SPATIAL EXPONENTIAL DECAY OF THE GROUND STATE OF THE RENORMALIZED NELSON MODEL BY FEYNMAN-KAC FORMULA

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1 Nelson model

This is the joint work [11] with Oliver Matte. In [11] we discuss the pointwise spatial decay of the ground state of the *renormalized* Nelson model [15, 9, 14]. In this article we review only the standard Nelson model. I.e., we study the Nelson model with a cutoff. When a cutoff is imposed, the problem becomes significantly easier. It is no exaggeration to say that the theory closely resembles that of Schrödinger operators. In contrast, the renormalized Nelson model with the cutoff removed becomes considerably more difficult. The main question addressed in [11] concerns the spatial decay estimates for the renormalized Nelson model.

Let $\Phi_{\rm g}$ be the ground state of the Nelson Hamiltonian. It is a vector involved in $L^2(\mathbb{R}^3;\mathcal{F})$. An upper bound of the spatial decay of $\|\Phi_{\rm g}(x)\|_{\mathcal{F}}$ has been already shown in [10]. In this article the lower bound is shown in terms of an Agmon type metric.

We apply stochastic methods to measure the spatial exponential localization. This type of arguments have been done for Schrödinger type operators in a large number of papers, e.g., [1, 2, 3, 6, 5, 4, 16, 17].

1.1 Quantum mechanical matters

The particle Hamiltonian is defined by the 3-dimensional Schrödinger operator with external potential V:

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V,$$

which acts in $L^2(\mathbb{R}^3)$. We introduce the Kato-decomposable class [2, Section 4] and [6].

Definition 1.1 *Let* $V : \mathbb{R}^d \to \mathbb{R}$.

(1) V *is a Kato-class potential if and only if*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{\mathrm{B}_r(x)} |\kappa_d(x-y)V(y)|\,\mathrm{d}y = 0$$

holds with function κ_d *depending on the dimension* d*:*

$$\kappa_d(x) = \begin{cases} |x|, & d=1, \\ -\log|x|, & d=2, \\ |x|^{2-d}, & d\geq 3. \end{cases}$$

The set of Kato-class potentials is denoted by $\mathcal{K}(\mathbb{R}^d)$.

(2) $V \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ *if and only if* $\mathbb{1}_K V \in \mathcal{K}(\mathbb{R}^d)$ *for any compact set* $K \subset \mathbb{R}^d$.

(3) V *is Kato-decomposable if and only if* $V = V_+ - V_-$ *with* $V_+(x) = \max\{V(x), 0\}$ *and* $V_-(x) = \max\{-V(x), 0\}$ *satisfy that* $V_+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $V_- \in \mathcal{K}(\mathbb{R}^d)$. *The set of Kato-decomposable potentials is denoted by* $\mathcal{K}_d$.

The self-adjoint operator of the form $H_{\mathrm{p}} = -\frac{1}{2}\Delta + V$ with Kato-decomposable potential V is defined through a Feynman-Kac formula. Let $(B_t)_{t\geq 0}$ be 3-dimensional Brownian motion on a probability space $(\mathscr{X}, \mathcal{B}, \mathcal{W}^x)$, which starts from $x \in \mathbb{R}^3$ at $t = 0$. The expectation value with respect to the probability measure $\mathcal{W}^x$ is denoted by $\mathbb{E}^x[\ldots]$. In particular we set $\mathbb{E}$ for $\mathbb{E}^0$ for notational simplicity. Let V be bounded. Then H_{p} is self-adjoint on $D(\Delta)$ and we have

$$(f, e^{-tH_{\mathrm{p}}} g)_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s} \overline{f(B_0)} g(B_t)]\mathrm{d}x.$$

Replacing V on the right-hand side above with Kato-decomposable potentials, one can also see that the right-hand side is finite for any $f, g \in L^2(\mathbb{R}^3)$ and, by Riesz representation theorem, one defines a strongly continuous one-parameter semigroup S_t, $t \geq 0$ such that

$$(f, S_t g)_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s} \overline{f(B_0)} g(B_t)]\mathrm{d}x.$$

By the Stone theorem for semigroups, there exists the self-adjoint operator H_{p} such that $S_t = e^{-tH_{\mathrm{p}}}$ for $t \geq 0$. This is the definition of H_{p} with Kato-decomposable potentials V.

1.2 Nelson Hamiltonian

Let us define the quantum field part. Let $\mathcal{F}$ be the boson Fock space over $L^2(\mathbb{R}^3)$ defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_{(n)}$$

with n particle subspace $\mathcal{F}_{(n)} = L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$ for $n \geq 1$ and $\mathcal{F}_{(0)} = \mathbb{C}$. Then $\Phi \in \mathcal{F}$ is denoted by $\Phi = \oplus_{n=0}^{\infty} \Phi^{(n)}$. The vector $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}$ is called the Fock vacuum. Let $a^\dagger(g)$ and $a(g)$ be the creation operator and the annihilation operator smeared by $g \in L^2(\mathbb{R}^3)$, respectively, acting in $\mathcal{F}$. They satisfy that $a(g)^* = a^\dagger(\bar{g})$, $[a(g), a^\dagger(f)] = (\bar{g}, f)_{L^2(\mathbb{R}^3)}$ and $[a(g), a(f)] = 0 = [a^\dagger(g), a^\dagger(f)]$. Let $\omega(k) = |k|$ be the relativistic energy of a single massless boson with momentum $k \in \mathbb{R}^3$. The free field Hamiltonian H_{f} acting in $\mathcal{F}$ is given by

$$H_{\mathrm{f}} = \mathrm{d}\Gamma(\omega),$$

where

$$\begin{aligned}
&\left(\mathrm{d}\Gamma(\omega)\Phi\right)^{(n)}(k_1, \ldots, k_n) = \left(\sum_{j=1}^{n} \omega(k_j)\right) \Phi^{(n)}(k_1, \ldots, k_n), \quad n \geq 1, \\
&\mathrm{d}\Gamma(\omega)\Omega = 0.
\end{aligned}$$

The total Hilbert space $\mathcal{H}$ for the Nelson model is defined by

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

Now let us define the Nelson Hamiltonian with a cutoff $\hat{\varphi}$. Let $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^3)$ be the set of real-valued Schwarz distributions on $\mathbb{R}^3$ and $\hat{\varphi}$ the Fourier transform of φ in the sense of distribution.

Assumption 1.2 *Let $\varphi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^3)$. We suppose that $\hat{\varphi} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$ and $\hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$.*

Throughout this paper we assume Assumption 1.2. Let $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})\check{\ }$, where $\check{f}$ denotes the inverse Fourier transform of f. The linear interaction H_{I} is defined by

$$H_{\mathrm{I}} = \int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{I}}(x) \mathrm{d}x$$

with the action:

$$(H_{\mathrm{I}}\Phi)(x) = \phi(x)\Phi(x) \quad a.e. x \in \mathbb{R}^3.$$

Here for each $x \in \mathbb{R}^3$, the field operator $\phi(x)$ is given by

$$\phi(x) = \frac{1}{\sqrt{2}} \left\{ a^\dagger \left(\frac{\hat{\varphi}}{\sqrt{\omega}} e^{-ik\cdot x} \right) + a \left(\frac{\tilde{\hat{\varphi}}}{\sqrt{\omega}} e^{ik\cdot x} \right) \right\}.$$

Here $\tilde{\hat{\varphi}}(k) = \hat{\varphi}(-k)$. Then the Nelson Hamiltonian with ultraviolet cutoff $\hat{\varphi}$ and Kato-decomposable potential V is defined by

$$H = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{I}}.$$

Under Assumption 1.2, H_{I} is symmetric and infinitesimally small with respect to the self-adjoint operator $H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}$. Then H is self-adjoint on $D(H_{\mathrm{p}} \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_{\mathrm{f}})$ by the Kato–Rellich theorem.

1.3 FKF for e^{-tH}

Let us define the bounded operator $\mathrm{J}_{[0,t]}$ by

$$\mathrm{J}_{[0,t]} = \overline{e^{\frac{1}{2}W} e^{a^\dagger(U)} e^{-tH_{\mathrm{f}}} e^{a(\tilde{U})}},$$

where $\overline{\{\ldots\}}$ denotes the operator closure and

$$U(k) = -\int_0^t \frac{e^{-|s|\omega(k)}\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e^{-ik\cdot B_s} \mathrm{d}s, \quad \tilde{U}(k) = -\int_0^t \frac{e^{-|s-t|\omega(k)}\hat{\varphi}(-k)}{\sqrt{2\omega(k)}} e^{ik\cdot B_s} \mathrm{d}s.$$

The exponent W is given by

$$W = \frac{1}{2} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}r \int_{\mathbb{R}^3} \frac{e^{-|s-r|\omega(k)}|\hat{\varphi}(k)|^2}{\omega(k)} e^{-ik\cdot(B_s - B_r)} \mathrm{d}k.$$

Note that $e^{a^\sharp(f)} = \sum_{n=0}^\infty a^\sharp(f)^n/n!$ is an unbounded operator. On e can see that $\|\mathrm{J}_{[0,t]}\| \le C_{\hat{\varphi}}(t)$, where

$$C_{\hat{\varphi}}(t) = \begin{cases} 2\exp\left\{ \dfrac{t}{2} \|\hat{\varphi}/\omega\|^2 + 2t(t \vee 1)(\|\hat{\varphi}/\sqrt{\omega}\|^2 + \|\hat{\varphi}/\omega\|^2) \right\} & \hat{\varphi}/\omega^{3/2} \notin L^2(\mathbb{R}^3), \\ 2\exp\left\{ t\left(\dfrac{3}{2}\|\hat{\varphi}/\omega\|^2 + \|\hat{\varphi}/\omega^{3/2}\|^2 \vee \|\hat{\varphi}/\sqrt{\omega}\|^2 \right) \right\} & \hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3). \end{cases}$$

In particular we have

$$|(\Psi, \mathrm{J}_{[0,t]}\Phi)_{\mathcal{F}}| \le C_{\hat{\varphi}}(t) \|\Psi\|_{\mathcal{F}} \|\Phi\|_{\mathcal{F}}. \tag{1.1}$$

It is important to see that $\mathrm{J}_{[0,t]}$ depends on $w \in \mathscr{X}$ but the right-hand sides of (1.1) are independent of $w \in \mathscr{X}$. Let $V \in \mathcal{K}_3$ and suppose Assumption 1.2. Then we have FKF:

$$(F, e^{-tH}G)_{\mathcal{H}} = \int_{\mathbb{R}^3} \mathbb{E}^x [e^{-\int_0^t V(B_s)\mathrm{d}s} (F(B_0), \mathrm{J}_{[0,t]} G(B_t))_{\mathcal{F}}] \mathrm{d}x.$$

We refer to e.g., [14, 10].

1.4 Ground state

The next proposition guarantees the existence and the uniqueness of the ground state of the Nelson Hamiltonian H.

Proposition 1.3 *Suppose that* $\hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3)$ *and* $V \in \mathcal{K}_3$*. Assume that the binding condition holds true. Then the ground state of* H *exists and it is unique.*

Proof: See [8, (3) and Theorem 3.1] for the binding condition and [18, 7] for the existence of the ground state. □

Example 1.4 *Let* V *be such that* $\lim_{|x|\to\infty} V(x) = \infty$*. Then* V *satisfies the binding condition, and then the ground state of* H *exists and unique.*

Let $\Phi_{\rm b}$ be a bound state of H such that $H\Phi_{\rm b} = E_{\rm b}\Phi_{\rm b}$. Let $\Psi \in \mathcal{F}$ and $t \geq 0$. Since $e^{-tH}\Phi_{\rm b} = e^{-tE_{\rm b}}\Phi_{\rm b}$, we have

$$(\Psi, \Phi_{\rm b}(x))_{\mathcal{F}} = \mathbb{E}^x[e^{-\int_0^t (V(B_s)-E_{\rm b}){\rm d}s}(\Psi, {\rm J}_{[0,T]}\Phi_{\rm b}(B_t))_{\mathcal{F}}] \quad a.e. x \in \mathbb{R}^3. \tag{1.2}$$

Let $\Phi_{\rm g}$ be the ground state of H such that $H\Phi_{\rm g} = E_{\rm g}\Phi_{\rm g}$, where $E_{\rm g}$ denotes the infimum of the spectrum of H. We set

$$\ell_\Omega(x) = \mathbb{E}^x[e^{-\int_0^t (V(B_s)-E_{\rm g}){\rm d}s}(\Omega, {\rm J}_{[0,T]}\Phi_{\rm g}(B_t))_{\mathcal{F}}].$$

Lemma 1.5 *Let* $V \in \mathcal{K}_3$*. Then* $\ell_\Omega(x)$ *is continuous in* x *and* $\ell_\Omega(x) > 0$ *for all* $x \in \mathbb{R}^3$*.*

Proof: The continuity is shown in [13, 11] and the positivity in [14, 11]. □

By Lemma 1.5 and (1.2), ℓ_Ω can be regarded as the continuous version of $(\Omega, \Phi_{\rm g}(\cdot))_{\mathcal{F}}$.

2 Pointwise bounds

By using an Agmon metric type argument [1, 5], we can estimate the lower bound of $\|\Phi_{\rm g}(x)\|_{\mathcal{F}}$.

2.1 Geodesic distance by Agmon metric

We introduce a geometric object which we will use to measure the spatial exponential decay of the ground state.

Assumption 2.1 *Suppose that* V *is continuous,* $V(x) \geq \varepsilon$ *for all* $x \in \mathbb{R}^3$ *with some* $\varepsilon > 0$ *and* $\lim_{|x|\to\infty} V(x) = \infty$*.*

Suppose Assumption 2.1. Let us set

$$W = V_{\sup}.$$

W is also continuous and satisfies that $W(x) \geq \varepsilon$ for all $x \in \mathbb{R}^3$ and $\lim_{|x|\to\infty} W(x) = \infty$. We fix $T > 0$. We estimate $\|\Phi_{\rm g}(x)\|_{\mathcal{F}}$ from below in terms of the exponent of an Agmon type metric. We define two C^1-path spaces:

$$\mathcal{C}^* = \{{\rm q} \in C^1([0,T];\mathbb{R}^3) \mid {\rm q}(0) = x, {\rm q}(T) = 0\},$$
$$\mathcal{C} = \{\gamma \in C^1([0,T];\mathbb{R}^3) \mid \gamma(0) = 0, \gamma(T) = x\}.$$

Let

$$\mathscr{A}({\rm q},T) = \int_0^T \left(W({\rm q}(s)) + \frac{1}{2}|\dot{\rm q}(s)|^2 \right) {\rm d}s, \quad {\rm q} \in \mathcal{C}^*,$$
$$\mathscr{L}(\gamma,T) = \int_0^T \sqrt{2W(\gamma(s))}|\dot\gamma(s)|{\rm d}s, \quad \gamma \in \mathcal{C}.$$

In the differential geometry, $\mathscr{A}({\rm q},T)$ describes the energy and $\mathscr{L}(\gamma,T)$ the distance. We set $\gamma^{\rm q}(s) = {\rm q}(T-s)$ for ${\rm q} \in \mathcal{C}^*$ and ${\rm q}^\gamma(s) = \gamma(T-s)$ for $\gamma \in \mathcal{C}$. Then $\gamma^{\rm q} \in \mathcal{C}$, ${\rm q}^\gamma \in \mathcal{C}^*$ and

$$\mathscr{L}(\gamma^{\rm q},T) \leq \mathscr{A}({\rm q},T),$$
$$\mathscr{L}(\gamma,T) \leq \mathscr{A}({\rm q}^\gamma,T)$$

follow for any ${\rm q} \in \mathcal{C}^*$ and $\gamma \in \mathcal{C}$ by the arithmetic and geometric inequality. We are interested in the existence of a minimizer γ^* of the length $\mathscr{L}(\gamma,T)$. This kind of problem is very standard in the differential geometry and it is related to geodesic arcs. W is however not a C^∞-function. Then we shall approximate W by a C^∞-function.

Lemma 2.2 *For $b > 0$ there exists $Y \in C^\infty(\mathbb{R}^3)$ such that*

$$(1-b)W(x) \leq Y(x) \leq (1+b)W(x).$$

Proof: Let $\{f_n\}_n$ be a partition of unity such that $f_n \in C_0^\infty(\mathbb{R}^3)$ and ${\rm supp} f_n \cap K$ is empty for all but finitely many n for any compact set K. Let $\rho \in C_0^\infty(\mathbb{R}^3)$ be such that $\rho(x) \geq 0$ and $\int_{\mathbb{R}^3} \rho(x){\rm d}x = 1$. Let $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon)$. Let K_n be a compact set such that ${\rm supp}\rho_n \subset K_n$. Set $r_n = 2^{-n} b \inf_{x\in K_n} |W(x)|$. Suppose that $0 < \varepsilon_n < 1/n$ so that ${\rm supp}\rho_{\varepsilon_n} * f_n W \subset K_n$ and $\sup_x |(\rho_{\varepsilon_n} * f_n W)(x) - f_n(x)W(x)| \leq r_n$. Then

$$-r_n \leq (\rho_{\varepsilon_n} * f_n W)(x) - f_n(x)W(x) \leq r_n$$

and we have

$$f_n(x)W(x) - r_n \leq (\rho_{\varepsilon_n} * f_n W)(x) \leq f_n(x)W(x) + r_n$$

Define $Y = \sum_{n=1}^\infty (\rho_{\varepsilon_n} * f_n W)(x)$. We have

$$(1-b)W(x) \leq Y(x) \leq (1+b)W(x).$$

□

For $x, y \in \mathbb{R}^3$ we define the geodesic distance for W by

$$\varrho(x,y) = \inf\left\{\mathscr{L}(\gamma, T) \mid \gamma \in \mathcal{C}^1([0,T];\mathbb{R}^3), \gamma(0) = x, \gamma(T) = y\right\}.$$

ϱ defines a metric on $\mathbb{R}^3$. Suppose Assumption 2.1. Let Y be given by Lemma 2.2. Set

$$W_b = \frac{1}{1-b}Y$$

for simplicity. Then $W(x) \leq W_b(x)$. Let

$$\mathscr{L}_b(\gamma, T) = \int_0^T \sqrt{2W_b(\gamma(s))}|\dot{\gamma}(s)|\mathrm{d}s.$$

Then $W_b \geq \varepsilon/(1-b) > 0$, $W_b \in C^\infty(\mathbb{R}^3)$ and $\lim_{|x|\to\infty} W_b(x) = \infty$. Let

$$\varrho_b(x,y) = \inf\left\{\mathscr{L}_b(\gamma, T) \mid \gamma \in C^1([0,T];\mathbb{R}^3), \gamma(0) = x, \gamma(T) = y\right\}.$$

Let $\varrho_b(0, X) = \inf_{x\in X} \varrho_b(0,x)$ be the distance from 0 to X. One point compactification of $\mathbb{R}^3$ is denoted by $\mathbb{R}^3 \cup \{\infty\}$. We define the distance from 0 to $\{\infty\}$ by

$$\varrho_b(0, \{\infty\}) = \sup_{\substack{K\subset\mathbb{R}^3 \\ compact}} \varrho_b(0, \mathbb{R}^3 \setminus K).$$

Lemma 2.3 *ϱ_b is geodesically complete.*

Proof: It is known in [1, Lemma A1.2] that ϱ_b is geodesically complete if and only if $\varrho_b(x, \{\infty\}) = \infty$ for all $x \in \mathbb{R}^3$. By the triangle inequality, $\varrho_b(x, \{\infty\}) = \infty$ for all $x \in \mathbb{R}^3$ if and only if $\varrho_b(y, \{\infty\}) = \infty$ for some $y \in \mathbb{R}^3$. We shall show that $\varrho_b(0, \{\infty\}) = \infty$. Let $N > 0$. Since $\lim_{|x|\to\infty} W_b(x) = \infty$, there exists δ_N such that $W_b(x) > N$ for any $|x| > \delta_N$. We can suppose that $\delta_N - \delta_{N/2} > 1$. Let B_{δ_N} be the closed ball centered at 0 with radius δ_N. We see that

$$\varrho_b(0, \{\infty\}) \geq \varrho_b(0, \mathbb{R}^3 \setminus B_{\delta_N}).$$

There exists $y_\varepsilon \in \mathbb{R}^3 \setminus B_{\delta_N}$ such that

$$\begin{aligned} &\varrho_b(0, \{\infty\}) + \varepsilon \\ &\geq \inf\left\{ \int_0^T \sqrt{2W_b(\gamma^\varepsilon(s))}|\dot{\gamma}^\varepsilon(s)|\mathrm{d}s \,\middle|\, \gamma^\varepsilon \in C^1([0,T];\mathbb{R}^3), \gamma^\varepsilon(0) = x, \gamma^\varepsilon(T) = y_\varepsilon \right\}. \end{aligned}$$

Then there exists $\eta^\varepsilon \in C^1([0,T];\mathbb{R}^3)$ such that $\eta^\varepsilon(0) = x$ and $\eta^\varepsilon(T) = y_\varepsilon$ and

$$\varrho_b(0, \{\infty\}) \geq \int_0^T \sqrt{2W_b(\eta^\varepsilon(s))}|\dot{\eta}^\varepsilon(s)|\mathrm{d}s - 2\varepsilon.$$

Let us define $T_N = \sup\{s \mid |\eta^\varepsilon(s)| \leq \delta_{N/2}\}$. Hence

$$\begin{aligned} \varrho_b(0, \{\infty\}) &\geq \int_{T_N}^T \sqrt{2W_b(\eta^\varepsilon(s))}|\dot{\eta}^\varepsilon(s)|\mathrm{d}s - 2\varepsilon \geq \sqrt{N} \int_{T_N}^T |\dot{\eta}^\varepsilon(s)|\mathrm{d}s - 2\varepsilon \\ &\geq \sqrt{N}(\delta_N - \delta_{N/2}) - 2\varepsilon \geq \sqrt{N} - 2\varepsilon. \end{aligned}$$

Since N is arbitrary, the lemma is proven. □

Lemma 2.4 *There exists a minimizer* $\gamma^* \in C^\infty([0,T];\mathbb{R}^3)$ *of* $\mathscr{L}_b(\gamma)$.

Proof: The geodesic completeness proved in Lemma 2.3 implies that there exists a length minimizing geodesic connecting any two points by Hopf-Rinow theorem [12, Theorem 6.13 and Corollary 6.15]. Then the minimizer γ^* exists. $\square$

Let $g(x) = (g_{ij}(x))_{1\le i,j\le 3}$ be

$$g_{ij}(x) = W_b(x)\delta_{ij}, \quad 1 \le i,j \le 3,\ x \in \mathbb{R}^3.$$

Let $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ and $\dot\gamma_i(s) = \frac{d\gamma_i}{ds}(s)$ for $1 \le i \le 3$. Then

$$\int_0^T \sqrt{W_b(\gamma(s))}|\dot\gamma(s)|\mathrm{d}s = \int_0^T \sqrt{\sum_{i,j=1}^3 g_{ij}(\gamma(s))\dot\gamma_i(s)\dot\gamma_j(s)}\mathrm{d}s. \tag{2.1}$$

The minimizer γ^* in Lemma 2.4 satisfies the geodesic equation:

$$\ddot\gamma_k + \sum_{i,j=1}^3 \Gamma^k_{ij}(\gamma)\dot\gamma_i\dot\gamma_j = 0, \quad 1 \le k \le 3. \tag{2.2}$$

Here Γ^k_{ij} denotes the Christoffel symbol given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^3 g^{kl}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right),$$

where $g^{kl}(x) = (g^{-1}(x))_{kl} = \frac{1}{W_b(x)}\delta_{kl}$. Since $W_b(x) \ge \varepsilon/(1-b)$, g^{-1} exists for any $x \in \mathbb{R}^3$. Note that $\Gamma^k_{ij} = \Gamma^k_{ji}$ by symmetries.

Lemma 2.5 *Let* γ^* *be the minimizer of* $\mathscr{L}_b(\gamma)$. *Then* $|\dot\gamma^*(s)| \ne 0$ *for* $s \in [0,T]$.

Proof: Let $W_k(x) = \frac{\partial W_b}{\partial x_k}(x)$ for $1 \le k \le 3$. It follows that

$$\Gamma^k_{nn} = \begin{cases} -\dfrac{1}{2}\dfrac{W_k}{W_b}, & k \ne n, \\[2ex] \dfrac{1}{2}\dfrac{W_k}{W_b}, & k = n. \end{cases}$$

On the other hand when k, n and m are different from each other, $\Gamma^k_{nm} = 0$. Moreover we see that

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}\frac{W_2}{W_b}, \quad \Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2}\frac{W_3}{W_b},$$
$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2}\frac{W_3}{W_b}, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2}\frac{W_1}{W_b},$$
$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}\frac{W_1}{W_b}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}\frac{W_2}{W_b}.$$

Together with them, (2.2) is written as

$$\begin{cases} W_b(\gamma)\ddot{\gamma}_1 + W_1(\gamma)|\dot{\gamma}_1|^2 - \frac{1}{2}W_1(\gamma)|\dot{\gamma}|^2 + W_2(\gamma)\dot{\gamma}_1\dot{\gamma}_2 + W_3(\gamma)\dot{\gamma}_1\dot{\gamma}_3 = 0, \\ W_b(\gamma)\ddot{\gamma}_2 + W_2(\gamma)|\dot{\gamma}_2|^2 + W_1(\gamma)\dot{\gamma}_1\dot{\gamma}_2 - \frac{1}{2}W_2(\gamma)|\dot{\gamma}|^2 + W_3(\gamma)\dot{\gamma}_2\dot{\gamma}_3 = 0, \\ W_b(\gamma)\ddot{\gamma}_3 + W_3(\gamma)|\dot{\gamma}_3|^2 + W_1(\gamma)\dot{\gamma}_1\dot{\gamma}_3 + W_2(\gamma)\dot{\gamma}_2\dot{\gamma}_3 - \frac{1}{2}W_3(\gamma)|\dot{\gamma}|^2 = 0. \end{cases} \tag{2.3}$$

From (2.3) it follows that

$$\frac{d}{ds}W_b(\gamma)|\dot{\gamma}|^2 = \sum_{k=1}^{3}\left(W_k(\gamma)|\dot{\gamma}|^2 + 2W_b(\gamma)\ddot{\gamma}_k\right)\dot{\gamma}_k = 0,$$

which implies that $W_b(\gamma(s))|\dot{\gamma}(s)|^2$ is a nonzero constant. In particular $|\dot{\gamma}(s)| \neq 0$ for $0 \leq s \leq T$ since $W_b(\gamma(s)) > 0$. □

For a fixed $T > 0$ we define

$$\mathscr{A}(\mathrm{q}, S) = \int_0^S \left(W_b(\mathrm{q}(s)) + \frac{1}{2}|\dot{\mathrm{q}}(s)|^2\right)\mathrm{d}s.$$

We shall connect two minima:

$$\inf\left\{\mathscr{L}_b(\gamma, T) \mid \gamma(0) = 0, \gamma(T) = 0, \gamma \in C^\infty([0,T];\mathbb{R}^3)\right\},$$
$$\inf\{\mathscr{A}_b(\mathrm{q}, S) \mid S > 0, \mathrm{q}(0) = x, \mathrm{q}(s) = 0, \mathrm{q} \in C^\infty([0,S];\mathbb{R}^3)\}.$$

$\mathscr{L}_b(\gamma, T) = \int_0^T \sqrt{2W_b(\gamma(s))}|\dot{\gamma}(s)|\mathrm{d}s$ is invariant under re-parametrization: $\gamma \to \gamma \circ \phi$ by any smooth bijection $\phi : [0,T] \to [0,T]$. On the other hand in general $\mathscr{A}_b(\mathrm{q}, S)$ is not invariant. From this property one can construct a bijection ϕ such that

$$\sqrt{2W_b(\gamma \circ \phi(s))}|\gamma \dot{\circ} \phi(s)| = W_b(\gamma \circ \phi(s)) + \frac{1}{2}|\gamma \dot{\circ} \phi(s)|^2.$$

Then we have the lemma.

Lemma 2.6 *Suppose Assumption 2.1. Let* $b > 0$ *and* Y *be given by Lemma 2.2. Then there exists minimizer* $(\mathrm{q}^*, S^*) \in C^\infty([0,T];\mathbb{R}^3) \times (0,\infty)$ *of* $\mathscr{A}_b(\mathrm{q}, S)$ *and it holds that* $\mathscr{L}_b(\gamma^*, T) = \mathscr{A}_b(\mathrm{q}^*, S^*)$. *Moreover let* $\gamma_*(s) = \mathrm{q}^*(S^* - s)$. *Then*

$$\mathscr{L}_b(\gamma^*, T) = \int_0^{S^*} \sqrt{2W_b(\gamma_*(s))}|\dot{\gamma}_*(s)|\mathrm{d}s = \mathscr{A}_b(\mathrm{q}^*, S^*). \tag{2.4}$$

Proof: Note that $\mathscr{L}_b(\gamma, T)$ is invariant under reparametrization of path γ. Thus for any bijective C^1 function $\phi : [0,T] \to [0,T]$, by $\gamma_\phi(s) = \gamma(\phi(s))$, $\mathscr{L}_b(\gamma, T) = \mathscr{L}_b(\gamma_\phi, T)$. Let γ^* be the minimizer of $\mathscr{L}_b(\gamma, T)$. Then

$$\mathscr{L}_b(\gamma^*, T) = \int_0^1 \sqrt{2W_b(\hat{\gamma}(s))}|\dot{\hat{\gamma}}(s)|\mathrm{d}s,$$

where $\hat{\gamma}(s) = \gamma^*(s/T)$. Since $2\sqrt{ab} = a + b$ if and only if $a = b$, it follows that

$$\sqrt{2W_b(\gamma(s))} = |\dot{\gamma}(s)|, \quad s \in [0, 1],$$

if and only if

$$\int_0^1 \sqrt{2W_b(\gamma(s))}|\dot{\gamma}(s)|\mathrm{d}s = \int_0^1 \left(W_b(\gamma(s)) + \frac{1}{2}|\dot{\gamma}(s)|^2 \right) \mathrm{d}s.$$

Since $W_b \in C^\infty(\mathbb{R}^3)$, $W_b(x) > 0$ for $x \in \mathbb{R}^3$, $\hat{\gamma} \in C^\infty([0, 1])$ and $|\dot{\hat{\gamma}}(t)| > 0$ for $0 \leq t \leq 1$, we see that

$$t \mapsto F(t) = \frac{\sqrt{2W_b(\hat{\gamma}(t))}}{|\dot{\hat{\gamma}}(t)|}, \quad t \in [0, 1]$$

is Lipschitz continuous. Hence there exists the solution ϕ to the ordinary differential equation

$$\dot{\phi}(s) = F(\phi(s)), \quad s \in [0, S^*].$$

Here $\phi : [0, S^*] \to [0, 1]$ is bijective with $\phi(S^*) = 1$ and $\dot{\phi}(s) > 0$ for $s \in [0, S^*]$. Let $\hat{\gamma}_\phi = \hat{\gamma} \circ \phi$. Then $|\dot{\hat{\gamma}}_\phi(s)| = |\dot{\hat{\gamma}}(s)|\dot{\phi}(s)$. We then see that $\hat{\gamma}_\phi \in \mathcal{C}$ satisfies that

$$\sqrt{2W_b(\hat{\gamma}_\phi(s))} = |\dot{\hat{\gamma}}_\phi(s)|, \quad s \in [0, S^*].$$

This implies that

$$\begin{aligned}
\mathscr{L}_b(\gamma^*, T) &= \int_0^1 \sqrt{2W_b(\hat{\gamma}(s))}|\dot{\hat{\gamma}}(s)|\mathrm{d}s \\
&= \int_0^{S^*} \sqrt{2W_b(\hat{\gamma}_\phi(s))}|\dot{\hat{\gamma}}_\phi(s)|\mathrm{d}s \\
&= \int_0^{S^*} \left(W_b(\hat{\gamma}_\phi(s)) + \frac{1}{2}|\dot{\hat{\gamma}}_\phi(s)|^2 \right) \mathrm{d}s \\
&= \int_0^{S^*} \left(W_b(\mathrm{q}^{\hat{\gamma}_\phi}(s)) + \frac{1}{2}|\dot{\mathrm{q}}^{\hat{\gamma}_\phi}(s)|^2 \right) \mathrm{d}s \\
&= \mathscr{A}_b(\mathrm{q}^{\hat{\gamma}_\phi}, S^*).
\end{aligned}$$

For any $\mathrm{q} \in \mathcal{C}$ and $S > 0$, we have

$$\begin{aligned}
\mathscr{A}_b(\mathrm{q}, S) &\geq \int_0^S \sqrt{2W_b(\gamma(s))}|\dot{\gamma}(s)|\mathrm{d}s \\
&\geq \mathscr{L}_b(\gamma^*, T) = \mathscr{A}_b(\mathrm{q}^{\hat{\gamma}_\phi}, S^*).
\end{aligned}$$

This implies that $\mathscr{A}_b(\mathrm{q}^{\hat{\gamma}_\phi}, S^*)$ is a minimizer. We set $\mathrm{q}^* = \mathrm{q}^{\hat{\gamma}_\phi}$ and reset $\gamma_* = \hat{\gamma}_\phi$. Then $\mathscr{L}_b(\gamma^*, T) = \mathscr{A}_b(\mathrm{q}^*, S^*)$ and (2.4) hold true. □

2.2 Exponential decay

Let $K \subset \mathbb{R}^3$ be a compact set. Since $\ell_\Omega(\cdot)$ is continuous and strictly positive on K, we can set $\chi_K = \inf_{y \in K} \ell_\Omega(y) > 0$. Let $\Phi_\infty = \sup\limits_{y \in \mathbb{R}^3} \|\Phi_{\mathrm{g}}(y)\|_{\mathcal{F}}$.

Lemma 2.7 *Let $T > 0$. Then there exists $\tau > 0$ such that for any $\mathrm{q} \in \mathcal{C}^*$,*

$$\ell_\Omega(x) \geq \chi_K e^{-\int_0^T \left(V_{\sup}(\mathrm{q}(s)) + \frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2\right) \mathrm{d}s} e^{-T\tau} e^{T(E_{\mathrm{g}} - \frac{\|\hat{\varphi}/\omega\| \Phi_\infty}{\sqrt{2}\chi_K})}. \tag{2.5}$$

Proof: By Jensen's inequality, we have

$$\begin{aligned} \ell_\Omega(x) &= \mathbb{E}\left[e^{-\int_0^T (V(B_s+x) - E_{\mathrm{g}}) \mathrm{d}s} (\Omega, e^{-\phi_{\mathcal{E}}(\int_0^T \mathrm{j}_s \tilde{\varphi}(\cdot - B_s) \mathrm{d}s)} \mathrm{J}_t \Phi_{\mathrm{g}}(B_T + x))_{\mathcal{F}}\right] \\ &\geq \mathbb{E}\left[e^{-\int_0^T (V(B_s+x) - E_{\mathrm{g}}) \mathrm{d}s} \ell_\Omega(B_T + x) e^{-\frac{\|\hat{\varphi}/\omega\| \Phi_\infty}{\sqrt{2}\ell_\Omega(B_T + x)}}\right]. \end{aligned}$$

Let $\mathrm{q} \in \mathcal{C}^*$ and we define ξ by

$$\xi = e^{-\int_0^T \dot{\mathrm{q}}(s) \cdot \mathrm{d}B_s - \frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2 \mathrm{d}s}.$$

Thus $\mathbb{E}[\xi] = 1$. By the Girsanov theorem, we see that

$$\mathbb{E}\left[e^{-\int_0^T (V(B_s+x) - E_{\mathrm{g}}) \mathrm{d}s} \ell_\Omega(B_T + x) e^{-\frac{\|\hat{\varphi}/\omega\| \Phi_\infty}{\sqrt{2}\ell_\Omega(B_T + x)}}\right] = \mathbb{E}\left[\xi e^{-\int_0^T (V(B_s + \mathrm{q}(s)) - E_{\mathrm{g}}) \mathrm{d}s} \ell_\Omega(B_T) e^{-T\frac{\|\hat{\varphi}/\omega\| \Phi_\infty}{\sqrt{2}\ell_\Omega(B_T)}}\right].$$

Let $M = \{|B_s| \leq 1, 0 \leq s \leq T\}$ and K be the unit closed ball. Thus $\ell_\Omega(B_T) \geq \chi_K$ on M and

$$\ell_\Omega(x) \geq \chi_K e^{-\int_0^T V_{\sup}(\mathrm{q}(s)) \mathrm{d}s} e^{T(E_{\mathrm{g}} - \frac{\|\hat{\varphi}/\omega\| \Phi_\infty}{\sqrt{2}\chi_K})} \mathbb{E}\left[\xi \mathbb{1}_M\right].$$

By Jensen's inequality again, we have

$$\mathbb{E}\left[\xi \mathbb{1}_M\right] \geq \chi_K e^{-\frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2 \mathrm{d}s} \mathbb{E}[\mathbb{1}_M] e^{\frac{\mathbb{E}\left[\mathbb{1}_M\left(-\int_0^T \dot{\mathrm{q}}(s) \cdot \mathrm{d}B_s\right)\right]}{\mathbb{E}[\mathbb{1}_M]}} = \chi_K e^{-\frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2 \mathrm{d}s} \mathbb{E}[\mathbb{1}_M].$$

Note that $\mathbb{E}[\mathbb{1}_M] \geq e^{-T\tau}$ with some $\tau > 0$ [5], where τ is the infimum of the spectrum of $-\Delta/2$ on the unit ball with Dirichlet boundary condition. Thus (2.5) follows. □

Theorem 2.8 ([11]) *Let $\gamma \in \mathcal{C}$ and $\varepsilon > 0$. Then there exists $R > 0$ such that*

$$\chi_K e^{-(1+\varepsilon)\int_0^T \sqrt{2V_{\sup}(\gamma(s))}|\dot{\gamma}(s)| \mathrm{d}s} \leq \|\Phi_{\mathrm{g}}(x)\|_{\mathcal{F}}, \quad |x| \geq R.$$

Proof: By Lemma 2.6, there exists minimizer $(\mathrm{q}^*, S^*) \in C^\infty([0,T];\mathbb{R}^3) \times (0,\infty)$ of $\mathscr{A}_b(\mathrm{q}, S)$ and $\gamma^* \in C^\infty([0,T];\mathbb{R}^3)$ of $\mathscr{L}_b(\gamma, T)$. It also holds that $\mathscr{L}_b(\gamma^*, T) = \mathscr{A}_b(\mathrm{q}^*, S^*)$. It can be shown in Lemma 2.10 below that there exists $R_{\delta,b}$ such that

$$S^* \leq \delta \int_0^{S^*} \left(W_b(\mathrm{q}^*(s)) + \frac{1}{2}\int_0^T |\dot{\mathrm{q}}^*(s)|^2\right) \mathrm{d}s \tag{2.6}$$

for $|x| \geq R_{\delta,b}$. Hence putting $T = S^*$ and $q = q^*$ in Lemma 2.7, we have for $|x| \geq R_{\delta,b}$,

$$\begin{aligned}\ell_\Omega(x) &\geq \chi_K \exp\left\{-(1+\delta)\mathscr{A}_b(\mathrm{q}^*, S^*)\right\} = \chi_K \exp\left\{-(1+\delta)\mathscr{L}_b(\gamma^*, T)\right\} \\ &\geq \chi_K \exp\left\{-(1+\delta)\sqrt{\frac{1+b}{1-b}}\mathscr{L}(\gamma, T)\right\}.\end{aligned}$$

Choose δ and b such that $1+\varepsilon = (1+\delta)\sqrt{\frac{1+b}{1-b}}$. Then the theorem follows. □

Corollary 2.9 ([11]) *(1) Suppose Assumption 2.1. Let* $\varepsilon > 0$*. Then there exists* R *such that*

$$\chi_K e^{-(1+\varepsilon)|x|\int_0^T \sqrt{2V_{\mathrm{sup}}(sx)}\mathrm{d}s} \leq \|\Phi_{\mathrm{b}}(x)\|_{\mathcal{F}}, \quad |x| \geq R.$$

(2) Assume that V *obeys the lower bound*

$$V(x) \geq \frac{a^2}{2}|x|^{2n} - b, \quad |x| \geq R$$

for some $a, b, n > 0$ *and* $R > 0$ *so that* $a^2R^{2n}/2 - b > 0$*. Then for all* $\varepsilon > 0$*, there exist* $c_\varepsilon > 0$ *and* $C_\varepsilon > 0$ *such that*

$$C_\varepsilon e^{-(1+\varepsilon)\frac{a}{n+1}|x|^{n+1}} \leq \|\Phi_{\mathrm{b}}(x)\|_{\mathcal{F}} \leq c_\varepsilon e^{-(1-\varepsilon)\frac{a}{n+1}|x|^{n+1}}.$$

Proof: (1) Put $\gamma(s) = sx$ in Theorem 2.8. Then (1) follows. (2) The lower bound follows from (1) and the upper bound from [10]. □

Lemma 2.10 (2.6) *holds true.*

Proof: The proof of (2.6) below is a minor modification of that of [5, Proposition 2.5 (iii)]. Note that $(1-b)W \leq Y \leq (1+b)W$ with $W = V_{\mathrm{sup}}$, $Y \in C^\infty(\mathbb{R}^3)$ and $W_b = Y/(1-b)$. Let $\rho(x) = \inf\{\mathscr{A}_b(\mathrm{q}, S) \mid S > 0, \mathrm{q}(0) = x, \mathrm{q}(s) = 0, \mathrm{q} \in C^\infty([0, S]; \mathbb{R}^3)\}$. Then we have

$$\rho(x) = \int_0^{S^*} \left(W_b(\mathrm{q}^*(s)) + \frac{1}{2}|\dot{\mathrm{q}}^*(s)|^2\right) \mathrm{d}s.$$

Note that $W_b \geq V_{\mathrm{sup}}$. Since $\sqrt{2W_b(\mathrm{q}^*(s))} = |\dot{\mathrm{q}}^*(s)|$, we see that $\rho(x) = 2\int_0^{S^*} W_b(\mathrm{q}^*(s))\mathrm{d}s$. Then

$$\rho(x) \geq 2\delta S^*, \tag{2.7}$$

where $\delta = \inf_x V_{\mathrm{sup}}(x)$. Let $a(R) = \min_{|x|\geq R} V_{\mathrm{sup}}(x)$ and s_R be the first time that $|\mathrm{q}^*(s_R)| = R$. Let $\mathrm{q}^*(s_R) = z$ and we set $\eta(s) = \mathrm{q}^*(s + s_R)$, which satisfies that $\sqrt{2W_b(\eta(s))} = |\dot{\eta}(s)|$ for $s \in [0, S^* - s_R]$. Then the minimum $\rho(z) = \inf\{\mathscr{A}_b(\mathrm{q}, S) \mid S > 0, \mathrm{q}(0) = z, \mathrm{q}(s) = 0, \mathrm{q} \in C^\infty([0, S]; \mathbb{R}^3)\}$ is given by

$$\rho(z) = \int_0^{S^*-s_R} \left(V_{\mathrm{sup}}(\eta(s)) + \frac{1}{2}\int_0^T |\dot{\eta}(s)|^2\right) \mathrm{d}s.$$

In a similar manner to (2.7) we have

$$\sup_{|y|=R} \rho(y) \geq \rho(z) \geq 2\delta(S^* - s_R).$$

Since $|\mathrm{q}^*(s)| \geq R$ for $s \in [0, s_R]$, we have $\rho(x) \geq 2s_R a(R)$. Thus

$$S^* \leq s_R + \frac{1}{2\delta} \sup_{|y|=R} \rho(y) \leq \frac{1}{2a(R)} \rho(x) + \frac{1}{2\delta} \sup_{|y|=R} \rho(y)$$

and

$$\frac{S^*}{\rho(x)} \leq \frac{1}{2a(R)} + \frac{1}{2\delta} \frac{\sup_{|y|=R} \rho(y)}{\rho(x)}$$

Let R be sufficiently large such that $1/2a(R) < \varepsilon$. Then take $|x| \to \infty$. We have

$$\lim_{|x|\to\infty} \frac{S^*}{\rho(x)} \leq \varepsilon.$$

Then (2.6) follows. □

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References

[1] S. Agmon. *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations:Bound on Eigenfunctions of N-body Schrödinger Operators.* Princeton University Press, 1982.

[2] M. Aizenman and B. Simon. Brownian motion and Harnack's inequality for Schrödinger operators. *Commun. Pure Appl. Math.*, 35:209–271, 1982.

[3] R. Carmona. Pointwise bounds for Schrödinger eigenstates. *Commun. Math. Phys.*, 62:97–106, 1978.

[4] R. Carmona, W. C. Masters, and B. Simon. Relativistic Schrödinger operators: Asymptotic behavior of the eigenfunctions. *J. Funct. Anal.*, 91:117–142, 1990.

[5] R. Carmona and B. Simon. Pointwise bounds on eigenfunctions and wave packets in N-body quantum system. *Commun. Math. Phys.*, 80:59–98, 1981.

[6] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators. With Application to Quantum Mechanics and Global Geometry.* Springer, 1987.

[7] C. Gérard. On the existence of ground states for massless Pauli-Fierz Hamiltonians. *Ann. H. Poincaré*, 1:443–459, 2000.

[8] M. Griesemer, E. Lieb, and M. Loss. Ground states in non-relativistic quantum electrodynamics. *Invent. Math.*, 145:557–595, 2001.

[9] M. Gubinelli, F. Hiroshima, and J. Lörinczi. Ultraviolet renormalization of the Nelson Hamiltonian through functional integration. *J. Funct. Anal.*, 267:3125–3153, 2014.

[10] F. Hiroshima and O. Matte. Ground states and their associated Gibbs measures in the renormalized Nelson model. *Rev. Math.Phys.*, 33:2250002 (84 pages), 2021.

[11] F. Hiroshima and O. Matte. Two-sided bounds on the point-wise spatial decay of ground states in the renormalized Nelson model with confining potentials. *arXiv:2501.10704*, preprint, 2024.

[12] J.M. Lee. *Riemann manifolds.* Springer, 1997.

[13] O. Matte. Continuity properties of the semi-group and its integral kernel in non-relativistic qed. *Rev. Math. Phys.*, 28:1650011, 90 pp., 2016.

[14] O. Matte and J. Møller. Feynman-Kac formulas for the ultra-violet renormalized Nelson model. *Astérisque*, 404:vi+110, 2018.

[15] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.*, 5:1990–1997, 1964.

[16] B. Simon. *Functional Integrations and Quantum Physics.* Academic Press, 1979, 2nd ed. 2005.

[17] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc.*, 7:447–526, 1982.

[18] H. Spohn. Ground state of a quantum particle coupled to a scalar boson field. *Lett. Math. Phys.*, 44:9–16, 1998.