A remark on Feynman propagators in Dereziński-Siemssen's paper

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1 Introduction

The purpose of this note is to rewrite a construction of the Feynman propagator from Dereziński-Siemssen's paper [2] in a simplified and more explicit manner.

The Feynman propagators play an important role in interacting QFTs on Minkowski spacetimes. It is non-trivial to generalize them to curved spacetimes. Due to the work of Duistermaat-Hörmander [1], they can be defined by the inverse of the Klein-Gordon operator such that a wavefront condition is satisfied (see for example, [3, Definition 1.1]).

However, the Feynman propagators are not always unique (although they are uniquely defined modulo smoothing operators) so it is crucial to understand which ones can be identified. On perturbations of Minkowski spacetimes, Gérard-Wrochna ([3] [4]) constructed Feynman propagators in the massive case using the scattering theory and appropriate function spaces. In [5], the author showed that the (anti-) Feynman propagators in Gérard-Wrochna are equal to the outgoing resolvents of the Klein-Gordon operators, which also can be identified with the ones constructed by Vasy [6]. On the other hand, in an earlier work of Dereziński-Siemssen [2], the Feynman propagators are constructed and are identified with the outgoing resolvents but in static spacetimes. Although their proof is very simple, the function spaces and operators used there seem not explicit (at least for the author). In this paper, we follow the argument in [2] on ultra-static spacetimes and provide a more explicit proof. Finally, we obtain an explicit formula for the Feynman propagator (see Theorem 2.10).

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2 Feynman propagators in Dereziński-Siemssen's paper

2.1 Assumptions on spacetimes

Let (M, g) be an ultra-static spacetime, that is, there exists a Riemannian manifold (Σ, g_Σ) such that

$$M = \mathbb{R}_t \times \Sigma, \quad g = -dt^2 + g_\Sigma.$$

Then, the corresponding Klein-Gordon operator is given by

$$\Box_g = -\partial_t^2 + \Delta_\Sigma,$$

where Δ_Σ is the Laplace-Beltrami operator on (Σ, g_Σ). We define

$$K := -\Box_g + m^2 = \partial_t^2 - \Delta_\Sigma + m^2, \quad L := -\Delta_\Sigma + m^2$$

as in [2, §2], where the mass term $m > 0$ is a positive constant. Here we note that the massless case $m = 0$ is not dealt with [2] due to [2, Assumption 3.3].

We write $L^2(M)$ (resp. $L^2(\Sigma)$) as the L^2-space on M (resp. Σ) induced from the metric g (resp. g_Σ). Then the operator K with a domain $C_c^\infty(M)$ (resp. the operator L with a domain $C_c^\infty(\Sigma)$) is symmetric on $L^2(M)$ (resp. $L^2(\Sigma)$).

According to [2, Assumption 2.3], we assume the following :

Assumption A. L is essentially self-adjoint on $C_c^\infty(\Sigma)$ with respect to $L^2(\Sigma)$. We denote its closure by the same symbol L.

Remark 2.1. This assumption is satisfied if the Riemannian manifold (Σ, g_Σ) is geodesically complete (for example, Σ is compact and boundaryless).

2.2 Reduction to a first order system

Now we rewrite the Klein-Gordon equation to a first order system as usual. Set

$$\pi_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := u_2, \quad \iota_2 u := \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad \rho u := \begin{pmatrix} u \\ -D_t u \end{pmatrix} \tag{2.1}$$

and

$$B := \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad H = QB = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix} \tag{2.2}$$

as in [2, §3].

Then we have the following:

Lemma 2.2.

$$K = \pi_2(D_t + iB)\rho.$$

In particular, we have

$$Ku = f \Leftrightarrow (D_t + B)(\rho u) = \iota_2 f.$$

Remark 2.3. In [2, §4], it is stated that $K = -\pi_2(D_t + iB)\rho$ (where we note that $\beta = 1$ in our case). But this seems a mistake.

Proof. We note $K = -D_t^2 + L$. We calculate as

$$D_t(\rho u) = \begin{pmatrix} D_t u \\ -D_t^2 u \end{pmatrix} = \begin{pmatrix} D_t u \\ -Lu \end{pmatrix} + \begin{pmatrix} 0 \\ Ku \end{pmatrix} = -\begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix} \begin{pmatrix} u \\ -D_t u \end{pmatrix} + \begin{pmatrix} 0 \\ Ku \end{pmatrix}$$

and hence $(D_t + B)(\rho u) = \iota_2(Ku)$. This proves the lemma.

□

2.3 The energy space and the spectral projections

Since L is positive definite as a self-adjoint operator on $L^2(\Sigma)$ (due to the assumption $m > 0$), we can consider the form domain $\mathcal{H}_{en}$ of H with the metric

$$(\mathbf{u}, \mathbf{v})_{en} := (\mathbf{u}, H\mathbf{v})_{L^2(\Sigma)\oplus L^2(\Sigma)} \quad \text{for} \quad \mathbf{u}, \mathbf{v} \in \mathcal{H}_{en}.$$

In our case, the energy space $\mathcal{H}_{en}$ is just

$$\mathcal{H}_{en} = H^1(\Sigma) \oplus L^2(\Sigma)$$

as a set, where $H^1(\Sigma)$ is the Sobolev space: $H^1(\Sigma) := \{u \in L^2(\Sigma) \mid \mathrm{grad}(u) \in L^2(\Sigma; T\Sigma)\}$.

Lemma 2.4. *[2, Proposition 3.7] The operator B (defined in* (2.2)*) is essentially self-adjoint on $C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$ with respect to $\mathcal{H}_{en}$.*

Remark 2.5. In our case, the domain of the unique self-adjoint extension is given by $H^2(\Sigma) \oplus L^2(\Sigma)$, where $H^2(\Sigma) = \{u \in L^2(\Sigma) \mid \Delta_\Sigma u \in L^2(\Sigma)\}$. This essentially follows from Assumption A.

We denote the self-adjoint extension of B by the same symbol. As in [2, §6], we define the spectral projectors $\Pi^\pm$ (acting on $\mathcal{H}_{en}$) onto the positive and negative part of the spectrum of B. We want to write them in an explicit way.

Lemma 2.6.

$$\Pi^\pm = \frac{1}{2}\begin{pmatrix} I & \pm L^{-\frac{1}{2}} \\ \pm L^{\frac{1}{2}} & I \end{pmatrix},$$

where $L^{\frac{1}{2}} = (-\Delta_\Sigma + m^2)^{\frac{1}{2}}, L^{-\frac{1}{2}} = (-\Delta_\Sigma + m^2)^{-\frac{1}{2}}$ are defined via the spectral theorem for the self-adjoint operator $-\Delta_\Sigma$.

Remark 2.7. This operator is same as $c_{free}^{\pm,vac}$ in [5, §4.1] for the Minkowski spacetime. This operator may appear in [3, (1.4)], but they write $c_{free}^{\pm,vac} = \frac{1}{2}\begin{pmatrix} I & \pm L^{\frac{1}{2}} \\ \pm L^{\frac{1}{2}} & I \end{pmatrix}$ (the $(1,2)$-element is different from the above). The author believes that this is just a typo (actually, the operator [3, (1.4)] is not a projection).

Proof. We consider a unitary operator $U : \mathcal{H}_{en} \to L^2(\Sigma) \oplus L^2(\Sigma)$ defined by

$$U = \begin{pmatrix} L^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix}.$$

We write $\tilde{B} := UBU^{-1}$, which is a self-adjoint operator on $L^2(\Sigma) \oplus L^2(\Sigma)$ and denote its spectrum projections onto the positive and negative part of the spectrum by $\tilde{\Pi}^{\pm}$. Since U is unitary, we have

$$\tilde{\Pi}^{\pm} = U\Pi^{\pm}U^{-1}. \tag{2.3}$$

By the definitions of B and U, we have

$$\tilde{B} = \begin{pmatrix} L^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix} \begin{pmatrix} L^{-\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & L^{\frac{1}{2}} \\ L^{\frac{1}{2}} & 0 \end{pmatrix} = L^{\frac{1}{2}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Since the projections of the 2×2-matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the positive and negative part of the spectrum is given by $\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$, we have

$$\tilde{\Pi}^{\pm} = \frac{1}{2}\begin{pmatrix} I & \pm I \\ \pm I & I \end{pmatrix}.$$

Hence

$$\Pi^{\pm} = U^{-1}\tilde{\Pi}^{\pm}U = \frac{1}{2}\begin{pmatrix} I & \pm L^{-\frac{1}{2}} \\ \pm L^{\frac{1}{2}} & I \end{pmatrix}$$

as desired.

□

2.4 Feynman propagators

Following [2, §6], we introduce the Feynman propagator of the first order operator $\partial_t + iB$ as

$$(E^F\mathbf{f})(t,y) := \int_{-\infty}^{t} e^{-i(t-s)B}\Pi^{+}\mathbf{f}(s,y)ds - \int_{t}^{\infty} e^{-i(t-s)B}\Pi^{-}\mathbf{f}(s,x)ds, \quad (t,y) \in M = \mathbb{R} \times \Sigma$$

for $\mathbf{f} \in \langle t \rangle^{-r} L^2(\mathbb{R}; \mathcal{H}_{en})$ with $r > \frac{1}{2}$. We can easily to see that

$$(\partial_t + iB)(E^F\mathbf{f})(t,y) = \mathbf{f}(t,y)$$

due to the relationship $\Pi^+ + \Pi^- = I$.

Lemma 2.8. *We have*

$$e^{-itB}\Pi^{\pm} = e^{\mp itL^{\frac{1}{2}}}\Pi^{\pm} \quad \textit{for} \quad t \in \mathbb{R}.$$

Remark 2.9. The operator e^{-itB} is matrix-valued but $e^{-itL^{\frac{1}{2}}}$ is scalar-valued.

Proof. We recall from the proof of Lemma 2.6 that

$$UBU^{-1} = \tilde{B} := \begin{pmatrix} 0 & L^{\frac{1}{2}} \\ L^{\frac{1}{2}} & 0 \end{pmatrix}, \quad U\Pi^{\pm}U^{-1} = \tilde{\Pi}^{\pm} := \frac{1}{2}\begin{pmatrix} I & \pm I \\ \pm I & I \end{pmatrix}.$$

Since

$$e^{-it\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix},$$

we have

$$e^{-it\tilde{B}} = \begin{pmatrix} \cos(tL^{\frac{1}{2}}) & -i\sin(tL^{\frac{1}{2}}) \\ -i\sin(tL^{\frac{1}{2}}) & \cos(tL^{\frac{1}{2}}) \end{pmatrix}$$

and hence

$$Ue^{-i(t-s)B}\Pi^{\pm}U^{-1} = e^{-i(t-s)\tilde{B}}\tilde{\Pi}^{\pm} = e^{\mp i(t-s)L^{\frac{1}{2}}}\tilde{\Pi}^{\pm}.$$

Using the fact that U commutes with L, we obtain

$$e^{-itB}\Pi^{\pm} = U^{-1}e^{\mp i(t-s)L^{\frac{1}{2}}}\tilde{\Pi}^{\pm}U = e^{\mp i(t-s)L^{\frac{1}{2}}}U^{-1}\tilde{\Pi}^{\pm}U = e^{\mp itL^{\frac{1}{2}}}\Pi^{\pm}.$$

□

We define the Feynman propagator of K by

$$G^F := i\pi_2 QE^F\iota_2 \tag{2.4}$$

following [2, (4.5)], where π_2 and ρ are defined in (2.1) and Q is defined in (2.2). Here we note that G^F is defined by $G^F = -i\pi_2 QE^F\iota_2$ in [2, (4.5)] buy its signature may be mistaken.

Now we deduce an explicit expression the Feynaman propagator G^F.

Theorem 2.10.

$$G^F f(t,y) = \frac{i}{2}L^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-i|t-s|L^{\frac{1}{2}}}f(s,y)ds.$$

for $f \in \langle t\rangle^{-r}L^2(M) = \langle t\rangle^{-r}L^2(\mathbb{R};L^2(\Sigma))$ with $r > \frac{1}{2}$.

Proof. By Lemmas 2.6 and 2.8, we have

$$\begin{aligned}\pi_2 Qe^{-i(t-s)B}\Pi^{\pm}\iota_2(f(s,\cdot)) =& \pi_2 Qe^{\mp i(t-s)L^{\frac{1}{2}}}\Pi^{\pm}\begin{pmatrix} 0 \\ f(s,\cdot) \end{pmatrix} = \frac{1}{2}e^{\mp i(t-s)L^{\frac{1}{2}}}\pi_2 Q\begin{pmatrix} \pm L^{-\frac{1}{2}}f(s,\cdot) \\ f(s,\cdot) \end{pmatrix} \\ =& \frac{1}{2}e^{\mp i(t-s)L^{\frac{1}{2}}}\pi_2\begin{pmatrix} f(s,\cdot) \\ \pm L^{-\frac{1}{2}}f(s,\cdot) \end{pmatrix} = \pm\frac{1}{2}L^{-\frac{1}{2}}e^{\mp i(t-s)L^{\frac{1}{2}}}f(s,\cdot).\end{aligned}$$

Thus we have

$$\begin{aligned}G^F f(t,y) =& i\left(\frac{1}{2}L^{-\frac{1}{2}}\int_{-\infty}^{t} e^{-i(t-s)L^{\frac{1}{2}}}f(s,y)ds + \frac{1}{2}L^{-\frac{1}{2}}\int_{t}^{\infty} e^{i(t-s)L^{\frac{1}{2}}}f(s,y)ds\right) \\ =& \frac{i}{2}L^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-i|t-s|L^{\frac{1}{2}}}f(s,y)ds.\end{aligned}$$

□

References

[1] J. Duistermaat, L. Hörmander, Fourier integral operators. II. Acta Math, **128**, (1972), 183–269.

[2] J. Dereziński, D. Siemssen, Feynman propagators on static spacetimes, Rev. Math. Phys. **30**, (2018), 1850006.

[3] C. Gérard, M. Wrochna, The massive Feynman propagator on asymptotically Minkowski spacetimes, Amer. J. Math. **141**, (2019), 1501–1546.

[4] C. Gérard, M. Wrochna, The massive Feynman propagator on asymptotically Minkowski spacetimes II, Int. Math. Res. Notices. **2020**, (2020), 6856–6870.

[5] K. Taira, Limiting absorption principle and equivalence of Feynman propagators on asymptotically Minkowski spacetimes, Commun. Math. Phys. **388** (2021), 625–655.

[6] A. Vasy, Essential self-adjointness of the wave operator and the limiting absorption principle on Lorentzian scattering spaces, J. Spectr. Theory. **10**, (2020), 439–461.