Kan extension in Micro-Macro duality scheme

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Research Origin for Dressed Photons

Kan extension and Micro-Macro duality schemes

In the following we examine the possibility to embed the categorical tool known as the **Kan extension** [1] in a suitable position in the Micro-Macro duality scheme [2] which describes a physical system in a material hierarchy. As a conclusion, we see the above theoretical possibility can be materialized in our mathematical framework.
In this attempt, we notice such a special property of the "classifying map" in the Micro-Macro duality scheme from the "classifying space" to the non-commutative algebra of physical quantities as its **injectivity** which has been clarified explicitly for the first time only through the attempt to embed the Kan extenstion into the Micro-Macro duality scheme.
At the end, there will remain as an unsolved problem as to which version(s) of the right Kan extension, or left one, or both, are relevant to the context.
Once a satisfactory goal in the above problem has been achieved, what is asserted boldly by Mac Lane in his famous book, "Categories for the Working Mathematicians" in such a form that every concept is a **Kan extension** [2] will gain a new persuasive support.

Three-term structure in Kan extension vs four-term one in Micro-Macro duality scheme

For those who have sufficient knowledge about the category theory, it would be unnecessary to explain a Kan extension: in the simplest form, it can be seen as a problem to find the unknown third functor $c : \mathcal{E} \dashrightarrow \Omega$ in such a context that two functors $\omega : \mathcal{D} \longrightarrow \mathcal{E}$ and $\gamma : \mathcal{D} \longrightarrow \Omega$ among three categories $\mathcal{D}, \mathcal{E}$ and Ω are known to exist to satisfy the relation:

$$\begin{array}{ccc} \mathcal{E} & \overset{c}{\dashrightarrow} & \Omega \\ \omega \uparrow & \nearrow \gamma & \\ \mathcal{D} & & \end{array}$$

Namely, we want to examine whether the image category $\mathcal{E}$ of the functor $\omega : \mathcal{D} \longrightarrow \mathcal{E}$ can be extended to Ω via the third functor $c : \mathcal{E} \dashrightarrow \Omega$ so that the composite functor $c \circ \omega$ becomes identical with the given functor $c \circ \omega = \gamma : \mathcal{D} \longrightarrow \Omega$.

On the other hand, the Micro-Macro duality scheme can be depicted by the following diagram consisting of quadrality scheme:

$$\begin{array}{cccc} & \mathcal{E}(\mathcal{A}) & \longrightarrow & \Omega \cong \hat{G} \\ & \uparrow & \swarrow & \\ \swarrow & \uparrow & & \\ G & \curvearrowright & \mathcal{A} & \end{array}$$

To related this quadrality scheme with the Kan extension, the former four terms should be reduced to the latter three terms. For this purpose, we identify the dynamical system $G \curvearrowright \mathcal{A}$ with the crossed product $\mathcal{D} = \mathcal{A} \triangleleft G$ consisting of an algebra$\mathcal{A}$ of physical quantities acted on by a group G in the above quadrality scheme, through which the essence of the latter Micro-Macro duality can be represented by the following three-term diagram:

$$\begin{array}{ccc} \mathcal{E} = E_{\mathcal{A}} & \overset{\omega}{\longrightarrow} & \hat{G} \cong \Omega \\ \uparrow & \swarrow & \\ \mathcal{D} = \mathcal{A} \triangleleft G & & \end{array}$$

Here, $\mathcal{E} = E_{\mathcal{A}}$ is the totality of states ω of $\mathcal{A}$ defined as normalized positive linear functionals on the (C*-)algebra $\mathcal{A}$ of physical quantities, the classifying space $\Omega \cong \hat{G}$ as the group dual $\hat{G}$ representing the inequivalent situations of this dynamical system $\mathcal{D} = \mathcal{A} \triangleleft G$.

For instance, in the case that the group G describes a flat Minkowski spacetime (x^μ), $\hat{G} \cong \Omega$ represents the space of energy-momentum spectrum (p_μ) as its dual. In the above correspondence, what is to be remarked is the opposite directions of the functors $\gamma : \mathcal{D} \longrightarrow \Omega$ appearing in the Kan extension:

$$\begin{array}{ccc} & & \Omega \\ & \nearrow \gamma & \\ \mathcal{D} & & \end{array}$$

and of $g : \Omega \longrightarrow \mathcal{D}$ appearing in the Micro-Macro duality:

$$\begin{array}{ccc} & & \Omega \\ & \swarrow g & \\ \mathcal{D} = \mathcal{A} \triangleleft G & & \end{array}$$

It would be concise to relate $g = \gamma^{-1} : \Omega = (p_\mu) = (\hat{x_\mu}) \longrightarrow \mathcal{D} = \mathcal{A} \triangleleft G$ to the quantum gravity which is defined on the dual of the domain Ω $[(x^\mu) = \hat{\Omega} \longrightarrow \mathcal{D} = \mathcal{A} \triangleleft G]$ and which takes the value in the non-commutative (=quantum) algebra $\mathcal{A}$ or $\mathcal{D} = \mathcal{A} \triangleleft G$.

When the algebra $\mathcal{A}$ acted by G is neglected, it is natural that the morphisms γ, g are mutually inverse, and hence, it is unexpectedly concluded that the quantum gravity g defined on the classical spacetime $\hat{\Omega}$ with values in the non-commutative algebra $\mathcal{A}$ or $\mathcal{D}$ is injective. At the end, we recall that there remains an unsolved question whether the Kan extension appearing above is the right Kan extension $Ran_\omega(\gamma)$ or the left Kan extension $Lan_\omega(\gamma)$ characterized by the directions of the relevant natural transformations as follows:
the right Kan extension $Ran_\omega(\gamma) : \mathcal{E} \longrightarrow \Omega$ of γ satisfying

$$Nat(d \longrightarrow Ran_\omega(\gamma)) \cong Nat(d\omega \longrightarrow \gamma)$$

or the left Kan extension $Lan_\omega(\gamma) : \mathcal{E} \longrightarrow \Omega$ of γ satisfying

$$Nat(Lan_\omega(\gamma) \longrightarrow d) \cong Nat(\gamma \longrightarrow d\omega)$$

for arbitrary functor $d : \mathcal{E} \longrightarrow \Omega$. The question is as to which of the above two possibilities $Ran_\omega(\gamma)$ or $Lan_\omega(\gamma)$ is realized and as to what is the difference in their physical meaning?

[1] Mac Lane, S., Categories for the working mathematician, Springer-Verlag, 1971.

[2] Ojima, I., A unified scheme for generalized sectors based on selection criteria –Order parameters of symmetries and of thermal situations and physical meanings of classifying categorical adjunctions–, Open Sys. Info. Dyn. 10, 235-279 (2003).