FEYNMAN–KAC FORMULA FOR FIBER HAMILTONIANS IN THE RELATIVISTIC NELSON MODEL IN TWO SPATIAL DIMENSIONS

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Abstract. In this proceeding we consider a translation invariant Nelson type model in two spatial dimensions modeling a scalar relativistic particle in interaction with a massive radiation field. As is well-known, the corresponding Hamiltonian can be defined with the help of an energy renormalization. First, we review a Feynman–Kac formula for the semigroup generated by this Hamiltonian proven by the authors in a recent preprint (where several matter particles and exterior potentials are treated as well). After that, we employ a few technical key relations and estimates obtained in our preprint to present an otherwise self-contained derivation of new Feynman–Kac formulas for the fiber Hamiltonians attached to fixed total momenta of the translation invariant system. We conclude by inferring an alternative derivation of the Feynman–Kac formula for the full translation invariant Hamiltonian.

1. Introduction

The original Nelson model describes a conserved number of non-relativistic quantum mechanical matter particles linearly coupled to a quantized radiation field (boson field). Its crucial feature is its comparatively simple renormalizability. In fact, the heuristic matter-radiation interaction term in the Hamiltonian is *a priori* ill-defined, as its behavior for large boson momenta is too singular. Imposing an ultraviolet cutoff in the interaction term and adding explicitly given cutoff dependent renormalization energies, we obtain, however, a well-defined family of Hamiltonians converging in the norm resolvent sense to a unique *renormalized* Hamiltonian as the cutoff parameter goes to infinity. This has been demonstrated by Nelson almost sixty years ago [Nel64a, Nel64b] and in later technical improvements by several authors (norm instead of strong resolvent convergence has been observed first by Ammari [Amm00]). Ever since the spectral and probabilistic analysis of Nelson's model and variants thereof has been a popular topic in mathematical quantum field theory.

A modification of Nelson's model, where the (scalar) matter particles are relativistic as well, has already been studied in the beginning of the 1970's. Working in spatial dimension three, as Nelson did, Gross [Gro73] was able to prove the existence of renormalized Hamiltonians by procedures more elaborate than Nelson's involving implicit particle mass renormalizations, a passage to a non-Fock representation and compactness arguments; whether Gross' renormalized Hamiltonian is unique is still unclear. Sloan [Slo74] treated the relativistic version of Nelson's model in spatial dimension two and was able to prove resolvent convergence, in the strong sense and along a subsequence of a given sequence of cutoff parameters at least. A few years ago only, Schmidt [Sch19] provided a new construction of Sloan's renormalized Hamiltonian. Employing the recently developed method of interior boundary conditions (see [LS19] and the references therein), Schmidt proved proper norm resolvent convergence and obtained explicit formulas for the domain of the renormalized Hamiltonian and its action on it.

In our work we are interested in the probabilistic analysis of Nelson type models and in particular in deriving Feynman–Kac formulas for the semigroups generated by the (semibounded) renormalized Hamiltonians. While Nelson obtained probabilistic representations of certain matrix elements of the unitary group [Nel64b], Feynman–Kac formulas for the semigroup in the original Nelson model were found in [GHL14, MM18]. The methods used in [MM18] also apply *mutatis mutandis* to the Hamiltonian constructed by Sloan and Schmidt, that we refer to as the relativistic Nelson Hamiltonian in two spatial dimensions. For the N matter particle version of that Hamiltonian including exterior potentials, the present authors proved Feynman–Kac formulas in [HM23]. For earlier work on Feynman–Kac formulas for ultraviolet regularized Nelson type and related models (such as the Pauli-Fierz model) and numerous applications we refer to the textbook [HL20] and the references given there.

In this proceeding we discuss the translation invariant (no external potential) relativistic Nelson Hamiltonian for one matter particle in two space dimensions. This Hamiltonian is unitarily equivalent to a direct integral of fiber Hamiltonians, each attached to a fixed total momentum of the entire matter-radiation system. After reviewing the Feynman–Kac formula for the full Hamiltonian from [HM23], we shall derive Feynman–Kac formulas for the fiber Hamiltonians, employing only a few key estimates and relations from [HM23] as starting points. It would also be possible to explicitly fiber-decompose the probabilistic side of the Feynman–Kac formula for the full Hamiltonian and spend a little bit of work to argue that the so-obtained families of operators define a semigroup for every fixed total momentum, that must be generated by a corresponding renormalized fiber Hamiltonian; compare, e.g., [MM18, Chapter 7] for the non-relativistic case. Here we favor, however, the more independent derivation only based on the technical key inputs from [HM23]. For in this way, the reader can see proof strategies from [HM23] at work in a slightly different setting.

Structure of the proceeding. After using the remaining part of this introduction to clarify our notation for operators in bosonic Fock space, we briefly explain the construction of the Hamiltonian H for the translation invariant relativistic Nelson model in two spatial dimensions in Section 2. In Section 3 we introduce some stochastic processes employed throughout the proceeding and present Feynman–Kac formulas found in [HM23] for the semigroups generated by H and its versions H_Λ containing ultraviolet cutoff interaction terms. By means of a Lee-Low-Pines transformation, we shall turn H_Λ into a direct integral of fiber Hamiltonians $\widehat{H}_\Lambda(\xi)$ attached to total momenta $\xi \in \mathbb{R}^2$ of the matter-radiation system in Section 4. The objective of Section 5 is to present the crucial technical ingredients from [HM23] applied in the remaining parts of the text, which otherwise are fairly self-contained. Our derivation of Feynman–Kac formulas for fiber Hamiltonians starts in Section 6, where the corresponding Feynman–Kac integrands and semigroups are analyzed first. The Feynman–Kac formulas themselves are established in Section 7, first for the ultraviolet regularized fiber operators $\widehat{H}_\Lambda(\xi)$ and afterwards for their renormalized versions, i.e., the norm resolvent limits $\widehat{H}(\xi) := \lim_{\Lambda\to\infty} \widehat{H}_\Lambda(\xi)$. In fact, as a byproduct of our method, we shall obtain an independent proof for the existence of these limits, improving on [Slo74] where only strong resolvent convergence along subsequences is proven. (While Schmidt treated the full Hamiltonian explicitly in [Sch19], norm resolvent convergence of fiber Hamiltonians can probably be inferred from his results, too, see [DH22] for an approach along these lines.) Clearly as expected, it turns out that H is the direct integral of the renormalized fiber operators $\widehat{H}(\xi)$ after a Lee-Low-Pines transformation. This is verified in Section 8, where we also present an alternative derivation of the Feynman–Kac formulas for the full translation invariant Hamiltonians, based on the ones for fiber operators.

Operators in bosonic Fock space. All fiber Hamiltonians alluded to above act in the bosonic Fock space over $L^2(\mathbb{R}^2)$ defined by

$$\mathscr{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{2n}).$$

In the above relation $L^2_{\mathrm{sym}}(\mathbb{R}^{2n})$ is the closed subspace of $L^2(\mathbb{R}^{2n})$ comprising all functions $\phi_n \in L^2(\mathbb{R}^{2n})$ satisfying $\phi_n(k_1, \ldots, k_n) = \phi_n(k_{\pi(1)}, \ldots, k_{\pi(n)})$ a.e. for every permutation π of $\{1, \ldots, n\}$; here $k_j \in \mathbb{R}^2$ for every $j \in \{1, \ldots, n\}$.

Throughout this proceeding we use standard notation for the following operators acting in $\mathscr{F}$ (see, e.g., [Ara18, Par92] for their constructions and basic properties):

For every $f \in L^2(\mathbb{R}^2)$, the symbols $a^\dagger(f)$ and $\varphi(f)$ denote the corresponding creation and field operators, respectively. Thus, $\varphi(f)$ is selfadjoint and equal to the closure of $a^\dagger(f) + a^\dagger(f)^*$. If V is a unitary operator on $L^2(\mathbb{R}^2)$, then $\Gamma(V)$ denotes its second quantization, which is a unitary operator on $\mathscr{F}$. We shall sometimes use that $\Gamma(V_1)\Gamma(V_2) = \Gamma(V_1V_2)$ for unitary operators V_1 and V_2 on $L^2(\mathbb{R}^2)$. Finally, if A is a selfadjoint operator on $L^2(\mathbb{R}^2)$, then $\mathrm{d}\Gamma(A)$ denotes its differential second quantization. That is, $\mathrm{d}\Gamma(A)$ is the unique selfadjoint operator on $\mathscr{F}$ satisfying $\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(A)} = \Gamma(\mathrm{e}^{-\mathrm{i}tA})$, $t \in \mathbb{R}$.

2. The relativistic Nelson model in two spatial dimensions

In this section we first introduce the Hamiltonian H_Λ for the total particle-radiation system with an ultraviolet cutoff interaction term and finally the renormalized Hamiltonian H. Both are selfadjoint operators in the Hilbert space $L^2(\mathbb{R}^2, \mathscr{F})$.

The matter particle is assumed to have a non-negative mass $m_{\mathrm{p}} \geqslant 0$ and dispersion relation

$$\psi(\xi) := (|\xi|^2 + m_{\mathrm{p}}^2)^{1/2} - m_{\mathrm{p}}, \quad \xi \in \mathbb{R}^2. \tag{2.1}$$

Since the model would be unstable otherwise, the bosons have a strictly positive mass $m_{\mathrm{b}} > 0$. The dispersion relation for a single boson is

$$\omega(k) := (|k|^2 + m_{\mathrm{b}}^2)^{1/2}, \quad k \in \mathbb{R}^2. \tag{2.2}$$

The coupling function for the matter-radiation interaction is given by

$$v := g\omega^{-1/2} \notin L^2(\mathbb{R}^2), \quad \text{with a coupling constant } g \in \mathbb{R}\backslash\{0\}. \tag{2.3}$$

Since v is not square-integrable, an energy renormalization will be necessary to define the Hamiltonian H for our model. That is, we first introduce Hamiltonians containing the ultraviolet cutoff coupling functions

$$v_\Lambda := \chi_{B_\Lambda} v \in L^2(\mathbb{R}^2), \quad \Lambda \in [0, \infty).$$

Here χ_{B_Λ} is the indicator function of the two-dimensional open ball of radius Λ about the origin B_Λ, with the understanding that $B_0 = \varnothing$. Abbreviating

$$e_x(k) := \mathrm{e}^{-\mathrm{i}k\cdot x}, \quad k \in \mathbb{R}^2,$$

for every $x \in \mathbb{R}^2$, and introducing renormalization energies

$$E_\Lambda^{\mathrm{ren}} := \int_{B_\Lambda} \frac{v^2(k)}{\omega(k) + \psi(k)} \mathrm{d}k, \quad \Lambda \in [0, \infty), \tag{2.4}$$

we define the relativistic Nelson operator with ultraviolet cutoff at $\Lambda \in [0, \infty)$ by

$$(H_\Lambda \Phi)(x) := (\psi(-\mathrm{i}\nabla)\Phi)(x) + \mathrm{d}\Gamma(\omega)\Phi(x) + \varphi(e_x v_\Lambda)\Phi(x) + E_\Lambda^{\mathrm{ren}}\Phi(x),$$

for a.e. $x \in \mathbb{R}^2$ and every

$$\Phi \in \mathcal{D}(H_\Lambda) = \mathcal{D}(H_0) := H^1(\mathbb{R}^2, \mathcal{F}) \cap L^2(\mathbb{R}^2, \mathcal{D}(\mathrm{d}\Gamma(\omega))).$$

Here and henceforth, $\mathcal{D}(\cdot)$ stands for domains of selfadjoint operators equipped with their graph norms. The Sobolov space $H^1(\mathbb{R}^2, \mathcal{F})$ is defined via the $\mathcal{F}$-valued Fourier transformation F, that is given by Bochner-Lebesgue integrals

$$(F\Phi)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\xi\cdot x} \Phi(x) \mathrm{d}x, \quad \xi \in \mathbb{R}^2,$$

whenever $\Phi \in L^1(\mathbb{R}^2, \mathcal{F}) \cap L^2(\mathbb{R}^2, \mathcal{F})$, and isometric extension to $L^2(\mathbb{R}^2, \mathcal{F})$. The selfadjoint operator $\psi(-\mathrm{i}\nabla)$ is defined by means of F as well, i.e., by definition,

$$(F\psi(-\mathrm{i}\nabla)\Phi)(\xi) = \psi(\xi)(F\Phi)(\xi), \quad \text{a.e. } \xi \in \mathbb{R}^2,$$

for every $\Phi \in \mathcal{D}(\psi(-\mathrm{i}\nabla)) = H^1(\mathbb{R}^2, \mathcal{F})$. Employing the Kato-Rellich theorem and the standard relative bound

$$\|\varphi(e_x v_\Lambda)\phi\| \leqslant 2^{1/2} \|(\omega^{-1/2} \vee 1)v_\Lambda\| \|(1 + \mathrm{d}\Gamma(\omega))^{1/2}\phi\|, \quad x \in \mathbb{R}^2, \tag{2.5}$$

which is available for all ϕ in the form domain of $\mathrm{d}\Gamma(\omega)$, we can indeed verify selfadjointness of every H_Λ with $\Lambda \in (0, \infty)$ on $\mathcal{D}(H_0)$.

Finally, the renormalized relativistic Nelson operator in two space dimensions is given by

$$H := H_\infty := \underset{\Lambda\to\infty}{\text{norm-resolvent-lim}}\, H_\Lambda. \tag{2.6}$$

Existence of the above limit has been established in [Slo74, Sch19]. It has been reproven in [HM23] as an automatic byproduct of the proof strategy for the Feynman–Kac formula established there; see Section 8 for yet another proof.

3. Feynman–Kac formulas for the full Hamiltonians

Throughout this proceeding we fix a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, \mathbb{P})$ satisfying the usual hypotheses as well as a $(\mathfrak{F}_t)_{t\geqslant 0}$-Lévy process X whose Lévy symbol is $-\psi$ and all whose paths are càdlàg. Expectations with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$, and we put $X_{t-} := \lim_{s\uparrow t} X_s$ for all $t > 0$. We recall that X has characteristics $(0, 0, \nu)$, where its Lévy measure ν has an explicitly known density with respect to the Lebesgue-Borel measure; see, e.g., [HM23, §2.2].

Next, we define the stochastic processes appearing in our Feynman–Kac integrands: For every $\Lambda \in [0, \infty]$, we introduce the following well-defined $L^2(\mathbb{R}^2)$-valued Bochner-Lebesgue integrals,

$$U^-_{\Lambda,t} := \int_0^t \mathrm{e}^{-s\omega} e_{X_s} v_\Lambda \mathrm{d}s, \quad U^+_{\Lambda,t} := \int_0^t \mathrm{e}^{-(t-s)\omega} e_{X_s} v_\Lambda \mathrm{d}s, \quad t \geqslant 0. \tag{3.1}$$

For both choices of the sign, $(U^\pm_{\Lambda,t})_{t\geqslant 0}$ is a continuous and adapted $L^2(\mathbb{R}^2)$-valued process [HM23, Appendix B]. For finite Λ, the analogue of Feynman's complex action in our model is given by

$$u_{\Lambda,t} := \int_0^t \langle U^+_{\Lambda,s} | e_{X_s} v_\Lambda \rangle \mathrm{d}s - tE^{\mathrm{ren}}_\Lambda, \quad t \geqslant 0, \ \Lambda \in [0, \infty). \tag{3.2}$$

It defines a real-valued continuous and adapted process. In [HM23] and, in a slightly more sketchy fashion, in Lemma 5.1, we re-write this expression employing Itô's formula and obtain a more regular one where the ultraviolet cutoff can be dropped. This results in the following formula for the limiting complex action: Setting

$$\beta := (\omega + \psi)^{-1} v \in L^2(\mathbb{R}^2),$$

we define

$$u_{\infty,t} := \int_{(0,t]\times\mathbb{R}^2} \langle U^+_{\infty,s}|e_{X_{s-}}(e_z-1)\beta\rangle \mathrm{d}\widetilde{N}(s,z) - \langle U^+_{\infty,t}|e_{X_t}\beta\rangle, \quad t \geqslant 0. \tag{3.3}$$

Here the integral is an isometric stochastic integral with respect to the martingale valued measure $\widetilde{N}$ of X; see, e.g., [App09] for the nomenclature used here and detailed explanations. The corresponding stochastic integral process is a càdlàg L^2-martingale. (In fact, the paths of u_∞ are $\mathbb{P}$-a.s. continuous [HM23, Corollary 6.9].)

The last building blocks in our Feynman–Kac integrands are the operator norm convergent series

$$F_t(h) := \sum_{n=0}^{\infty} \frac{1}{n!} a^\dagger(h)^n \mathrm{e}^{-t\mathrm{d}\Gamma(\omega)}, \quad t > 0,\ h \in L^2(\mathbb{R}^2),$$

which define analytic maps $F_t : L^2(\mathbb{R}^2) \to \mathscr{B}(\mathscr{F})$. For these maps and their derivatives we have the bounds [GMM17]

$$\|F_t(h)\| \leqslant \mathcal{S}(\|h\|_t), \quad \|F_t'(h)\tilde{h}\| \leqslant 4\|\tilde{h}\|_t \mathcal{S}(\|h\|_t), \quad h, \tilde{h} \in L^2(\mathbb{R}^2), \tag{3.4}$$

where $\|h\|_t^2 := \|h\|^2 + \|(t\omega)^{-1/2}h\|^2$ and $\mathcal{S}(z) := \sum_{n=0}^\infty (n!)^{-1/2}(2z)^n$, $z \in \mathbb{C}$.

We are now in a position to introduce the Fock space operator-valued parts of the Feynman–Kac integrands for the entire matter-radiation system. For all $\Lambda \in [0,\infty]$ and $x \in \mathbb{R}^2$, they are given by the adjoints of

$$W_{\Lambda,t}(x) := \mathrm{e}^{u_{\Lambda,t}} F_{t/2}(-e_x U^+_{\Lambda,t}) F_{t/2}(-e_x U^-_{\Lambda,t})^* = \Gamma(e_x) W_{\Lambda,t}(0) \Gamma(e_{-x}), \tag{3.5}$$

whenever $t > 0$, and $W_{\Lambda,0}(x) := \mathbb{1}_{\mathscr{F}}$.

The next theorem is a special case of [HM23, Theorem 2.1]. Departing from a few technical key ingredients presented in Section 5 we shall obtain an otherwise independent proof of the asserted formula (3.6) (for a.e. x) at the end of Section 8.

Theorem 3.1 (Feynman–Kac formulas for the entire system). *Let $\Lambda \in [0,\infty]$, $\Phi \in L^2(\mathbb{R}^2, \mathscr{F})$ and $t > 0$. Then $\mathrm{e}^{-tH_\Lambda}\Phi$ has a unique continuous representative which is given by*

$$(\mathrm{e}^{-tH_\Lambda}\Phi)(x) = \mathbb{E}\left[W_{\Lambda,t}(x)^*\Phi(x+X_t)\right], \quad x \in \mathbb{R}^2. \tag{3.6}$$

4. Lee-Low-Pines transformation and fiber Hamiltonians

The Hamiltonians H_Λ and H are invariant under translations of the entire matter-radiation system in space and can therefore be represented as direct integrals with respect to the system's total momentum of selfadjoint fiber Hamiltonians. This is implemented by the Lee-Low-Pines transformation in two dimensions defined by

$$\mathscr{U} := F \int_{\mathbb{R}^2}^{\oplus} \Gamma(e_{-x}) \mathrm{d}x. \tag{4.1}$$

We shall briefly discuss the transformation by $\mathscr{U}$ of the ultraviolet regularized Hamiltonians H_Λ with $\Lambda \in [0,\infty)$:

For $i \in \{1,2\}$, we let K_i denote the maximal operator of multiplication with k_i on $L^2(\mathbb{R}^2)$, i.e., $(K_i f)(k) = k_i f(k)$, a.e. $k \in \mathbb{R}^2$, $f \in \mathscr{D}(K_i)$. Further, we put $\mathrm{d}\Gamma(K) := (\mathrm{d}\Gamma(K_1), \mathrm{d}\Gamma(K_2))$ and $\mathscr{D}(\mathrm{d}\Gamma(K)) := \bigcap_{i=1}^2 \mathscr{D}(\mathrm{d}\Gamma(K_i))$. Then the fiber Hamiltonian $\widehat{H}_\Lambda(\xi)$ with $\Lambda \in [0,\infty)$ attached to the total momentum $\xi \in \mathbb{R}^2$ turns out to be

$$\widehat{H}_\Lambda(\xi) := \psi(\xi - \mathrm{d}\Gamma(K)) + \mathrm{d}\Gamma(\omega) + \varphi(v_\Lambda) + E_\Lambda^{\mathrm{ren}}.$$

In view of (2.5) (with $x = 0$) this operator is selfadjoint on $\mathscr{D}(\widehat{H}_\Lambda(\xi)) = \mathscr{D}(\mathrm{d}\Gamma(\omega))$.

In fact, it is straightforward to verify that

$$\mathscr{U} H^1(\mathbb{R}^2,\mathcal{F}) = \Big\{\Psi \in L^2(\mathbb{R}^2,\mathcal{F}) \,\Big|\, \Psi(\xi) \in \mathcal{D}(\mathrm{d}\Gamma(K)) \text{ a.e. } \xi, \text{ and } \int_{\mathbb{R}^2} \|\psi(\xi - \mathrm{d}\Gamma(K))\Psi(\xi)\|^2 \mathrm{d}\xi < \infty\Big\},$$

and, for all $\Phi \in H^1(\mathbb{R}^2,\mathcal{F})$,

$$(\mathscr{U}\psi(-\mathrm{i}\nabla)\Phi)(\xi) = \psi(\xi - \mathrm{d}\Gamma(K))(\mathscr{U}\Phi)(\xi), \quad \text{a.e. } \xi \in \mathbb{R}^2.$$

Moreover, $\mathscr{U}$ maps $L^2(\mathbb{R}^2,\mathcal{D}(\mathrm{d}\Gamma(\omega)))$ into itself and

$$\mathscr{U}\mathrm{d}\Gamma(\omega)\Phi = \mathrm{d}\Gamma(\omega)\mathscr{U}\Phi, \quad \Phi \in L^2(\mathbb{R}^2,\mathcal{D}(\mathrm{d}\Gamma(\omega))).$$

Finally, we have the well-known commutation relations

$$\Gamma(e_{-x})\varphi(e_x v_\Lambda)\phi = \varphi(v_\Lambda)\Gamma(e_{-x})\phi, \quad x \in \mathbb{R}^2,\ \Lambda \in [0,\infty), \tag{4.2}$$

for, e.g., all $\phi \in \mathcal{D}(\mathrm{d}\Gamma(\omega))$. Putting these remarks together and observing strong resolvent measurability of the family $(\widehat{H}_\Lambda(\xi))_{\xi\in\mathbb{R}^2}$, we infer indeed that

$$\mathscr{U} H_\Lambda \mathscr{U}^* = \int_{\mathbb{R}^2}^{\oplus} \widehat{H}_\Lambda(\xi)\mathrm{d}\xi, \quad \Lambda \in [0,\infty). \tag{4.3}$$

An analogous relation for H is derived in Corollary 8.2 below.

5. Main technical ingredients

In this section we collect the main technical ingredients from [HM23] that we shall employ in the remaining part of this proceeding to give an otherwise fairly self-contained derivation of Feynman–Kac formulas for fiber Hamiltonians.

We start by explaining where our formula (3.3) for the complex action $u_{\infty,t}$ originates from. Notice that both terms in the definition (3.2) of $u_{\Lambda,t}$ with finite Λ become ill-defined when the cutoff at Λ is dropped. We can, however, exploit the presence of the oscillating terms e_{X_s} under the integral in (3.2) to arrive at a new formula for $u_{\Lambda,t}$ comprising more regular terms. This is done with the help of Itô's formula:

Lemma 5.1. *Let $\Lambda \in [0,\infty)$. Then, $\mathbb{P}$-a.s.,*

$$u_{\Lambda,t} = \int_{(0,t]\times\mathbb{R}^2} \langle U^+_{\Lambda,s}|e_{X_{s-}}(e_z - 1)\beta\rangle \mathrm{d}\widetilde{N}(s,z) - \langle U^+_{\Lambda,t}|e_{X_t}\beta\rangle, \quad t \geqslant 0. \tag{5.1}$$

As the reader will notice, in (3.3) we turn the identity (5.1) satisfied for finite Λ into a definition of $u_{\infty,t}$. Recall that $U^+_{\infty,t}$ is well-defined right away. Further, it is not difficult to check that the stochastic integral in (5.1) is meaningful for $\Lambda = \infty$ as well.

Sketch of the proof of Lemma 5.1. The first step is to observe the integral equation

$$U^+_{\Lambda,t} = \int_0^t (e_{X_s} v_\Lambda - \omega U^+_{\Lambda,s})\mathrm{d}s, \quad t \geqslant 0, \tag{5.2}$$

which can be derived with the help of (3.1) and the fundamental theorem of calculus for the Lebesgue integral [HM23, Lemma 4.1]. Since $\mathbb{R}^2 \ni z \mapsto \chi_{B_\Lambda}\mathrm{e}^{-\mathrm{i}K\cdot z}\beta \in$

$L^2(\mathbb{R}^2)$ is bounded and smooth with bounded partial derivatives of any order, and since $U^+_{\Lambda,t} = \chi_{B_\Lambda} U^+_{\Lambda,t}$, we can combine (5.2) with Itô's formula to $\mathbb{P}$-a.s. get

$$\begin{aligned}\langle U^+_{\Lambda,t} | e_{X_t}\beta\rangle &= \int_0^t \langle e_{X_s} v_\Lambda | e_{X_s}\beta\rangle \mathrm{d}s \\ &\quad - \int_0^t \langle \omega U^+_{\Lambda,s} | e_{X_s}\beta\rangle \mathrm{d}s - \int_0^t \langle \psi U^+_{\Lambda,s} | e_{X_s}\beta\rangle \mathrm{d}s \\ &\quad + \int_{(0,t]\times\mathbb{R}^2} \langle U^+_{\Lambda,s} | e_{X_{s-}}(e_z - 1)\beta\rangle \mathrm{d}\widetilde{N}(s,z), \quad t \geqslant 0.\end{aligned}$$

Here the integral in the first line of the right hand side equals $tE^{\mathrm{ren}}_\Lambda$; recall (2.4). Further, since $\chi_{B_\Lambda}(\omega + \psi)\beta = v_\Lambda$, the expression in the second line is equal to $-u_{\Lambda,t} - tE^{\mathrm{ren}}_\Lambda$; see (3.2). □

Employing the formulas for the complex action in (3.3) and (5.1) it is possible to derive the bounds and convergence relations of the next lemma, whose proof can be found in [HM23, §6]. In fact, the second members on the right hand sides of (3.3) and (5.1) can be estimated trivially. The main abstract ingredients used to deal with the stochastic integrals in (3.3) and (5.1) are Kunita's inequality and an exponential tail estimate for Lévy type stochastic integrals due to Applebaum and Siakalli [App09, Sia09].

Lemma 5.2 (**Exponential moment bound and convergence).** *Let $p \in [1,\infty)$. Then there exists $a_p \in (0,\infty)$, also depending on the model parameters m_{p}, m_{b} and g, such that*

$$\sup_{\Lambda\in[0,\infty]} \mathbb{E}\Big[\sup_{s\in[0,t]} \mathrm{e}^{pu_{\Lambda,s}}\Big] \leqslant \mathrm{e}^{a_p(1+t)}, \quad t \geqslant 0.$$

Furthermore,

$$\mathbb{E}\Big[\sup_{s\in[0,t]} \big|\mathrm{e}^{u_{\Lambda,s}} - \mathrm{e}^{u_{\infty,s}}\big|^p\Big] \xrightarrow{\Lambda\to\infty} 0, \quad t \geqslant 0.$$

The next result we shall apply without detailed proof is the flow relation (5.4) implied by [HM23, Lemma 7.9]. Since we only consider finite Λ in (5.4), its proof is, however, fairly elementary: Applying both sides of (5.4) to an exponential vector in Fock space, i.e., a vector of the form $\epsilon(h) := F_1(h)(1,0,0,\dots)$, the proof is reduced to three relations involving the integral processes u_Λ and $U^\pm_\Lambda$ that can be verified by straightforward substitutions. In fact, these computations are virtually identical to those in the proof of [MM18, Lemma 4.18].

For all $\Lambda \in [0,\infty]$, we denote by

$$W_{\Lambda,s,s+r}(x) = \Gamma(e_x) W_{\Lambda,s,s+r}(0)\Gamma(e_{-x}), \quad r,s \geqslant 0,\ x \in \mathbb{R}^2, \tag{5.3}$$

the $\mathscr{B}(\mathscr{F})$-valued random variables obtained by working on the filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_{s+r})_{r\geqslant 0}, \mathbb{P})$ and putting $(X_{s+r} - X_s)_{r\geqslant 0}$ in place of X in (3.1) to (3.3) and (3.5).

Lemma 5.3 (**Flow relation).** *Let $\Lambda \in [0,\infty)$. Then*

$$W_{\Lambda,t}(0) = W_{\Lambda,s,t}(X_s) W_{\Lambda,s}(0), \quad t \geqslant s \geqslant 0. \tag{5.4}$$

Finally, we shall need an integral equation involving $W_{\Lambda,t}(0)$ with finite Λ applied to a vector in the dense subset $\mathscr{C}$ of $\mathscr{F}$ given by

$$\mathscr{C} := \mathrm{span}\big\{\epsilon(f) \in \mathscr{F} \,\big|\, f \in \mathscr{D}(\omega^2)\big\} \subset \mathscr{D}(\mathrm{d}\Gamma(\omega)^2). \tag{5.5}$$

We shall abbreviate

$$h_\Lambda(x) := \mathrm{d}\Gamma(\omega) + \varphi(e_x v_\Lambda) + E^{\mathrm{ren}}_\Lambda, \quad x \in \mathbb{R}^2,$$

so that

$$\widehat{H}_\Lambda(\xi) = \psi(\xi - \mathrm{d}\Gamma(K)) + h_\Lambda(0), \quad \xi \in \mathbb{R}^2,\ \Lambda \in [0,\infty). \tag{5.6}$$

Lemma 5.4 (Integral equation). *Let $\Lambda \in [0,\infty)$ and $\phi \in \mathscr{C}$. Then, at every fixed elementary event, $W_{\Lambda,s}(0)\phi \in \mathscr{D}(\mathrm{d}\Gamma(\omega))$ for all $s \geqslant 0$, the path $[0,\infty) \ni s \mapsto h_\Lambda(X_s)W_{\Lambda,s}(0)\phi \in \mathscr{F}$ is càdlàg and*

$$W_{\Lambda,t}(0)\phi - \phi = -\int_0^t h_\Lambda(X_s)W_{\Lambda,s}(0)\phi\mathrm{d}s, \quad t \geqslant 0. \tag{5.7}$$

Sketch of the proof of Lemma 5.4. The first two statements hold in view of

$$W_{\Lambda,t}(0)\epsilon(f) = \mathrm{e}^{u_{\Lambda,t} - \langle U^-_{\Lambda,t}|f\rangle}\epsilon(\mathrm{e}^{-t\omega}f - U^+_{\Lambda,t}), \quad t \geqslant 0,\ f \in \mathscr{D}(\omega^2). \tag{5.8}$$

Moreover, after scalar-multiplying (5.8) with an exponential vector $\epsilon(f_1)$ with $f_1 \in \mathscr{D}(\omega)$, the proof of (5.7) is reduced to a straightforward computation making use of (5.2); see [HM23, Lemma 4.2] for details. □

6. Feynman–Kac integrands and semigroups for fixed total momentum

In the whole Section 6 we fix $\Lambda \in [0,\infty]$. We shall discuss the Feynman–Kac integrands given by

$$\widehat{W}_{\Lambda,t}(\xi) := \mathrm{e}^{-\mathrm{i}\xi\cdot X_t}\Gamma(e_{-X_t})W_{\Lambda,t}(0), \quad t > 0,\ \xi \in \mathbb{R}^2, \tag{6.1}$$

and $\widehat{W}_{\Lambda,0}(\xi) := \mathbb{1}_{\mathscr{F}}$, as well as associated semigroups.

Remark 6.1. For every $t > 0$, we have the alternative formulas

$$\begin{aligned}\widehat{W}_{\Lambda,t}(0) &= \mathrm{e}^{u_{\Lambda,t}}F_{t/2}(-e_{-X_t}U^+_{\Lambda,t})\Gamma(e_{-X_t})F_{t/2}(-U^-_{\Lambda,t})^* \\ &= \mathrm{e}^{u_{\Lambda,t}}F_{t/3}(-e_{-X_t}U^+_{\Lambda,t})\mathrm{e}^{\mathrm{i}K\cdot X_t - t\mathrm{d}\Gamma(\omega)/3}F_{t/3}(-U^-_{\Lambda,t})^*.\end{aligned}$$

We further know from [GMM17] that the map $(0,\infty) \times L^2(\mathbb{R}^2) \ni (s,h) \mapsto F_s(h) \in \mathscr{B}(\mathscr{F})$ is continuous, and since $|k| \leqslant \omega(k)$, $k \in \mathbb{R}^2$, the map $(0,\infty) \times \mathbb{R}^2 \ni (s,x) \mapsto \mathrm{e}^{\mathrm{i}K\cdot x - s\mathrm{d}\Gamma(\omega)/3} \in \mathscr{B}(\mathscr{F})$ is continuous as well. In conjunction with the separability of $L^2(\mathbb{R}^2)$ as well as the adaptedness and path regularity properties of u_Λ and $U^\pm_\Lambda$ these remarks reveal the following for every $\xi \in \mathbb{R}^2$:

(a) For every fixed $t \geqslant 0$, $\widehat{W}_{\Lambda,t}(\xi)$ is an $\mathfrak{F}_t$-measurable and separably valued $\mathscr{B}(\mathscr{F})$-valued random variable.
(b) At every fixed elementary event, the map $t \mapsto \widehat{W}_{\Lambda,t}(\xi) \in \mathscr{B}(\mathscr{F})$ is right-continuous on $(0,\infty)$ and it has left limits at every point of $(0,\infty)$ that we denote by $\widehat{W}_{\Lambda,t-}(\xi)$, $t > 0$.

Lemma 5.2 is the main ingredient for the next statement.

Proposition 6.2 (Moment bounds and convergence). *Let $p \in [1,\infty)$. Then there exists some $c_p \in (0,\infty)$, solely depending on p and the model parameters m_p, m_b and g, such that*

$$\sup_{\xi\in\mathbb{R}^2}\mathbb{E}\Big[\sup_{s\in[0,t]}\|\widehat{W}_{\Lambda,s}(\xi)\|^p\Big] = \mathbb{E}\Big[\sup_{s\in[0,t]}\|W_{\Lambda,s}(0)\|^p\Big] \leqslant \mathrm{e}^{c_p(1+t)}, \quad t \geqslant 0. \tag{6.2}$$

Furthermore,

$$\sup_{\xi\in\mathbb{R}^2}\mathbb{E}\Big[\sup_{s\in[0,t]}\|\widehat{W}_{\Lambda,s}(\xi) - \widehat{W}_{\infty,s}(\xi)\|^p\Big] \xrightarrow{\Lambda\to\infty} 0, \quad t \geqslant 0. \tag{6.3}$$

Remark 6.3. The norms under the expectations in (6.2) and (6.3) actually are ξ-independent.

Proof. Since $\Gamma(e_x)$ is unitary for every $x \in \mathbb{R}^2$, the first relation in (6.2) is obvious from (6.1). The second relation in (6.2) and (6.3) are implied by (3.4) and Lemma 5.2 and the elementary bound $\|\chi_{B_\Lambda \setminus B_\sigma} U^{\pm}_{\infty,t}\|^2_{t/2} \leqslant 6\pi g^2(\sigma^2 + m_{\mathrm{b}}^2)^{-1/2}$ valid for all $0 \leqslant \sigma < \Lambda \leqslant \infty$. □

We also have an analogue of Lemma 5.3 for fixed total momentum.

Proposition 6.4 (Flow equation). *Let $\xi \in \mathbb{R}^2$ and put*

$$\widehat{W}_{\Lambda,s,t}(\xi) := \mathrm{e}^{-\mathrm{i}\xi\cdot(X_t - X_s)}\Gamma(e_{-(X_t-X_s)})W_{\Lambda,s,t}(0), \quad t \geqslant s.$$

Then, $\mathbb{P}$-a.s.,

$$\widehat{W}_{\Lambda,t}(\xi) = \widehat{W}_{\Lambda,s,t}(\xi)\widehat{W}_{\Lambda,s}(\xi), \quad t \geqslant s. \tag{6.4}$$

Proof. For finite Λ, (6.4) is equivalent to (5.4) in view of (5.3) and (6.1). By virtue of (6.3) and its analogue for $(\widehat{W}_{\Lambda,s,s+r}(\xi))_{r\geqslant 0}$, (6.4) extends to $\Lambda = \infty$. □

In view of item (a) in Remark 6.1 as well as Proposition 6.2 the following $\mathscr{B}(\mathscr{F})$-valued expectations are well-defined:

$$\widehat{T}_{\Lambda,t}(\xi) := \mathbb{E}[\widehat{W}_{\Lambda,t}(\xi)^*], \quad t \geqslant 0,\ \xi \in \mathbb{R}^2. \tag{6.5}$$

Proposition 6.5 (Norm bound and convergence). *With c_1 denoting the Λ-independent constant appearing in Proposition 6.2, we have*

$$\sup_{\xi\in\mathbb{R}^2} \|\widehat{T}_{\Lambda,t}(\xi)\| \leqslant \mathrm{e}^{c_1(1+t)}, \quad t \geqslant 0. \tag{6.6}$$

Furthermore,

$$\sup_{\xi\in\mathbb{R}^2}\sup_{s\in[0,t]} \|\widehat{T}_{\Lambda,s}(\xi) - \widehat{T}_{\infty,s}(\xi)\| \xrightarrow{\Lambda\to\infty} 0, \quad t \geqslant 0. \tag{6.7}$$

Proof. Manifestly, (6.6) and (6.7) follow from (6.2) and (6.3), respectively. □

We continue by deriving a Markov property involving the family $(\widehat{T}_{\Lambda,t}(\xi))_{t\geqslant 0}$ that directly will entail its semigroup property.

Theorem 6.6 (Markov property). *Let $\xi \in \mathbb{R}^2$ and $t \geqslant s \geqslant 0$. Then, $\mathbb{P}$-a.s.,*

$$\mathbb{E}^{\mathfrak{F}_s}[\widehat{W}_{\Lambda,t}(\xi)^*] = \widehat{W}_{\Lambda,s}(\xi)^*\widehat{T}_{\Lambda,t-s}(\xi). \tag{6.8}$$

Proof. This follows upon taking adjoints on both sides of (6.4) and observing that $\widehat{W}_{\Lambda,s,t}(\xi)^*$ is $\mathfrak{F}_s$-independent while $\widehat{W}_{\Lambda,s}(\xi)^*$ is $\mathfrak{F}_s$-measurable. In fact, let $(\mathfrak{F}_r^s)_{r\geqslant 0}$ denote the (automatically right-continuous) completion of the natural filtration associated with $(X_{r+s} - X_s)_{r\geqslant 0}$. Applying Remark 6.1 to that filtration and the time-shifted Lévy process, we see that $(\widehat{W}_{\Lambda,s,s+r}(\xi)^*)_{r\geqslant 0}$ is adapted to $(\mathfrak{F}_r^s)_{r\geqslant 0}$. Since X is $(\mathfrak{F}_r)_{r\geqslant 0}$-Lévy, we know, however, that each $\mathfrak{F}_r^s$ with $r \geqslant 0$ and $\mathfrak{F}_s$ are independent. To get (6.8) we also exploit that $\widehat{W}_{\Lambda,s,t}(\xi)^*$ and $\widehat{W}_{\Lambda,t-s}(\xi)^*$ have the same distribution. □

Taking expectations in (6.8) with $t = r + s$ we arrive at the following result:

Corollary 6.7 (Semigroup property). *For all $\xi \in \mathbb{R}^2$ and $r, s \geqslant 0$,*

$$\widehat{T}_{\Lambda,s+r}(\xi) = \widehat{T}_{\Lambda,s}(\xi)\widehat{T}_{\Lambda,r}(\xi).$$

7. Feynman–Kac formulas for fiber Hamiltonians

We now turn to the derivation of the Feynman–Kac formulas for the fiber Hamiltonians. First, we shall do this for the ultraviolet regularized operators, by showing that the semigroup $(\widehat{T}_{\Lambda,t}(\xi))_{t\geqslant 0}$ is strongly continuous and identifying $\widehat{H}_\Lambda(\xi)$ as its generator. For the latter two tasks we require the stochastic differential equations derived in the next lemma. Recall the definition (5.5) of $\mathscr{C}$.

Lemma 7.1 (Stochastic differential equation with cutoff). *Let $\Lambda \in [0,\infty)$, $\xi \in \mathbb{R}^2$ and $\phi \in \mathscr{C}$. Then, $\mathbb{P}$-a.s.,*

$$\widehat{W}_{\Lambda,t}(\xi)\phi - \phi = -\int_0^t \widehat{H}_\Lambda(\xi)\widehat{W}_{\Lambda,s}(\xi)\phi \mathrm{d}s + \int_{(0,t]\times\mathbb{R}^2} (\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z}) - 1)\widehat{W}_{\Lambda,s-}(\xi)\phi \mathrm{d}\widetilde{N}(s,z), \quad t \geqslant 0.$$

Proof. Let $\eta \in \mathscr{D}(\mathrm{d}\Gamma(\omega)^2)$. Since $\mathbb{R}^2 \ni z \mapsto \mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot z}\eta \in \mathscr{F}$ is twice continuously differentiable and bounded with bounded first and second order partial derivatives, we $\mathbb{P}$-a.s. have the Itô formula

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_t}\eta - \eta \\ &= -\int_0^t \psi(\xi - \mathrm{d}\Gamma(K))\mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_s}\eta \mathrm{d}s \\ &\quad + \int_{(0,t]\times\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_{s-}}(\mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot z} - 1)\eta \mathrm{d}\widetilde{N}(s,z), \quad t \geqslant 0.\end{aligned}$$

In conjunction with Lemma 5.4 and Itô's product rule for scalar products it $\mathbb{P}$-a.s. implies

$$\begin{aligned}&\langle \mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_t}\eta | W_{\Lambda,t}(0)\phi\rangle - \langle \eta|\phi\rangle \\ &= -\int_0^t \langle \mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_s}\eta | (\psi(\xi - \mathrm{d}\Gamma(K)) + h_\Lambda(X_s))W_{\Lambda,s}(0)\phi\rangle \mathrm{d}s \\ &\quad + \int_{(0,t]\times\mathbb{R}^2} \langle \mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_{s-}}(\mathrm{e}^{\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot z} - 1)\eta | W_{\Lambda,s-}(0)\phi\rangle \mathrm{d}\widetilde{N}(s,z),\end{aligned}$$

for all $t \geqslant 0$. On account of (4.2) and (6.1) we further have

$$\mathrm{e}^{-\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_s} h_\Lambda(X_s)W_{\Lambda,s}(0)\phi = h_\Lambda(0)\widehat{W}_{\Lambda,s}(\xi)\phi, \quad s \geqslant 0,$$

as well as $\mathrm{e}^{-\mathrm{i}(\xi-\mathrm{d}\Gamma(K))\cdot X_{s-}}W_{\Lambda,s-}(0) = \widehat{W}_{\Lambda,s-}(\xi)$, $s > 0$. Since η can be chosen from a countable dense subset, these remarks and (5.6) $\mathbb{P}$-a.s. imply the asserted stochastic differential equation. □

We will also need the following bound. In its proof we argue similarly as in the proof of [GMM17, Lemma 10.9].

Lemma 7.2. *Let $\Lambda \in [0,\infty)$ and $\xi \in \mathbb{R}^2$. Then there exists a constant $b_\Lambda(\xi) \in (0,\infty)$, also depending on the model parameters m_p, m_b and g, such that*

$$\mathbb{E}\big[\|(1 + \mathrm{d}\Gamma(\omega))^{-1}(\widehat{W}_{\Lambda,t}(\xi)\phi - \phi)\|^2\big] \leqslant b_\Lambda(\xi)t\mathrm{e}^{b_\Lambda(\xi)t}\|\phi\|^2, \quad t \geqslant 0,\ \phi \in \mathscr{F}. \tag{7.1}$$

Proof. To start with we assume that $\phi \in \mathscr{C}$. Abbreviating $\theta := 1 + \mathrm{d}\Gamma(\omega)$ and $\eta_t := \theta^{-1}(\widehat{W}_{\Lambda,t}(\xi)\phi - \phi)$, $t \geqslant 0$, we then infer from (7.1) and Itô's product formula that, $\mathbb{P}$-a.s.,

$$\begin{aligned}\|\eta_t\|^2 &= -2\int_0^t \mathrm{Re}\langle\eta_s|\theta^{-1}\widehat{H}_\Lambda(\xi)\widehat{W}_{\Lambda,s}(\xi)\phi\rangle\mathrm{d}s \\ &\quad+\int_{(0,t]\times\mathbb{R}^2}\|\theta^{-1}(\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z})-1)\widehat{W}_{\Lambda,s-}(\xi)\phi\|^2\mathrm{d}N(s,z) \\ &\quad+2\int_{(0,t]\times\mathbb{R}^2}\mathrm{Re}\langle\eta_{s-}|\theta^{-1}(\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z})-1)\widehat{W}_{\Lambda,s-}(\xi)\phi\rangle\mathrm{d}\widetilde{N}(s,z), \quad t\geqslant 0,\end{aligned} \tag{7.2}$$

where N is the Poisson point measure defined by the jumps of X. Here

$$\|\theta^{-1}(\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z})-1)\| \leqslant \min\{|z|,2\}(1+|\xi|), \quad z\in\mathbb{R}^2,$$

and in view of (2.5) we know that the operator $\theta^{-1}\widehat{H}_\Lambda(\xi)$ is bounded. We further recall (6.2), from which we infer the *a priori* bound $\mathbb{E}[\sup_{s\in[0,t]}\|\eta_s\|^4] \leqslant \mathrm{e}^{4c(1+t)}\|\phi\|^4$, $t\geqslant 0$, with $c\in(0,\infty)$ solely depending on m_{p}, m_{b} and g. On account of these bounds, the stochastic integral in the last line of (7.2) is a martingale starting at 0 and in particular its expectation is 0. Furthermore, the expectation of the N-integral in the second line of (7.2) equals

$$\int_0^t\int_{\mathbb{R}^2}\mathbb{E}\big[\|\theta^{-1}(\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z})-1)\widehat{W}_{\Lambda,s-}(\xi)\phi\|^2\big]\mathrm{d}\nu(z)\,\mathrm{d}s.$$

Upon taking expectations on both sides of (7.2) we thus find

$$\begin{aligned}\mathbb{E}[\|\eta_t\|^2] &\leqslant 2t\mathrm{e}^{(c+c_2/2)(1+t)}\|\theta^{-1}\widehat{H}_\Lambda(\xi)\|\|\phi\|^2 \\ &\quad+t\mathrm{e}^{c_2(1+t)}(1+|\xi|)^2\int_{\mathbb{R}^2}\min\{|z|,2\}^2\mathrm{d}\nu(z)\,\|\phi\|^2, \quad t\geqslant 0.\end{aligned}$$

Here the ν-integral is finite because ν is a Lévy measure. Finally, we invoke the dominated convergence theorem and (6.2) to extend (7.1) to general $\phi\in\mathscr{F}$. □

We can now prove the Feynman–Kac formula for the ultraviolet regularized fiber Hamiltonians.

Theorem 7.3 (Feynman–Kac formula with cutoff). *Let* $\Lambda\in[0,\infty)$ *and* $\xi\in\mathbb{R}^2$. *Then*

$$\lim_{t\to s}\|(\widehat{T}_{\Lambda,t}(\xi)-\widehat{T}_{\Lambda,s}(\xi))(1+\mathrm{d}\Gamma(\omega))^{-1}\| = 0, \quad s\geqslant 0, \tag{7.3}$$

and in particular the semigroup $(\widehat{T}_{\Lambda,t}(\xi))_{t\geqslant 0}$ *is strongly continuous. Furthermore,*

$$\mathrm{e}^{-t\widehat{H}_\Lambda(\xi)} = \widehat{T}_{\Lambda,t}(\xi), \tag{7.4}$$

and in particular $\widehat{T}_{\Lambda,t}(\xi)$ *is selfadjoint for every* $t\geqslant 0$.

Proof. First, we prove (7.3) which together with (6.6) entails strong continuity. To that end it suffices to show that

$$\lim_{t\downarrow 0}\|(\widehat{T}_{\Lambda,t}(\xi)-\mathbb{1}_{\mathscr{F}})(1+\mathrm{d}\Gamma(\omega))^{-1}\| = 0, \tag{7.5}$$

by the semigroup property and (6.6). Using Cauchy–Schwarz inequalities and applying Lemma 7.2 we find, however,

$$\begin{aligned}&\|(\widehat{T}_{\Lambda,t}(\xi)-\mathbb{1}_{\mathscr{F}})(1+\mathrm{d}\Gamma(\omega))^{-1}\| \\ &= \sup_{\|\phi_1\|=\|\phi_2\|=1}\big|\mathbb{E}[\langle(1+\mathrm{d}\Gamma(\omega))^{-1}(\widehat{W}_{\Lambda,t}(\xi)\phi_1-\phi_1)|\phi_2\rangle]\big| \\ &\leqslant \sup_{\|\phi_1\|=1}\mathbb{E}\big[\|(1+\mathrm{d}\Gamma(\omega))^{-1}(\widehat{W}_{\Lambda,t}(\xi)\phi_1-\phi_1)\|^2\big]^{1/2} \leqslant (b_\Lambda(\xi)t\mathrm{e}^{b_\Lambda(\xi)t})^{1/2}, \quad t>0,\end{aligned}$$

which proves (7.5), of course.

By our results proven so far, we know that the semigroup $(\widehat{T}_{\Lambda,t}(\xi))_{t\geqslant 0}$ has a closed generator, call it $G_\Lambda(\xi)$, whose spectrum is contained in the half-space $\{z \in \mathbb{C} | \operatorname{Re}[z] \geqslant a\}$ for some $a \in \mathbb{R}$. We shall now show that $G_\Lambda(\xi) = \widehat{H}_\Lambda(\xi)$, which is equivalent to the validity of (7.4) for all $t \geqslant 0$. In fact, it suffices to show the inclusion $\widehat{H}_\Lambda(\xi) \subset G_\Lambda(\xi)$, because we then can pick some $\zeta \in \mathbb{C}$ belonging to the resolvent sets of both $\widehat{H}_\Lambda(\xi)$ and $G_\Lambda(\xi)$ (e.g., $\zeta = a - 1 + \mathrm{i}$) and apply the second resolvent identity to see that $(\widehat{H}_\Lambda(\xi) - \zeta)^{-1} = (G_\Lambda(\xi) - \zeta)^{-1}$.

So let $\eta \in \mathscr{D}(\widehat{H}_\Lambda(\xi)) = \mathscr{D}(\mathrm{d}\Gamma(\omega))$. Scalar-multiplying the SDE in Lemma 7.1 with η and taking expectations afterwards, we find

$$\langle \widehat{T}_{\Lambda,t}(\xi)\eta - \eta | \phi \rangle = -\int_0^t \langle \widehat{T}_{\Lambda,s}(\xi)\widehat{H}_\Lambda(\xi)\eta|\phi\rangle \mathrm{d}s + \mathbb{E}[M_t(\eta,\phi)],$$

for all $t \geqslant 0$ and $\phi \in \mathscr{C}$. Here the stochastic integral process given by

$$M_t(\eta,\phi) := \int_{(0,t]\times\mathbb{R}^2} \langle \eta | (\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z}) - 1)\widehat{W}_{\Lambda,s-}(\xi)\phi\rangle \mathrm{d}\widetilde{N}(s,z), \quad t \geqslant 0,$$

is a martingale starting at 0. This follows from (6.2) and the bound

$$\|(\mathrm{e}^{-\mathrm{i}\xi\cdot z}\Gamma(e_{-z}) - 1)^*\eta\| \leqslant \min\{|z|, 2\}(2\|\eta\| + |\xi|\|\eta\| + \|\mathrm{d}\Gamma(\omega)\eta\|), \quad z \in \mathbb{R}^2.$$

In particular $\mathbb{E}[M_t(\eta,\phi)] = 0$, $t \geqslant 0$. Since ϕ can be chosen in a dense subset of $\mathscr{F}$, we deduce that

$$\frac{1}{t}(\widehat{T}_{\Lambda,t}(\xi)\eta - \eta) = -\frac{1}{t}\int_0^t \widehat{T}_{\Lambda,s}(\xi)\widehat{H}_\Lambda(\xi)\eta \mathrm{d}s, \quad t > 0, \tag{7.6}$$

with an $\mathscr{F}$-valued Bochner-Lebesgue integral on the right hand side. The whole expression on the right hand side of (7.6) converges to $-\widehat{H}_\Lambda(\xi)\eta$, as $t \downarrow 0$, because $\widehat{T}_{\Lambda,s}(\xi)\widehat{H}_\Lambda(\xi)\eta \to \widehat{H}_\Lambda(\xi)\eta$, as $s \downarrow 0$, by strong continuity of the semigroup. Thus, $\eta \in \mathscr{D}(G_\Lambda(\xi))$ with $G_\Lambda(\xi)\eta = \widehat{H}_\Lambda(\xi)\eta$. □

Using the convergence statements proven in Section 6 it is not hard to deduce our main result for the fiber Hamiltonians.

Theorem 7.4 (Renormalization; Feynman–Kac formula without cutoff). *Let $\xi \in \mathbb{R}^2$. Then the following holds:*

(i) *Statement (7.3) holds for $\Lambda = \infty$ as well.*
(ii) *$(\widehat{T}_{\infty,t}(\xi))_{t\geqslant 0}$ is a strongly continuous semigroup of selfadjoint operators satisfying $\|\widehat{T}_{\infty,t}(\xi)\| \leqslant \mathrm{e}^{c(1+t)}$ for all $t \geqslant 0$ and some $c \in (0,\infty)$.*
(iii) *Denote by $\widehat{H}(\xi)$ the selfadjoint, lower semibounded generator of $(\widehat{T}_{\infty,t}(\xi))_{t\geqslant 0}$, so that*

$$\mathrm{e}^{-t\widehat{H}(\xi)} = \widehat{T}_{\infty,t}(\xi) = \mathbb{E}[\widehat{W}_{\infty,t}(\xi)^*], \quad t \geqslant 0. \tag{7.7}$$

Then $\widehat{H}_\Lambda(\xi)$ converges in the norm resolvent sense to $\widehat{H}(\xi)$ as $\Lambda \to \infty$.

Proof. Part (i) is a consequence of (7.3) and the uniform convergence on compact time intervals in (6.7). Strong continuity of $(\widehat{T}_{\infty,t}(\xi))_{t\geqslant 0}$ follows from (i) and (6.6). Each $\widehat{T}_{\infty,t}(\xi)$ with $t \geqslant 0$ is selfadjoint since by (6.7) it is the norm limit as $\Lambda \to \infty$ of the selfadjoint operators $\widehat{T}_{\Lambda,t}(\xi)$; recall the last statement of Theorem 7.3. The norm bounds in (ii) have already been stated in (6.6). By (ii) and the Hille–Yosida theorem, an operator $\widehat{H}(\xi)$ as in (iii) exists and is unique. The norm resolvent convergence $\widehat{H}_\Lambda(\xi) \to \widehat{H}(\xi)$, $\Lambda \to \infty$, is known to be equivalent to the norm convergence $\mathrm{e}^{-t\widehat{H}_\Lambda(\xi)} \to \mathrm{e}^{-t\widehat{H}(\xi)}$, $\Lambda \to \infty$, for every $t \geqslant 0$. The latter holds due to (6.7), (7.4) and (7.7), which proves (iii). □

Remark 7.5. For all $\Lambda \in [0,\infty]$ and $t \geqslant 0$, the map $\mathbb{R}^2 \ni \xi \mapsto \widehat{T}_{\Lambda,t}(\xi) \in \mathscr{B}(\mathscr{F})$ is continuous. This follows from (6.1), (6.5), the $\mathbb{P}$-integrability of $\|W_{\Lambda,t}(0)\|$ and the dominated convergence theorem for the Bochner-Lebesgue integral. In view of (7.7) we may conclude that the familiy $(\widehat{H}(\xi))_{\xi\in\mathbb{R}^2}$ is strongly resolvent measurable and in particular its direct integral is a well-defined selfadjoint operator in $L^2(\mathbb{R}^2,\mathscr{F})$.

More is true for strictly positive particle masses:

Remark 7.6. Let $\Lambda \in [0,\infty]$ and $t \geqslant 0$. Assume that $m_{\mathrm{p}} > 0$ and set $S(m_{\mathrm{p}}) := \{z \in \mathbb{C}^2 \,|\, |\mathrm{Im}[z]| < m_{\mathrm{p}}\}$. Then the expectations

$$\widehat{T}_{\Lambda,t}(\zeta) := \mathbb{E}[\mathrm{e}^{\mathrm{i}\zeta\cdot X_t}\widehat{W}_{\Lambda,t}(0)^*], \quad \zeta \in S(m_{\mathrm{p}}),$$

are well-defined and extend the previously considered family $(\widehat{T}_{\Lambda,t}(\xi))_{\xi\in\mathbb{R}^2}$ to $S(m_{\mathrm{p}})$. Moreover, the map $S(m_{\mathrm{p}}) \ni \zeta \mapsto \widehat{T}_{\Lambda,t}(\zeta) \in \mathscr{B}(\mathscr{F})$ is analytic. This follows easily from Hölder's inequality and (6.2) since $\mathbb{E}[\mathrm{e}^{|\zeta||X_t|}] < \infty$ whenever $|\zeta| < m_{\mathrm{p}}$.

8. The full Hamiltonian revisited

We wish to verify that the renormalized operators $\widehat{H}(\xi)$, $\xi \in \mathbb{R}^2$, give rise to a fiber decomposition of the renormalized full Hamiltonian. In what follows $\mathscr{U}$ again denotes the Lee-Low-Pines tranformation of Section 4. The next corollary actually provides an independent existence proof for the norm resolvent limit of the family $(H_\Lambda)_{\Lambda\in[0,\infty)}$, based on the key ingredients collected in Section 5:

Corollary 8.1. *As Λ tends to infinity, $\mathscr{U} H_\Lambda \mathscr{U}^*$ converges in the norm resolvent sense to $\int_{\mathbb{R}^2}^{\oplus} \widehat{H}(\xi)\mathrm{d}\xi$.*

Proof. On account of (4.3) and the Feynman–Kac formulas (7.4) and (7.7), the statement is equivalent to the operator norm convergences

$$\int_{\mathbb{R}^2}^{\oplus} \widehat{T}_{\Lambda,t}(\xi)\mathrm{d}\xi \xrightarrow{\Lambda\to\infty} \int_{\mathbb{R}^2}^{\oplus} \widehat{T}_{\infty,t}(\xi)\mathrm{d}\xi, \quad t > 0,$$

which follow from the ξ-uniform convergence in (6.7). □

With H denoting the norm resolvent limit of $(H_\Lambda)_{\Lambda\in[0,\infty)}$ we thus arrive at:

Corollary 8.2. $\mathscr{U} H \mathscr{U}^* = \int_{\mathbb{R}^2}^{\oplus} \widehat{H}(\xi)\mathrm{d}\xi$.

Finally, we fulfill a promise we gave at the end of Section 3:

Alternative proof of the Feynman–Kac formula (3.6). Let $t > 0$. We assume that $\Psi \in L^2(\mathbb{R}^2,\mathscr{F})$ has an integrable Fourier transform $\widehat{\Psi}$ and set $\Phi(x) := \Gamma(e_x)\Psi(x)$, a.e. $x \in \mathbb{R}^2$. In view of

$$\mathrm{e}^{-tH_\Lambda}\Phi = \mathscr{U}^* \int_{\mathbb{R}^2}^{\oplus} \widehat{T}_{\Lambda,t}(\xi)\mathrm{d}\xi\, \mathscr{U}\Phi$$

as well as (4.1), (6.1) and (6.5) we find

$$\begin{aligned}(\mathrm{e}^{-tH_\Lambda}\Phi)(x) &= \Gamma(e_x)\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\xi\cdot X_t} W_{\Lambda,t}(0)^*\Gamma(e_{X_t})\big]\widehat{\Psi}(\xi)\mathrm{d}\xi \\ &= \mathbb{E}\big[\Gamma(e_x)W_{\Lambda,t}(0)^*\Gamma(e_{-x})\Gamma(e_{x+X_t})\Psi(x+X_t)\big] \\ &= \mathbb{E}[W_{\Lambda,t}(x)^*\Phi(x+X_t)], \quad \text{a.e. } x,\end{aligned}$$

where we applied the Fubini and Fourier inversion theorems in the second step and (3.5) in the third one. This proves (3.6) for all Φ in a dense subset of $L^2(\mathbb{R}^2,\mathscr{F})$.

Since, by (3.5) and (6.2),

$$\int_{\mathbb{R}^2} \|\mathbb{E}[W_{\Lambda,t}(x)^*\Phi(x+X_t)]\|^2 \mathrm{d}x \leqslant \mathrm{e}^{c_2(1+t)} \int_{\mathbb{R}^2} \mathbb{E}[\|\Phi(x+X_t)\|^2]\mathrm{d}x = \mathrm{e}^{c_2(1+t)}\|\Phi\|^2, \quad \Phi \in L^2(\mathbb{R}^2, \mathscr{F}),$$

it is clear that (3.6) extends to all $\Phi \in L^2(\mathbb{R}^2, \mathscr{F})$ by approximation. □

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