $(i \geq 0)$ has i inflow lines from the left and $(i+1)$ outflow lines to the right, where one new line starts from the middle point on the ground level. For each $j \geq 1$, the inflow line of the j-th level will flow out at the $(j+1)$-st level without any crossing. We give the weight $\sqrt{\lambda}$ to the card C_i. The creation card of level i

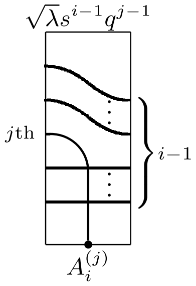


represents the operation

$$\big(\sqrt{\lambda}\, a^{\dagger}_{(q,s)}\big)\, \xi^{\otimes i} = \sqrt{\lambda}\, \xi^{\otimes (i+1)}, \quad i \geq 0.$$

3.2 Annihilation Cards

The annihilation card $A_i^{(j)}$ ($1 \leq j \leq i$, $i \geq 1$) has i inflow lines from the left and $(i-1)$ outflow lines to the right. On the card $A_i^{(j)}$, only the inflow line of the j-th level goes down to the middle point on the ground level and ends. The lines inflowed at lower than the j-th level keep their levels. Hence $(j-1)$ crossings will appear. The line inflowed at the ℓ-th level ($\ell > j$, higher than the j-th level) will flow out at the $(\ell-1)$-st level (one-decreased level) without any crossing. We shall give the weight $\sqrt{\lambda}\, s^{i-1} q^{j-1}$ to the card $A_i^{(j)}$, where the parameter q encodes the number of crossings and the parameter s encodes the number of through out lines on the card. The annihilation cards of level i,



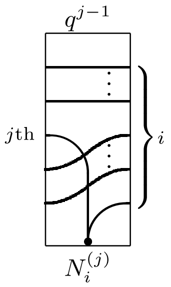
$j = 1, 2, \ldots i,$

represent the operation

$$\begin{aligned}\big(\sqrt{\lambda}\, a_{(q,s)}\big)\, \xi^{\otimes i} &= \underbrace{\sqrt{\lambda}\, s^{i-1}\, \xi^{\otimes (i-1)}}_{A_i^{(1)}} + \underbrace{\sqrt{\lambda}\, s^{i-1} q\, \xi^{\otimes (i-1)}}_{A_i^{(2)}} \\ &\quad + \underbrace{\sqrt{\lambda}\, s^{i-1} q^2\, \xi^{\otimes (i-1)}}_{A_i^{(3)}} + \cdots + \underbrace{\sqrt{\lambda}\, s^{i-1} q^{i-1}\, \xi^{\otimes (i-1)}}_{A_i^{(i)}} \\ &= \sqrt{\lambda}\, s^{i-1} [\, i\,]_q\, \xi^{\otimes (i-1)}, \qquad i \geq 1.\end{aligned}$$

3.3 Intermediate Cards

The intermediate card $N_i^{(j)}$ ($1 \leq j \leq i$, $i \geq 1$) has i inflow lines and the same number of outflow lines. On the card $N_i^{(j)}$, only the line inflowed at the j-th level goes down to the middle point on the ground and it will continue as the first lowest outflow line. The inflow line at the ℓ-th level ($\ell < j$, lower than the jth level) will flow out at the $(\ell+1)$-st level (one-increased level), and the inflow lines of higher than the jth level will keep their levels. Hence we have $(j-1)$ crossings. We shall give the weight q^{j-1} to the card $N_i^{(j)}$, where the number of crossings is encoded by the parameter q. The intermediate cards of level i,



$j = 1, 2, \ldots i,$

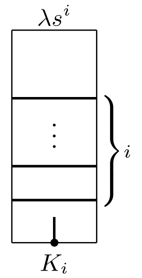
represent the operation

$$\begin{aligned}\big(n_q\big)\, \xi^{\otimes i} &= \underbrace{\xi^{\otimes i}}_{N_i^{(1)}} + \underbrace{q\, \xi^{\otimes i}}_{N_i^{(2)}} + \underbrace{q^2\, \xi^{\otimes i}}_{N_i^{(3)}} + \cdots + \underbrace{q^{i-1}\, \xi^{\otimes i}}_{N_i^{(i)}} \\ &= [\, i\,]_q\, \xi^{\otimes i}, \qquad i \geq 1.\end{aligned}$$

Remark 3.1. It should be noted that the middle point on the ground of level i for $i \geq 1$ is not the last element of the block of a partition π. Therefore, the intermediate cards of level i, $N_i^{(j)}$ for $j = 1, 2, \ldots i$, do not contain the weight s^{i-1} encoding the number of throughout lines from the left to right without touching the middle point on the ground. Therefore, $n_q \neq a^{\dagger}_{(q,s)} a_{(q,s)}$ unless $s = 1$.

3.4 Scalar Cards

The scalar card K_i $(i \geq 0)$ has i horizontally parallel lines and the short pole at the middle point on the ground. We shall give the weight $\lambda\, s^i$ to the card K_i, where the parameter s encodes the number of through out lines on the card. The scalar card K_i of level i



represents the operation

$$(\lambda\, k_s)\, \xi^{\otimes i} = \lambda\, s^i\, \xi^{\otimes i},$$

which can be considered as a s-deformation of the identity operator $I = k_1$.

3.5 Combinatorial Moment Formula

In this section, we would like to show how the cards arrangement technique is applied to derive a moment formula from the viewpoint of combinatorics. More detail description is provided in Asai-Yoshida [5] .

Each card arrangement gives the set partition of $[n]$, where the blocks of the partition could be realized by the concatenation of the lines on the cards. In this expression, one can see that the creation and the annihilation cards correspond to the first (minimum) and the last (maximum) elements in the blocks of the size ≥ 2, respectively, and the intermediate cards correspond to the intermediate elements in blocks. Furthermore, *the weight of the arrangement* is given by the product of the weights of the cards used in the arrangement.

Now we will briefly explain relationships between the weight of the arrangement and the set partition statistics, that is, roles of three parameters, λ, s and q.

(1) First of all, one can see

$$\big(\sqrt{\lambda}\big)^{\#\{\text{creation cards}\}+\#\{\text{annihilation cards}\}} = \lambda^{\#\{\text{creation cards}\}} = \lambda^{\#\big\{B \,|\, B\in\pi,\, |B|\geq 2\big\}}$$

because the Motzkin path condition is satsified. In addition, one has

$$\lambda^{\#\{\text{scalar cards}\}} = \lambda^{\#\big\{B \,|\, B\in\pi,\, |B|=1\big\}}.$$

Hence we have

$$\lambda^{\#\big\{B \,|\, B\in\pi,\, |B|\geq 2\big\}}\lambda^{\#\big\{B \,|\, B\in\pi,\, |B|=1\big\}} = \lambda^{\#\big\{B \,|\, B\in\pi\big\}} = \lambda^{|\pi|}.$$

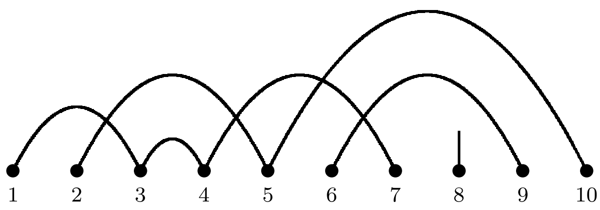
(2) Secondly, the parameter s encodes the total depth of blocks by the last elements, that is,

$$\prod_{B\in\pi} s^{\#\left\{ C \,\middle|\, \substack{C\in\pi,\, C \text{ covers } k, \text{ where} \\ k \text{ is the last element of } B} \right\}} = \prod_{B\in\pi} s^{\mathrm{dl}(B)} = s^{\mathrm{td}(\pi)}.$$

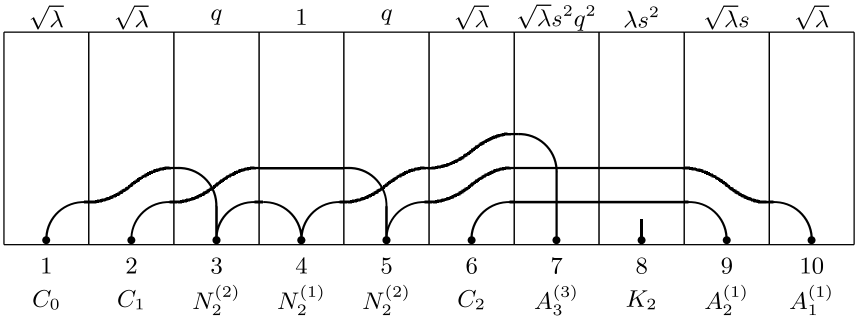
(3) Finally, the crossings appeared in the card arrangement are nothing but the restricted crossings. Therefore, the parameter q in the weight of the arrangement encodes the number of the restricted crossings in the corresponding partition, $q^{\mathrm{rc}(\pi)}$.

Let us explain the admissible card arrangement technique by an example. Then one may feel a flavor of our approach.

Example 3.2. Consider the partition $\pi = \big\{\{1,3,4,7\},\{2,5,10\},\{6,9\},\{8\}\big\} \in \mathcal{P}(10)$ in Example 2.4:



This partition can be realized in the following admissible card arrangement:



This arrangement is the realization of a contributor of length 10,

$$y = \underbrace{(\sqrt{\lambda}\, a_{(q,s)})}_{10} \underbrace{(\sqrt{\lambda}\, a_{(q,s)})}_{9} \underbrace{(\lambda\, k_s)}_{8} \underbrace{(\sqrt{\lambda}\, a_{(q,s)})}_{7} \underbrace{(\sqrt{\lambda}\, a^{\dagger}_{(q,s)})}_{6} \underbrace{(n_q)}_{5} \underbrace{(n_q)}_{4} \underbrace{(n_q)}_{3} \underbrace{(\sqrt{\lambda}\, a^{\dagger}_{(q,s)})}_{2} \underbrace{(\sqrt{\lambda}\, a^{\dagger}_{(q,s)})}_{1}.$$

As one can see from the above cards arrangement, the values of statistics are given by $|\pi| = 4$, $\mathrm{rc}(\pi) = 4$, $\mathrm{td}(\pi) = 5$ and the weight of card arrangement (the product of the weight of the cards) becomes $\mathrm{Wt}\big(A_\pi\big) = \lambda^4\, q^4\, s^5$.

Therefore, one can derive the following combinatorial moment formula of the (q, s)-Poisson distribution $\Pi_\lambda^{(q,s)}$.

Theorem 3.3 ([5])**.** *The n-th moment of the (q, s)-Poisson distribution $\Pi_\lambda^{(q,s)}$ is given by*

$$\varphi\Big(\big(\boldsymbol{p}_\lambda^{(q,s)}\big)^n\Big) = \sum_{\substack{contributor\, y\\ of\ length\ n}} \varphi(y) = \sum_{\pi \in \mathcal{P}(n)} \lambda^{|\pi|}\, q^{rc(\pi)}\, s^{td(\pi)}.$$

Remark 3.4. In [5], the special case $s = q$ is also discussed to see interesting connections with results [16].

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