On the evaluation of the multiplicity of the ground states for quantum field models

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[Abstract] We consider an abstract interaction system of Bose fields. The Hilbert space for the system is defined by a tensor product of a Hilbert space and boson Fock space. It is proven that if the total Hamiltonian has a ground state, its multiplicity is finite.

1 Main Result

In this article, we consider an abstract interaction system of Bose fields. The Hilbert space for the system is given by

$$\mathcal{H} = \mathcal{K} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d)),$$

where $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ denotes the boson Fock space over $L^2(\mathbb{R}^d)$. The free Hamiltonian is given by

$$H_0 = K \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma_{\mathrm{b}}(\omega),$$

where $d\Gamma_{\mathrm{b}}(\omega)$ is the second quantization of $\omega = \omega(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^d$. We assume that K and ω satisfy the condition below, respectively.

(A.1) K is self-adjoint and bounded from below.

(A.2) $\omega \in C(\mathbb{R}^d)$ and $\inf_{\mathbf{k} \in \mathbb{R}^d} \omega(\mathbf{k}) \geq 0$.

We denothe the annihilation operator by $a(f)$ and the creation operator by $a^\dagger(f)$. The creation operators and annihilation operators satisfy the canonical commutation relation

$$[a(f), a^\dagger(g)] = (f, g),$$
$$[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0.$$

The total Hamiltonian is defined by

$$H_\kappa = H_0 + \kappa H_{\mathrm{I}},$$

where $\kappa \in \mathbb{R}$ is called coupling constant.

We suppose the condition below.

(A.3) H_{I} is symmetric.

(A.4) For all $\kappa \in \mathbb{R}$, H_κ is self-adjoint.

For a self-adjoint operator H, we denote the lowest spectrum by $E_0(H) = \inf(\sigma(H))$, and we say that H has a ground state if $E_0(H)$ is an eigenvalue of H. We are interested in the multiplicity of the ground state of H_κ. The difficulty of the analysis is that the ground state is an eigenvalue which is embedded in continuous spectrum (refer to e.g., [1, 5]).

Suppose the condition below.

(A.5) K has a compact resolvent.

(A.6) There exists $Q_{\mathrm{I}} : \mathbb{R}^d \to \mathcal{L}(\mathcal{H})$ such that for all $\Phi, \Psi \in \mathcal{D}(H_0)$,

$$[H_{\mathrm{I}}, \mathbb{1} \otimes a(f)]^0(\Phi, \Psi) = \int_{\mathbb{R}^d} f(\mathbf{k})^* (\Phi, Q_{\mathrm{I}}(\mathbf{k})) d\mathbf{k}.$$

where $[X,Y]^0(\Phi,\Psi)$ is the weak commutator and $a(\mathbf{k})$ is the operator kernel of the annihilation operator. In addition, there exists $\zeta(\kappa) \geq 0$ and $\tau(\kappa) \geq 0$ such that for all $\Phi \in \mathcal{D}(H_0)$,

$$\int_{\mathbb{R}^d} \frac{1}{\omega(\mathbf{k})^2} \| Q_{\mathrm{I}}(\mathbf{k}) \Phi \|^2 d\mathbf{k} \leq \zeta(\kappa) \| H_\kappa \Phi \|^2 + \tau(\kappa) \| \Phi \|^2.$$

(A.7) $Q_{\mathrm{I}}(\mathbf{k})$ is strongly differentiable and there exists $\mu_l(\kappa) \geq 0$ and $\nu_l(\kappa) \geq 0$, for each $l = 0, \ldots d$, such that for all $\Phi \in \mathcal{D}(H_0)$,

$$\int_{\mathbb{R}^d} \frac{1}{\omega(\mathbf{k})^4} \| Q_{\mathrm{I}}(\mathbf{k}) \Phi \|^2 d\mathbf{k} \leq \mu_0(\kappa) \| H_\kappa \Phi \|^2 + \nu_0(\kappa) \| \Phi \|^2,$$

and

$$\int_{\mathbb{R}^d} \frac{1}{\omega(\mathbf{k})^2} \| \partial_{k_j} Q_{\mathrm{I}}(\mathbf{k}) \Phi \|^2 d\mathbf{k} \leq \mu_j(\kappa) \| H_\kappa \Phi \|^2 + \nu_j(\kappa) \| \Phi \|^2.$$

The following is the main result.

Suppose **(A.1)-(A.7)**. Then if H_κ has a ground state, its multiplicity is finite.

[Remark]

If we suppose $|\kappa|$ is sufficiently small, then it follows that dim ker $(H_\kappa - E_0(H_\kappa))$ $\leq$ dim ker $(K - E_0(K))$ from the main result in [6].

[Outline of Proof]

The strategy is to use a method considered in [7], which is a combined technics for showing the the existence of the ground state for all valued coupling constants in [3, 4] and it is proven that the multiplicity of the ground state for the quantum electrodynamics is finite for all values of coupling constants. Let us suppose that H_κ has a ground state with dim ker $(H_\kappa - E_0(H_\kappa)) = \infty$. Then there exists a ground state $\Omega_{\kappa,j}$, $j \in \mathbb{N}$, such that $H_\kappa \Omega_{\kappa,j} = E_0(H_\kappa)\Omega_{\kappa,j}$ and $\|\Omega_{\kappa,j}\| = 1$. We see that w-$\lim\limits_{j\to\infty} \Omega_{\kappa,j} = 0$. On the other hand, using the combined method above, we can prove w-$\lim\limits_{j\to\infty} \Omega_{\kappa,j} \neq 0$, and it yields a contradiction. Hence dim ker $(H_\kappa - E_0(H_\kappa)) < \infty$.

[Application]

We shortly introduce the generalized the spin-boson model [2]. The Hilbert space for the system is given by $\mathcal{H}_{\mathrm{GSB}} = \mathcal{K} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ and the total Hamiltonian by

$$H_{\mathrm{GSB}}(\kappa) \;=\; A \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma_{\mathrm{b}}(\omega) + \kappa \sum_{j=1} B_j \otimes \phi(f_j).$$

The precise conditions are not explained here, but the main result can be applied to this model.

謝辞

This work was supported by JSPS KAKENHI 20K03625 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

参考文献

[1] 新井朝雄、量子場の数理解析：歴史的概観と新しい漸近的摂動論、**数学**、**69** (2017) 255-279.

[2] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151**,455-503 (1997).

[3] C.Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, *Ann. H. Poincaré*, **1**, 443-459, (2000).

[4] M.Griesemer, E.Lieb and M.Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* **145**, 557-595, (2001).

[5] 廣島文生, 場の理論における埋蔵固有値の摂動問題, **数学**、**57** (2005) 70-92.

[6] F. Hiroshima, Multiplicity of ground states in quantum field models; applications of asymptotic fields, *J. Funct. Anal.* **224**, 431-470, (2005).

[7] T. Takaesu, Ground states of quantum electrodynamics with cutoffs, *Journal of Mathematical Physics* **59**, 072301, (2018).

[8] 高江洲俊光、相互作用する場の量子論の系の基底状態の多重度の評価について, 数理解析研究所考究録, **2257**, 54-56, (2023)

[9] T. Takaesu, On the evaluation of the multiplicity of the ground states for quantum field models, (in preparation)

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