SMOLUCHOWSKI-KRAMERS APPROXIMATION IN THE STOCHASTIC NONLINEAR DAMPED WAVE EQUATION IN 2D

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1. Introduction

This review is based on the published papers in collaboration with Masato Hoshino (Osaka University) and Takahisa Inui (Osaka University) [6, 7].

A stochastic force combined with a dissipation is used to model a temperature effect in the dynamics of partial/ordinal differential equations. In the paper [14], the authors study the $U(1)$-invariant relativistic complex field model in three dimensions, which, being added noise and dissipation, serves to describe, in various limits, properties at finite temperatures of super fluid systems, superconductors of type II, nematic liquid crystals, as well as relativistic bosons at finite chemical potential. The authors found a statistical universality among those models of different physical backgrounds.

More precisely, the under damped Langevin equation derived from the Lagrangian density for relativistic bosons with finite chemical potential in [14] reads the following damped nonlinear wave equation on $\mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$ driven by a complex-valued space-time white noise.

$$\frac{1}{c^2}\partial_t^2\psi + (\mu - i\gamma)\partial_t\psi - \Delta\psi - (\nu - |\psi|^2)\psi = \sqrt{\mu T}\xi, \tag{1.1}$$
$$\langle \xi(t,x)\rangle = 0, \quad \langle \xi^*(t,x)\xi(s,y)\rangle = 2\delta(t-s)\delta(x-y),$$

with $\mu, \gamma > 0$, $\nu \geq 0$ and $c > 0$. Here, $T > 0$ is the temperature. In the ultra relativistic and non relativistic limits, i.e. $\gamma \to 0$ and $c \to +\infty$ respectively, this Langevin equation approaches to

$$\frac{1}{c^2}\partial_t^2\psi + \mu\partial_t\psi - \Delta\psi - (\nu - |\psi|^2)\psi = \sqrt{\mu T}\xi \tag{1.2}$$

and

$$(\mu - i\gamma)\partial_t\psi - \Delta\psi - (\nu - |\psi|^2)\psi = \sqrt{\mu T}\xi \tag{1.3}$$

respectively. Eq.(1.2) and Eq.(1.3) are known as the Goldstone and the Gross-Pitaevskii models. The latter describes the dynamics of gaseous Bose-Einstein condensates ([8]). On the other hand, the former describes the dynamics of the Mott insulator phase with integer fillings ([2]).

By the numerical simulations, it is observed that all these models have the same statistical quantities around the equilibrium. What they call in [14] the equilibrium, can be described by the Gibbs measure ρ, which is written formally as:

$$\rho(d\psi d\phi) = \Gamma^{-1} e^{-\frac{H(\psi,\phi)}{T}} d\psi d\phi,$$

with

$$H(\psi, \phi) = \frac{1}{2c^2} \int |\phi|^2 dx + V(\psi),$$

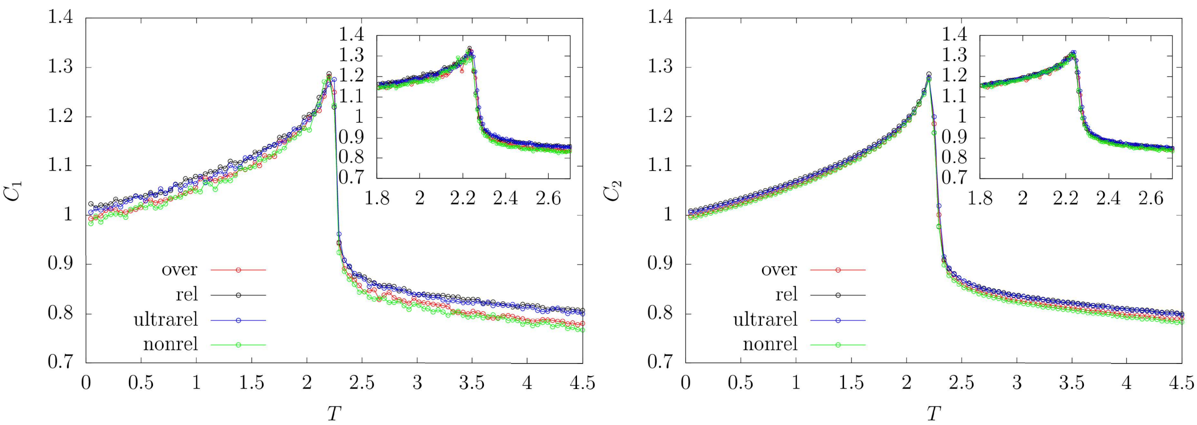
and

$$V(\psi) = \frac{1}{2} \int |\nabla \psi|^2 dx - \frac{\nu}{2} \int |\psi|^2 dx + \frac{1}{4} \int |\psi|^4 dx,$$

where Γ is a normalizing constant. Note that if we insert the wave functions $\psi = \psi_T$ of those models which depend on the temperature $T > 0$, $V(\psi_T)$ also.

$$V(\psi_T) = \frac{1}{2} \int |\nabla \psi_T|^2 dx - \frac{\nu}{2} \int |\psi_T|^2 dx + \frac{1}{4} \int |\psi_T|^4 dx.$$

The following figures were given by M. Kobayashi, one of the authors of [14].



The left simulation C_1 observes the variance of $V(\psi_T)$:

$$\langle V^2(\psi_T) \rangle - \langle V(\psi_T) \rangle^2 / T^2,$$

and the right figure C_2 shows the computation of the change rate, in terms of the temperature, of the average $\langle V(\Psi_T) \rangle$:

$$[\langle V(\psi_{T+\Delta T}) \rangle - \langle V(\psi_{T-\Delta T}) \rangle] / 2\Delta T.$$

In the above figures, 'rel' means Eq.(1.1), 'ultrarel' means Eq.(1.2) and 'nonrel' means Eq.(1.3). One can see clearly a similarity of those three models about the quantity at the object.

The measure ρ depends on the parameter c, but in fact we will see that the equilibrium does not depend on the parameters except ν, neither on the equations above, which, we believe, may explain the result of the numerical simulation.

We consider this equilibrium in the case of $d = 2$, setting $T = 2$, in this note. Unfortunately, we modify (1.1) and replace the term $-\Delta$ by $-\Delta + 1$ and set $\nu = 0$ in order to keep the positive definiteness of the linear part to define the corresponding Gaussian measure. Moreover, the ρ is a priori not well-defined, since L^4 is not in the support of ρ, but at this point we may give a sense using a renormalization technique as has been widely used by now.

Existence of solutions of all those models and the construction of the Gibbs measure have been established in [9, 10, 18, 12, 13, 19, 15]. Our interest is mainly in justifying the both limits $\gamma \to 0$ and $c \to +\infty$. The convergence as $c \to +\infty$ and the independence of Gaussian measure on c were already justified in [5] under the name of Smoluchowski-Kramers approximation in case of $d = 1$ with a space-time white noise, and for $d \geq 2$ with a colored noise under the Dirichlet boundary condition. See for example [3] and references therein for generalizations of this Smoluchowski-Kramers approximation issue.

2. Main results

In this section, we precisely mention our mathematical results on the equation explained in the previous section, setting $\varepsilon = \frac{1}{c} \in (0, 1]$ in (1.1). As a damping coefficient we consider more generally $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$, $\mathrm{Im}(\alpha) \neq 0$. For each $\varepsilon \in (0, 1]$, we consider the damped Klein-Gordon equation with an exterior force $f(t, x)$:

$$\begin{cases} \varepsilon^2 \partial_t^2 u_{\varepsilon,\alpha} + 2\alpha \partial_t u_{\varepsilon,\alpha} + (1 - \Delta) u_{\varepsilon,\alpha} = f, \quad t > 0, \ x \in \mathbb{T}^d, \\ (u_{\varepsilon,\alpha}, \varepsilon \partial_t u_{\varepsilon,\alpha})|_{t=0} = (\phi_0, \phi_1), \qquad x \in \mathbb{T}^d. \end{cases} \tag{2.1}$$

The spatial dimension d can be any $d \geq 1$ for the moment. To clarify the dependence on the parameter α in the estimates we will encounter, we use the following notation:

$$\mathbb{C}_+ = \{\alpha \in \mathbb{C}\,;\, \mathrm{Re}(\alpha) > 0\}, \qquad \mathbb{C}_+ \setminus (0, \infty) = \{\alpha \in \mathbb{C}_+\,;\, \mathrm{Im}(\alpha) \neq 0\}.$$

For any $s \in \mathbb{R}$, we denote the inhomogeneous Sobolev space by H^s, and we define $\mathcal{H}^s = H^s \times H^{s-1}$.

We will mainly focus on the non relativistic limit problem presenting the detailed analysis, and then we will mention our results briefly on the ultra relativistic limit which are similarly obtained. The first step is to derive the uniform energy estimate in ε for (2.1). For that purpose, we write the solution of (2.1) in the mild form.

$$u_{\varepsilon,\alpha}(t) = e^{t\lambda^+_{\varepsilon,\alpha}(\nabla)} \phi^+_{\varepsilon,\alpha} + e^{t\lambda^-_{\varepsilon,\alpha}(\nabla)} \phi^-_{\varepsilon,\alpha} + \int_0^t \Big(e^{(t-t')\lambda^+_{\varepsilon,\alpha}(\nabla)} f^+_{\varepsilon,\alpha}(t') + e^{(t-t')\lambda^-_{\varepsilon,\alpha}(\nabla)} f^-_{\varepsilon,\alpha}(t') \Big)\, dt', \tag{2.2}$$

where,

$$\lambda^\pm_{\varepsilon,\alpha}(\nabla) = \frac{-\alpha \pm \sqrt{\alpha^2 - \varepsilon^2 \langle \nabla \rangle^2}}{\varepsilon^2},$$

with the square root $\sqrt{\cdot}$ defined by

$$\sqrt{e^{i\theta}} = e^{i\frac{\theta}{2}}, \quad \theta \in (-\pi, \pi],$$

and

$$\phi^\pm_{\varepsilon,\alpha} = \frac{\mp \varepsilon^2 \lambda^\mp_{\varepsilon,\alpha}(\nabla) \phi_0 \pm \varepsilon \phi_1}{2\sqrt{\alpha^2 - \varepsilon^2 \langle \nabla \rangle^2}}, \quad f^\pm_{\varepsilon,\alpha}(t) = \pm \frac{1}{2\sqrt{\alpha^2 - \varepsilon^2 \langle \nabla \rangle^2}} f(t).$$

Theorem 1. *Let $d \geq 1$. For any $\sigma \in \mathbb{R}$,*

$$\|(u_{\varepsilon,\alpha}, \varepsilon\partial_t u_{\varepsilon,\alpha})\|_{L^\infty_T \mathcal{H}^\sigma_x(\mathbb{T}^d)} \lesssim_\alpha \|\phi_0\|_{H^\sigma(\mathbb{T}^d)} + \|\phi_1\|_{H^{\sigma-1}(\mathbb{T}^d)} + \|f\|_{L^2_T H^{\sigma-1}_x(\mathbb{T}^d)},$$

where the implicit proportional constants are locally bounded function of $\alpha \in \mathbb{C}_+ \setminus (0,\infty)$.

From now we restrict ourselves to the two dimensional case, and we pay attention to the equation in purpose, namely

$$\begin{cases} \varepsilon^2\partial_t^2\Psi_{\varepsilon,\alpha} + 2\alpha\partial_t\Psi_{\varepsilon,\alpha} + (1-\Delta)\Psi_{\varepsilon,\alpha} + |\Psi_{\varepsilon,\alpha}|^{2n}\Psi_{\varepsilon,\alpha} = 2\sqrt{\mathrm{Re}(\alpha)}\partial_t W, & t>0,\ x\in\mathbb{T}^2, \\ (\Psi_{\varepsilon,\alpha}, \varepsilon\partial_t\Psi_{\varepsilon,\alpha})|_{t=0} = (\psi,\phi), & x\in\mathbb{T}^2, \end{cases} \tag{2.3}$$

where $n \in \mathbb{N}$ and W is the cylindrical Wiener process as

$$W(t,x) = \sum_{k\in\mathbb{Z}^2} (\beta_{k,R}(t) + i\beta_{k,I}(t))e_k(x), \quad e_k(x) = \frac{1}{2\pi}e^{ik\cdot x}. \tag{2.4}$$

Here, $(\beta_{k,R}(t))_{t\geq 0}$ and $(\beta_{k,I}(t))_{t\geq 0}$ are sequences of independent real-valued Brownian motions on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. In all what follows, the notation $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$. For a probability measure μ defined on H^s, integration with respect to $\mu \otimes \mathbb{P}$ denoted by $\mathbb{E}_\mu$.

As was already proved in [9, 18], due to the space-time white noise, the solution of (2.3) have negative space regularity, and thus the nonlinear term $|\Psi|^{2n}\Psi$ is ill-defined. In order to make sense of this term, we will proceed as in [4, 9, 18], use the Wick product and renormalize the nonlinear term.

$$\begin{cases} \varepsilon^2\partial_t^2\Psi_{\varepsilon,\alpha} + 2\alpha\partial_t\Psi_{\varepsilon,\alpha} + (1-\Delta)\Psi_{\varepsilon,\alpha} + :\Psi_{\varepsilon,\alpha}^{n+1}\overline{\Psi_{\varepsilon,\alpha}}^n: = 2\sqrt{\mathrm{Re}(\alpha)}\partial_t W, & t>0,\ x\in\mathbb{T}^2, \\ (\Psi_{\varepsilon,\alpha}, \varepsilon\partial_t\Psi_{\varepsilon,\alpha})|_{t=0} = (\psi,\phi), & x\in\mathbb{T}^2. \end{cases} \tag{2.5}$$

The notation $:u^{n+1}\bar{u}^n:$ is the complex Wick product (For the definition see [6]).

Writing the solution $\Psi_{\varepsilon,\alpha} = U_{\varepsilon,\alpha} + Z_{\varepsilon,\alpha}$ with the stationary solution $Z_{\varepsilon,\alpha}$ for the linear stochastic equation

$$\varepsilon^2\partial_t^2 Z_{\varepsilon,\alpha} + 2\alpha\partial_t Z_{\varepsilon,\alpha} + (1-\Delta)Z_{\varepsilon,\alpha} = 2\sqrt{\mathrm{Re}(\alpha)}\partial_t W, \tag{2.6}$$

we find out the following random partial differential equation for $U_{\varepsilon,\alpha}$:

$$\begin{cases} \varepsilon^2\partial_t^2 U_{\varepsilon,\alpha} + 2\alpha\partial_t U_{\varepsilon,\alpha} + (1-\Delta)U_{\varepsilon,\alpha} + :(U_{\varepsilon,\alpha}+Z_{\varepsilon,\alpha})^{n+1}(\overline{U_{\varepsilon,\alpha}+Z_{\varepsilon,\alpha}})^n: = 0, \\ (U_{\varepsilon,\alpha}, \varepsilon\partial_t U_{\varepsilon,\alpha})|_{t=0} = (u,v) \end{cases} \tag{2.7}$$

with $(u,v) = (\psi,\phi) - (Z_\varepsilon, \varepsilon\partial_t Z_\varepsilon)|_{t=0}$.

Applying Theorem 1 to (2.7), we obtain the local existence of solution.

Corollary 2.1. *Fix any $T > 0$. Let $\sigma < 1$ be sufficiently close to 1 and let $\delta = 1 - \sigma$. For any compact subset K of $\mathbb{C}_+ \setminus (0,\infty)$, the equations (2.7) parametrized by $\varepsilon > 0$ and $\alpha \in K$ are uniformly well-posed: there exist a random time $T^*(\omega) > 0$, and a unique solution*

$$U_{\varepsilon,\alpha} \in C([0,T^*); H^\sigma) \cap C^1([0,T^*); H^{\sigma-1}), \quad a.s.$$

Here, T^* depends only on $\|(u,v)\|_{\mathcal{H}^\sigma}$ and $\sum_{k+\ell\leq 2n+1} \|:Z^k_{\varepsilon,\alpha}\overline{Z_{\varepsilon,\alpha}}^\ell:\|_{L^\infty_T W^{-\delta,\infty}}$.

Remark 2.1. *As we will see just below, we will consider the limit $\varepsilon \to 0$ (non relativistic limit) or $\mathrm{Im}(\alpha) \to 0$ (ultra relativistic limit). Therefore, the statement of Corollary 2.1 is precisely as follows: for each $\varepsilon > 0$ and $\alpha \in K$, there exist a random maximal existence time $T^*_{\varepsilon,\alpha}(\omega) > 0$, and a unique solution*

$$U_{\varepsilon,\alpha} \in C([0,T^*_{\varepsilon,\alpha}); H^\sigma) \cap C^1([0,T^*_{\varepsilon,\alpha}); H^{\sigma-1}), \quad a.s.,$$

and if we set $T^* := \mathrm{liminf}_{\varepsilon\to 0} T^*_{\varepsilon,\alpha}$, (or $T^{**} := \mathrm{liminf}_{\mathrm{Im}(\alpha)\to 0} T^*_{\varepsilon,\alpha}$,) then $T^*(\omega) > 0$ ($T^{**}(\omega) > 0$) a.s. and

$$U_{\varepsilon,\alpha} \in C([0,T^*(T^{**})); H^\sigma) \cap C^1([0,T^*(T^{**})); H^{\sigma-1}), \quad a.s.$$

The Gibbs measure has been constructed in [18], and can be made sense, again with the help of the renormalization:

$$\rho_{2n+2}(d\psi d\phi) = \Gamma^{-1} e^{-H(\psi,\phi)} d\psi d\phi$$

where,

$$H(\psi,\phi) = \frac{1}{2}\int |\phi|^2 dx + V(\psi),$$

and

$$V(\psi) = \frac{1}{2}\int |\nabla\psi|^2 dx + \frac{1}{2}\int |\psi|^2 dx + \frac{1}{2n+2}\int :|\psi|^{2n+2}: dx.$$

Γ is the normalizing constant. Using the Gaussian measures $\mu_0 = \mathcal{N}(0,(1-\Delta)^{-1})$ and $\mu_1 = \mathcal{N}(0,I)$ we may write

$$\begin{aligned}\rho_{2n+2}(d\psi d\phi) = \Gamma^{-1} e^{-H(\psi,\phi)} d\psi d\phi &= \Gamma^{-1} e^{-\frac{1}{2n+2}\int :|\psi|^{2n+2}:dx} \mu(d\psi d\phi)\\ &= \Gamma^{-1} e^{-\frac{1}{2n+2}\int :|\psi|^{2n+2}:dx} \mu_0(d\psi)\otimes\mu_1(d\phi).\end{aligned}$$

It is known that $\mu_0 = \mathcal{L}(Z_0)$ and $\mathrm{supp}\,\rho_{2n+2} = \mathrm{supp}\,\mu \subset \mathcal{H}^{-\delta}$ for any $\delta > 0$. Remark that the Gibbs measure does not depend on $\varepsilon > 0$, nor on $\alpha \in \mathbb{C}$.

We make use of the Gibbs measure to globalize the solution obtained above.

Proposition 1. *There exists a measurable set $\mathcal{O}_{\varepsilon,\alpha} \subset \mathcal{H}^{-\delta}$ such that $\rho_{2n+2}(\mathcal{O}_{\varepsilon,\alpha}) = 1$ and for $(\psi,\phi) \in \mathcal{O}_{\varepsilon,\alpha}$ the solution of* (2.5) *exists globally a.s..*

Finally we justify the non-relativistic limit. We may formally expect that when ε goes to 0, $\Psi_{\varepsilon,\alpha}$ converges to Ψ_α, which is the solution of the stochastic complex Ginzburg-Landau equation:

$$\begin{cases} 2\alpha\partial_t\Psi_\alpha + (1-\Delta)\Psi_\alpha + :\Psi_\alpha^{n+1}\overline{\Psi_\alpha}^n: = 2\sqrt{\mathrm{Re}(\alpha)}\partial_t W, \\ \Psi_\alpha|_{t=0} = \psi. \end{cases} \tag{2.8}$$

Theorem 2. *Let $\delta > 0$. Consider the solutions $\{\Psi_{\varepsilon(j),\alpha}\}_{j\in\mathbb{N}}$ of* (2.5) *according to the sequence $\varepsilon(j) = j^{-1}$. There exists a measurable set $\mathcal{A} \subset \mathcal{H}^{-\delta}$ such that $\mu(\mathcal{A}) = 1$ and for $(\psi,\phi) \in \mathcal{A}$, $\{\Psi_{\varepsilon(j),\alpha}\}_{j\in\mathbb{N}}$ converges to the solution of* (2.8) *in $L^\infty_\tau H^{-\delta}(\mathbb{T}^2)$ for any $\tau < T_{\mathrm{hmax}}$, almost surely, where T_{hmax} is the maximal existence time of the solution of (2.8). Moreover, this convergence holds globally in time for any $(\psi,\phi) \in \mathcal{A} \cap \bigcap_{j\in\mathbb{N}} \mathcal{O}_{\varepsilon(j)}$.*

In a similar way, we may prove the ultra relativistic limit. We fix $\varepsilon = 1$ and $\mathrm{Re}(\alpha) = \alpha_1 > 0$ and let $\mathrm{Im}(\alpha)$ go to 0. For the sake of simplicity, we write $\Psi_{1,\alpha_1+\alpha_2 i} = \Psi_{\alpha_2}$.

Corollary 2.2. *Consider the solutions* $\{\Psi_{\alpha_2(j)}\}_{j\in\mathbb{N}}$ *of* (2.5) *according to the sequence* $\alpha_2(j) = j^{-1}$. *There exists a measurable set* $\mathcal{B} \subset \mathcal{H}^{-\delta}$ *such that* $\mu(\mathcal{B}) = 1$ *and for* $(\psi, \phi) \in \mathcal{B}$, $\{\Psi_{\alpha_2(j)}\}_{j\in\mathbb{N}}$ *converges to the solution of* (2.3) *with* α *replaced by* α_1 *in* $L^\infty_\tau H^{-\delta}(\mathbb{T}^2)$ *for any* $\tau < T_{\mathrm{wmax}}$, *almost surely, where* T_{wmax} *is the maximal existence time of the solution of (2.3) with* α *replaced by* α_1. *Moreover, this convergence holds globally in time for any* $(\psi, \phi) \in \mathcal{B} \cap \bigcap_{j\in\mathbb{N}} \mathcal{O}_{\alpha_2(j)}$.

After this work has been published, we found that Zine [20] proved a same statement about the nonrelativistic limit, and that his result does not need a subsequence of the parameter. With his idea our results may be improved too, without taking a subsequence $\varepsilon(j) = j^{-1}$ and $\alpha_2(j) = j^{-1}$.

We remark that in [16] the authors consider the damped nonlinear wave equation with a regularized noise (without renormalization), and study possible limiting behavior of solutions as they remove the regularization. Such a triviality result is known for stochastic nonlinear heat and wave equations (see [1, 11, 16]).

3. Outline of the proofs

3.1. **local existence.** We solve (2.7) in the mild form, in the space

$$C([0,T], H^\sigma(\mathbb{T}^2)) \cap C^1([0,T], H^{\sigma-1}(\mathbb{T}^2)).$$

The proof of the local existence is quite similar to [18], but thanks to Theorem 1 we have the local existence uniformly in

$$\varepsilon \in (0,1], \qquad \alpha \in K,$$

for any fixed compact $K \subset \mathbb{C}_+ \setminus (0,\infty)$. To prove Theorem 1, Proposition 3 below is the base to understand the behaviour of the linear part. Combining Proposition 2 and Lemma 3.1 below, we can prove Theorem 1.

Proposition 2. *For any* $\sigma \in \mathbb{R}$, *we have*

$$\|e^{t\lambda^\pm_{\varepsilon,\alpha}(\nabla)}u\|_{L^\infty_T H^\sigma_x} \le \|u\|_{H^\sigma_x}, \tag{3.1}$$

$$\left\|\int_0^t e^{(t-s)\lambda^\pm_{\varepsilon,\alpha}(\nabla)} f(s)ds\right\|_{L^\infty_T H^\sigma_x} \lesssim_\alpha \varepsilon\|f\|_{L^2_T H^\sigma_x} + \|f\|_{L^2_T H^{\sigma-1}_x}. \tag{3.2}$$

In the second inequality, the implicit proportional constant is a locally bounded function of $\alpha \in \mathbb{C}_+$.

Lemma 3.1. *For any* $\sigma \in \mathbb{R}$ *and* $\theta \in [0,1]$,

$$\|\phi^\pm_{\varepsilon,\alpha}\|_{H^\sigma} \lesssim_\alpha \|\phi_0\|_{H^\sigma} + \|\phi_1\|_{H^{\sigma-1}},$$
$$\|f^\pm_{\varepsilon,\alpha}\|_{H^\sigma} \lesssim_\alpha \varepsilon^{-\theta}\|f\|_{H^{\sigma-\theta}},$$

where the implicit proportional constants are locally bounded function of $\alpha \in \mathbb{C}_+ \setminus (0,\infty)$.

Proposition 3. *Let* $\alpha \in \mathbb{C}$ *such that* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Im}(\alpha) \neq 0$.

(1) $0 < \mathrm{Re}\sqrt{\alpha^2 - s} \le \mathrm{Re}(\alpha)$ *for any* $s \ge 0$, *and the function* $[0,\infty) \ni s \mapsto \mathrm{Re}\sqrt{\alpha^2 - s}$ *is strictly decreasing.*

(2) $|\sqrt{\alpha^2 - s}| \geq \sqrt{2|\operatorname{Re}(\alpha)\operatorname{Im}(\alpha)|}$ *for any* $s \geq 0$.

(3) *For any* $s \geq 0$,

$$\left|\frac{\sqrt{s}}{\sqrt{\alpha^2 - s}}\right| \leq \sqrt{1 + \frac{|\alpha|^2}{2|\operatorname{Re}(\alpha)\operatorname{Im}(\alpha)|}}.$$

(4) *There exists* $C_\alpha > 0$ *such that* $\operatorname{Re}(-\alpha + \sqrt{\alpha^2 - s}) \leq -C_\alpha(s \wedge 1)$ *for any* $s \geq 0$. *Moreover,* C_α *is locally bounded from above and below in the region* $\mathbb{C}_+ := \{\alpha \in \mathbb{C}\,;\, \operatorname{Re}(\alpha) > 0\}$.

(5) *For any* $s \geq 0$ *we have*

$$\begin{cases} \sqrt{\alpha^2 - s} = \alpha - \dfrac{s}{2\alpha} + h(s, \alpha), \quad |h(s,\alpha)| \leq 8s^2/|\alpha|^3 & \text{if } s < |\alpha|^2/2, \\ \sqrt{\alpha^2 - s} = i\sqrt{s} + g(s,\alpha), \quad |g(s,\alpha)| \leq 6|\alpha|^2/\sqrt{s}, & \text{if } s > 2|\alpha|^2. \end{cases}$$

3.2. **Wick polynomials of the stationary solution.** We consider the linear stochastic equation (2.6). Since the results in this section are independent of ε and α, we write $Z = Z_{\varepsilon,\alpha}$ for simplicity. Setting $Y = \varepsilon\partial_t Z$, we have the system

$$\begin{cases} dZ(t) = \varepsilon^{-1}Y(t)dt, \\ dY(t) = \varepsilon^{-1}\left\{-2\alpha\varepsilon^{-1}Y(t) - (1-\Delta)Z(t)\right\}dt + 2\varepsilon^{-1}\sqrt{\operatorname{Re}(\alpha)}dW(t). \end{cases} \tag{3.3}$$

We define the Gaussian measure on $\mathcal{S}'(\mathbb{T}^2)^2$ by

$$\mu(dzdy) = \frac{1}{\Gamma_0}e^{-V_0(z,y)}dzdy,$$

with $V_0(z,y) = \frac{1}{2}\int_{\mathbb{T}^2}|z(x)|^2dx + \frac{1}{2}\int_{\mathbb{T}^2}|\nabla z(x)|^2dx + \frac{1}{2}\int_{\mathbb{T}^2}|y(x)|^2dx$ and with a normalizing constant Γ_0. Precisely, by identifying an element $\xi \in \mathcal{S}'(\mathbb{T}^2)$ with the sequence $(\hat{\xi}(k))_{k\in\mathbb{Z}^2}$, the measure μ is defined by the product

$$\mu = \bigotimes_{k\in\mathbb{Z}^2}\mathcal{N}_c(0, 2(1+|k|^2)^{-1}) \otimes \bigotimes_{\ell\in\mathbb{Z}^d}\mathcal{N}_c(0,2),$$

where $\mathcal{N}_c(0,r)$ denotes the complex normal distribution with mean zero and covariance $r > 0$. It is straightforward to see that μ is supported in $\mathcal{H}^{-\delta}$ for any $\delta > 0$. Also we may see that the Gaussian measure μ is invariant under the Markov process (Z, Y).

Let (Z, Y) be the stationary solution of (3.3) with the initial law $(Z,Y)|_{t=0} \sim \mu$. We consider the Wick polynomials of Z. Writing Π_N as the projection $\Pi_N : z \mapsto \sum_{|k|\leq N}\hat{z}(k)e_k$, by the stationarity,

$$C_N := \mathbb{E}[|\Pi_N Z(t,x)|^2] = \int |\Pi_N z|^2\mu(dzdy) = \sum_{|k|\leq N}\frac{2}{1+|k|^2}.$$

The same arguments as in [9, Proposition 2.1] imply the following result.

Proposition 4. *Let* $m, n \in \mathbb{N}$, $T > 0$, $p \in [1,\infty)$, *and* $\delta > 0$. *The sequence* $\{:(\Pi_N Z)^m(\overline{\Pi_N Z})^n:\}_{N\in\mathbb{N}}$ *is Cauchy in* $L^p(\Omega; C([0,T]; W^{-\delta,\infty}))$, *and converges* $\mathbb{P}_\mu$*-almost surely.*

We can get Corollary 2.1, similarly to [9, Proposition 3.5] and [18, Proposition 4.1], by the standard PDE argument applying Theorem 1 with $\phi_0 = u$, $\phi_1 = v$, and

$$f = -\sum_{k\leq 2n+1}P_k(U)\Xi_k,$$

where $P_k(U)$ is a k-th homogeneous polynomial of U and $\overline{U}$, and Ξ_k is an element of $L^\infty_T W^{-\delta,\infty}$, depends on the degree k.

3.3. **global existence a.e.** ρ_{2n+2}. For simplicity, we write $\Psi = \Psi_{\varepsilon,\alpha}$ and $\rho = \rho_{2n+2}$. We consider a finite dimensional approximation

$$\begin{cases} \varepsilon^2\partial_t^2\Psi^N + 2\alpha\partial_t\Psi^N + (1-\Delta)\Psi^N + \Pi_N : (\Pi_N\Psi^N)^{n+1}\overline{(\Pi_N\Psi^N)}^n : \\ = 2\sqrt{\mathrm{Re}(\alpha)}\partial_t W, & t>0,\ x\in\mathbb{T}^2, \\ (\Psi^N, \varepsilon\partial_t\Psi^N)|_{t=0} = (\psi,\phi), & x\in\mathbb{T}^2. \end{cases} \tag{3.4}$$

Setting $\Phi^N = \varepsilon\partial_t\Psi^N$, we have the system

$$\begin{cases} d\Psi^N(t) = \varepsilon^{-1}\Phi^N(t)dt, \\ d\Phi^N(t) = \varepsilon^{-1}\left\{-2\alpha\varepsilon^{-1}\Phi^N(t) - (1-\Delta)\Psi^N(t) - \Pi_N : (\Pi_N\Psi^N)^{n+1}\overline{(\Pi_N\Psi^N)}^n :\right\} dt \\ \qquad\qquad + 2\varepsilon^{-1}\sqrt{\mathrm{Re}(\alpha)}dW(t). \end{cases} \tag{3.5}$$

For any $N\in\mathbb{N}$, we define the truncated measure

$$\rho_N(d\psi d\phi) = \Gamma_N^{-1}e^{-H_N(\psi,\phi)}d\psi d\phi = \Gamma_N^{-1}e^{-\frac{1}{2n+2}\int_{\mathbb{T}^2}:(\Pi_N\Psi^N)^{n+1}\overline{(\Pi_N\Psi^N)}^{n+1}:dx}d\mu_0\otimes\mu_1.$$

where,

$$H_N(\psi,\phi) = \frac{1}{2}\int_{\mathbb{T}^2}|\phi|^2dx + V_N(\psi),$$

and

$$V_N(\psi) = \frac{1}{2}\int_{\mathbb{T}^2}|\nabla\psi|^2dx + \frac{1}{2}\int_{\mathbb{T}^2}|\psi|^2dx + \frac{1}{2n+2}\int_{\mathbb{T}^2} : (\Pi_N\Psi^N)^{n+1}\overline{(\Pi_N\Psi^N)}^{n+1} : dx.$$

We can apply Corollary 2.1 to Eq. (3.4) and prove the following result. The global well-posedness of (Ψ^N, Φ^N) follows from the energy estimate as in [18, Proposition 5.1].

Proposition 5. *Let $\delta>0$ and $T>0$. Let $(\psi,\phi)\in\mathcal{H}^{-\delta}$. Then, there exists a unique global solution in $C([0,T];\mathcal{H}^{-\delta})$ denoted by (Ψ^N,Φ^N) of (3.4). Moreover, there exists a measurable set $\mathcal{O}\subset\mathcal{H}^{-\delta}$ such that $\mu(\mathcal{O})=1$ with the following properties: for any $(\psi,\phi)\in\mathcal{O}$, $(\Psi^N,\Phi^N)\to(\Psi,\Phi)$ in $C([0,T];\mathcal{H}^{-\delta})$ in probability as $N\to\infty$ for any $T<T^*$, where $(\Psi,\Phi)=(U,\varepsilon\partial_t U)+(Z,\varepsilon\partial_t Z)$, U being the solution of (2.7) with zero initial values, given by Corollary 2.1 and T^* being its maximal existence time.*

We define the Feller transition semigroup $P_t^N f(\psi,\phi) = \mathbb{E}(f(\Psi^N,\Phi^N)(t,(\psi,\phi)))$ for $(\psi,\phi)\in\mathcal{O}$, and we can see easily that the measure ρ_N is invariant for $(P_t^N)_{t\geq 0}$.

The following lemma reflects some important properties of ρ_N, which are remarked in [18, Lemma 3.2].

Lemma 3.2. (1) *The normalizing constant Γ_N is bounded below independent of N.*

(2) *The sequence*

$$\left\{e^{-\frac{1}{2n+2}\int_{\mathbb{T}^2}H_{n+1,n+1}(\Pi_N\psi;C_N)dx}\right\}_{N\in\mathbb{N}}$$

is Cauchy in $L^p(d\mu_0)$ for $p\geq 1$, and we denote the limit by

$$e^{-\frac{1}{2n+2}\int_{\mathbb{T}^2}:|\psi|^{2n+2}:(x)dx}.$$

Lemma 3.3. *Let any $T > 0$ be fixed, and let $\sigma < 1$ be sufficiently close to 1 and $\delta = 1 - \sigma$ as in Corollary 2.1. There exists a constant C_T independent of N such that*

$$\sup_{N\in\mathbb{N}} \int_{\mathcal{H}^{-\delta}} \mathbb{E}\left[\sup_{0\le t\le T} \|(U^N(t), \varepsilon\partial_t U^N(t))\|_{\mathcal{H}^\sigma} \right] \rho_N(d\psi d\phi) \le C_T.$$

Proposition 1 follows from the next statement, using Proposition 5 and Fatou's lemma.

Theorem 3. *Fix any $T > 0$. Let σ and δ be as in lemma above. Then there exists a constant $C_T > 0$ such that*

$$\int_{\mathcal{H}^{-\delta}} \mathbb{E}\left[\sup_{0\le t< T\wedge T^*} \|(U(t), \varepsilon\partial_t U(t))\|_{\mathcal{H}^\sigma} \right] \rho(d\psi d\phi) \le C_T.$$

Hence $T^* = T$ a.s. for ρ-a.e. $(\psi, \phi) \in \mathcal{H}^{-\delta}$.

Thus, we can define the transition semigroup $P_t f(\psi,\phi) = \mathbb{E}(f(\Psi,\Phi)(t,(\psi,\phi)))$ for $(\psi,\phi) \in \overline{\mathcal{M}}$ and $t \ge 0$, and the invariance of measure ρ follows straightforward:

Corollary 3.4. *The measure ρ is invariant for $(P_t)_{t\ge 0}$.*

3.4. **non relativistic limit.** In this section, we fix α and let ε go to 0, so we omit the label α from underlined objects, e.g. we write $\Psi_\varepsilon = \Psi_{\varepsilon,\alpha}$.

3.4.1. *deterministic result.* We can expect that, the limit of the solution of (2.1) as $\varepsilon \to 0$ solves the linear heat equation

$$\begin{cases} 2\alpha\partial_t v + (1-\Delta)v = f, & (t,x) \in (0,\infty)\times\mathbb{T}^d, \\ v|_{t=0} = \phi_0, & x \in \mathbb{T}^d, \end{cases} \tag{3.6}$$

or equivalently,

$$v(t) = e^{\frac{t}{2\alpha}(\Delta-1)}\phi_0 + \frac{1}{2\alpha}\int_0^t e^{\frac{t-t'}{2\alpha}(\Delta-1)} f(t')dt'.$$

Theorem 4. *For any $\sigma \in \mathbb{R}$ and $\theta \in [0,1]$,*

$$\|u_\varepsilon - v\|_{L^\infty_T H^\sigma_x} \lesssim_\alpha \varepsilon^\theta \big(\|\phi_0\|_{H^{\sigma+\theta}} + \|\phi_1\|_{H^{\sigma-1+\theta}} + \|f\|_{L^2_T H^{\sigma-1+\theta}_x} \big),$$

where $u_\varepsilon = u_{\varepsilon,\alpha}$ is the solution of (2.1), and the proportional constants are locally bounded functions of $\alpha \in \mathbb{C}_+ \setminus (0,\infty)$.

proof. We decompose

$$u_\varepsilon - v = \mathbf{1}_{\{\varepsilon\langle\nabla\rangle \le |\alpha|/\sqrt{2}\}}(u_\varepsilon - v) + \mathbf{1}_{\{\varepsilon\langle\nabla\rangle > |\alpha|/\sqrt{2}\}}(u_\varepsilon - v).$$

For the latter part, we have the required estimate as a consequence of the ε-uniform estimates (Theorem 1), since

$$\|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle > |\alpha|/\sqrt{2}\}} w\|_{H^\sigma} \lesssim_\alpha \varepsilon^\theta \|w\|_{H^{\sigma+\theta}}.$$

for any $w \in H^{\sigma+\theta}$.

We consider the former part. First let $f=0$ and reorganize the term concerning initial values as follows.

$$\begin{aligned} u_\varepsilon - v &= e^{t\lambda_\varepsilon^+(\nabla)}(\phi_\varepsilon^+ - \phi_0) + e^{t\lambda_\varepsilon^-(\nabla)}\phi_\varepsilon^- + (e^{t\lambda_\varepsilon^+(\nabla)} - e^{\frac{t}{2\alpha}\Delta})\phi_0 \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

As for A_1 and A_2, we can ignore $e^{t\lambda_\varepsilon^\pm(\nabla)}$ because of the estimate (3.1). We write

$$\phi_\varepsilon^+ - \phi_0 = \frac{(\alpha - \sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2})\phi_0 + \varepsilon\phi_1}{2\sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2}}, \qquad \phi_\varepsilon^- = \frac{(-\alpha + \sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2})\phi_0 - \varepsilon\phi_1}{2\sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2}}.$$

By Proposition 3, we can estimate

$$\|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle\le|\alpha|/\sqrt{2}\}}(\phi_\varepsilon^+ - \phi_0)\|_{H^\sigma} + \|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle\le|\alpha|/\sqrt{2}\}}\phi_\varepsilon^-\|_{H^\sigma} \lesssim \varepsilon^2\|\phi_0\|_{H^{\sigma+2}} + \varepsilon\|\phi_1\|_{H^\sigma},$$

where we have used the estimate $\sqrt{\alpha^2 - \varepsilon^2\langle k\rangle^2} = \alpha - \frac{\varepsilon^2\langle k\rangle^2}{2\alpha} + O(\varepsilon^4\langle k\rangle^4)$ from Proposition 3-(5) for the term concerning ϕ_0, and use $\sqrt{\alpha^2 - \varepsilon^2\langle k\rangle^2} = O(1)$ for the term concerning ϕ_1. On the other hand, we can estimate

$$\|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle\le|\alpha|/\sqrt{2}\}}(\phi_\varepsilon^+ - \phi_0)\|_{H^\sigma} + \|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle\le|\alpha|/\sqrt{2}\}}\phi_\varepsilon^-\|_{H^\sigma} \lesssim \|\phi_0\|_{H^\sigma} + \|\phi_1\|_{H^{\sigma-1}},$$

by replacing $\sqrt{\alpha^2 - \varepsilon^2\langle k\rangle^2}$ by 1 or $\varepsilon\langle k\rangle$, from Proposition 3-(2) and (3). By the interpolation we have the required estimate for A_1 and A_2. As for A_3, we have

$$\|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle\le|\alpha|/\sqrt{2}\}}A_3\|_{H^\sigma} \lesssim \|\phi_0\|_{H^\sigma}$$

by the ε-uniform estimate (3.1). Since in the region $\varepsilon\langle k\rangle \le |\alpha|/\sqrt{2}$ we can estimate

$$\begin{aligned} |e^{t\lambda_\varepsilon^+(k)} - e^{-\frac{t}{2\alpha}\langle k\rangle^2}| &\lesssim t\big|\lambda_\varepsilon^+(k) + \frac{1}{2\alpha}\langle k\rangle^2\big| \left(|e^{t\lambda_\varepsilon^+(k)}| \vee |e^{-\frac{t}{2\alpha}\langle k\rangle^2}|\right) \\ &\lesssim t\varepsilon^2\langle k\rangle^4 e^{-ct\langle k\rangle^2} \\ &\lesssim \varepsilon^2\langle k\rangle^2 e^{-c't\langle k\rangle^2} \end{aligned}$$

for some constants $c, c' > 0$, so we have

$$\|\mathbf{1}_{\{\varepsilon\langle\nabla\rangle\le|\alpha|/\sqrt{2}\}}A_3\|_{H^\sigma} \lesssim \varepsilon^2\|\phi_0\|_{H^{\sigma+2}}.$$

By the interpolation, we have the required estimate for A_3.

Next we consider the case $\phi_0 = \phi_1 = 0$. We decompose

$$\begin{aligned} u_\varepsilon - v &= \int_0^t \frac{\alpha - \sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2}}{2\alpha\sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2}} e^{(t-s)\lambda_\varepsilon^+(\nabla)} f(s)ds \\ &\quad - \int_0^t \frac{1}{2\sqrt{\alpha^2 - \varepsilon^2\langle\nabla\rangle^2}} e^{(t-s)\lambda_\varepsilon^-(\nabla)} f(s)ds + \int_0^t \frac{1}{2\alpha}\big(e^{\frac{t-s}{2\alpha}(\Delta-1)} - e^{(t-s)\lambda_\varepsilon^+(\nabla)}\big) f(s)ds. \end{aligned}$$

Then we can estimate each term by a similar way to above. □

3.4.2. *probabilistic result.* We consider the solutions $\{\Psi_{\varepsilon(j)}\}_{j\in\mathbb{N}}$ according to $\varepsilon(j) = j^{-1}$. Theorem 2 is a consequence of the following proposition and Proposition 1.

Proposition 6. *Let $\delta > 0$. Let $(Z_{\varepsilon(j)}, Y_{\varepsilon(j)})$ be the solution of (3.3) with $(Z_{\varepsilon(j)}, Y_{\varepsilon(j)})|_{t=0} = (z, y)$, and let Z be the solution of*

$$2\alpha\partial_t Z + (1-\Delta)Z = 2\sqrt{\mathrm{Re}(\alpha)}\partial_t W$$

with $Z|_{t=0} = z$. Note that z and y are deterministic functions. Then there exists a measurable set $\mathcal{A} \subset \mathcal{H}^{-\delta}$ such that $\mu(\mathcal{A}) = 1$ and for any $(z, y) \in \mathcal{A}$, $m, n \in \mathbb{N}$, $T > 0$, and $p \in [1,\infty)$, the sequence of Wick products $: Z^m_{\varepsilon(j)} \overline{Z}^n_{\varepsilon(j)} :$ converges to $: Z^m \overline{Z}^n :$ in $C([0,T]; W^{-\delta,\infty})$ almost surely.

proof. We first note that the following (N, t, x)-uniform estimate can be shown:

$$\int_{\mathcal{H}^{-\delta}} \mathbb{E}\left[|\langle\nabla\rangle^{-\delta}(: \Pi_N Z^m_\varepsilon \overline{\Pi_N Z}^n_\varepsilon : - : \Pi_N Z^m \overline{\Pi_N Z}^n :)|^2\right] \mu(dzdy) \lesssim \varepsilon^\theta$$

for any $\theta \in (0,1]$. In a similar way to [9], we have that, for any $h \in [-1,1]$ and $\alpha_0 \in [0,1]$ such that $\alpha_0 < 2\delta$,

$$\mathbb{E}_\mu\left[|\delta_h\left\{\langle\nabla\rangle^{-\delta}(: \Pi_N Z^m_\varepsilon \overline{\Pi_N Z}^n_\varepsilon : - : \Pi_N Z^m \overline{\Pi_N Z}^n :)\right\}|^2\right] \lesssim |h|^{\alpha_0}\varepsilon^\theta,$$

where $\delta_h f(t) := f(t+h) - f(t)$. Thus, Nelson estimate, Sobolev embedding, and Kolmogorov criterion imply the convergence

$$\mathbb{E}_\mu\left[\| : \Pi_N Z^m_\varepsilon \overline{\Pi_N Z}^n_\varepsilon : - : \Pi_N Z^m \overline{\Pi_N Z}^n : \|^p_{C_T W^{-\delta,p}}\right] \lesssim \varepsilon^{\theta p/2}$$

for any $p \in [1,\infty)$. Moreover, by Proposition 4, as $N \to \infty$

$$\mathbb{E}_\mu\Big[\Big\| : Z^m_\varepsilon \overline{Z}^n_\varepsilon : - : Z^m \overline{Z}^n : \Big\|^p_{C_T W^{-\delta,\infty}}\Big] \lesssim \varepsilon^{\theta p/2}.$$

As for the almost sure convergence result, we use a similar argument to [17, Proposition 3.2]. For any $k \in \mathbb{N}$ and $\varepsilon > 0$, set

$$\mathbb{A}^k_\varepsilon = \Big\{((z,y),\omega) \in \mathcal{H}^{-\delta} \times \Omega\ ;\ \Big\| : Z^m_\varepsilon \overline{Z}^n_\varepsilon : - : Z^m \overline{Z}^n : \Big\|_{C_T W^{-\delta,\infty}} < k^{-1}\Big\}.$$

By Markov inequality,

$$\begin{aligned}
\mu \otimes \mathbb{P}((\mathbb{A}^k_\varepsilon)^c) &= \mu \otimes \mathbb{P}\Big[\Big\| : Z^m_\varepsilon \overline{Z}^n_\varepsilon : - : Z^m \overline{Z}^n : \Big\|_{C_T W^{-\delta,\infty}} \geq k^{-1}\Big] \\
&\lesssim k^p \mathbb{E}_\mu\Big[\Big\| : Z^m_\varepsilon \overline{Z}^n_\varepsilon : - : Z^m \overline{Z}^n : \Big\|^p_{C_T W^{-\delta,\infty}}\Big] \\
&\lesssim k^p \varepsilon^{\theta p/2}.
\end{aligned}$$

We then have, taking a subsequence $\varepsilon(j) = j^{-1}$ and setting p to be $\theta p > 2$,

$$\sum_{j=1}^\infty \mu \otimes \mathbb{P}((\mathbb{A}^k_{\varepsilon(j)})^c) \lesssim k^p \sum_{j=1}^\infty j^{-\theta p/2} < \infty.$$

By Borel-Cantelli lemma, the event $\mathbb{A}^k := \bigcup_{j=1}^{\infty} \bigcap_{i\geq j} \mathbb{A}^k_{\varepsilon(i)}$ has probability one, and for any $((z,y),\omega) \in \mathbb{A}^k$,

$$\limsup_{j\to\infty} \big\| : Z^m_{\varepsilon(j)} \overline{Z}^n_{\varepsilon(j)} : - : Z^m \overline{Z}^n : \big\|_{C_T W^{-\delta,\infty}} \leq k^{-1}.$$

Therefore, the event $\mathbb{A} := \bigcap_{k\in\mathbb{N}} \mathbb{A}^k$ also has probability one, and for any $((z,y),\omega) \in \mathbb{A}$,

$$\lim_{j\to\infty} \big\| : Z^m_{\varepsilon(j)} \overline{Z}^n_{\varepsilon(j)} : - : Z^m \overline{Z}^n : \big\|_{C_T W^{-\delta,\infty}} = 0,$$

i.e. $\{: Z^m_{\varepsilon(j)} \overline{Z}^n_{\varepsilon(j)} :\}_j$ converges in $C_T W^{-\delta,\infty}$, almost surely, for almost every initial condition (z,y).

□

Combining the deterministic result and the probabilistic result, the proof of Theorem 2 is completed.

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