DEFINABLE MORSE FUNCTIONS ON DEFINABLY COMPACT MANIFOLDS IN D-MINIMAL STRUCTURES

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Abstract. Consider a definable complete d-minimal expansion $(F, <, +, \cdot, 0, 1, \dots,)$ of an oredered field F. Let M be a definably compact definable C^rsubmanifold of F^n and $2 \leq r < \infty$. We prove that the set of definable Morse functions is open and dense in the set of definable C^r functions on M with respect to the definable C^2 topology.

1. Introduction

In Morse theory the topological deta of a space can be described by Morse functions defined on the space. We refer the readers to a famous book by J. Milnor [11] for Mose theory on compact C^∞ manifolds.

Let $\mathcal{M} = (F, +, \cdot, <, \dots)$ be a d-minimal expansion of an ordered field F. Everything is considered in $\mathcal{M}$, the term "definable" is used in the sense of "definable with parameters in $\mathcal{M}$", each definable map is assumed to be continuous and $2 \leq r < \infty$.

Definable C^r Morse functions in an o-minimal expansion of the standard structure of a real closed field are considered in [12], [7].

In this paper we consider definable C^r Morse frunctions on a definably compact definable submanifold in d-minimal expansions of an ordered field when $2 \leq r < \infty$. These structures are generalizations of o-minimal expansions of real closed fields. Definable compactness is a generalization of compactness. If $F = \mathbb{R}$, then for a definable subset of $\mathbb{R}^n$, it is definably compact if and only if it is compact.

Definable C^r manifolds are studied in [12], [1] [7].

Theorem 1.1 (Theorem 2.14) is our main result.

Theorem 1.1. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a definably compact $\mathcal{D}^2$ submanifold of F^n. Then the set of all definable Morse functions on M is open and dense in the set $\mathcal{D}^2(M)$ of $\mathcal{D}^2$.functions of M.*

Theorem 1.1 is a generalization of [7].

2. Preliminary and Proof of Theorem 1.1

Recall the definitions of d-minimality and definably completeness.

Definition 2.1. An expansion of a dense linear order without endpoints $\mathcal{F} = (F, <, \dots)$ is *definably complete* if any definable subset X of F has the supremum and infimum in $F \cup \{\pm\infty\}$ ([9]). A definably complete expansion $\mathcal{F}$ is *d-minimal* if for

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every m and definable subset A of F^{m+1}, there exists an $N \in \mathbb{N}$ such that for every $x \in F^m$ the set $\{y \in F | (x,y) \in A\}$ has non-empty interior or a union of at most N discrete sets ([3], [10]).

The definition of dimension is found in [5, Definition 3.1].

Definition 2.2 (Dimension)**.** Consider an expansion of a densely linearly order without endpoints $\mathcal{F} = (F, <, \ldots)$. Let M be a nonempty definable subset of F^n. The dimension of M is the maximal nonnegative integer d such that $\pi(M)$ has a nonempty interior for some coordinate projection $\pi : F^n \to F^d$. We consider that F^0 is a singleton with the trivial topology. We set $\dim(M) = -\infty$ when M is an empty set.

Definition 2.3 ([6])**.** Consider a definably complete expansion $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$ of an ordered field. Let r be $1 \leq r < \infty$.

(1) A definable subset X of F^n is a *d-dimensional $\mathcal{D}^r$ submanifold* of F^n if each point $x \in X$ there exist an open box U_x of x in F^n and a $\mathcal{D}^r$ diffeomorphism ϕ_x from U_x to some open box V_x of F^n such that $\phi_x(x) = 0$ and $U_x \cap Y = \phi_x^{-1}(F^d \cap V_x)$, where $F^d \subset F^n$ is the vectors whose last $(n-d)$ components are zero.

(2) A function $f : X \to F$ is a *$\mathcal{D}^r$ function* if for any $x \in X$ and the open box in (1), $f \circ \phi_x^{-1}$ is a $\mathcal{D}^r$ fuinction.

Definition 2.4. For a set X, a family $\mathcal{K}$ of subsets of X is called a *filtered collection* if, for any $B_1, B_2 \in \mathcal{K}$, there exists $B_3 \in \mathcal{K}$ with $B_3 \subseteq B_1 \cap B_2$.

Consider an expansion of a dense linear order without endpoints $\mathcal{F} = (F; <, \ldots)$. Let X and T be $\mathcal{D}^r$ manifolds. The parameterized family $\{S_t\}_{t \in T}$ of definable subsets of X is called *definable* if the union $\bigcup_{t \in T} \{t\} \times S_t$ is definable in $T \times X$.

A parameterized family $\{S_t\}_{t \in T}$ of definable subsets of X is a *definable filtered collection* if it is simultaneously definable and a filtered collection.

A definable space X is *definably compact* if every definable filtered collection of closed nonempty subsets of X has a nonempty intersection. This definition is found in [8, Section 8.4].

Definition 2.5. Consider a d-minimal expansion of an ordered field whose universe is F. Let $\pi : F^n \to F^d$ be a coordinate projection. A $\mathcal{D}^r$ submanifold M of F^n of dimension d is called a *$\mathcal{D}^r$ multi-valued graph (with respect to π)* if, for any $x \in M$, there exist an open box U in F^n containing the point x and a $\mathcal{D}^r$ map $\tau : \pi(U) \to F^n$ such that $M \cap U = \tau(\pi(U))$ and $\pi \circ \tau$ is the identity map defined on $\pi(U)$.

Lemma 2.6. *Consider a d-minimal expansion of an ordered field whose universe is F. Let M be a $\mathcal{D}^r$ submanifold M of dimension d. Let $\Pi_{n,d}$ be the set of coordinate projections from F^n onto F^d. Let U_π be the set of points x at which there exist an open box U in F^n containing the point x and a $\mathcal{D}^r$ map $\tau : \pi(U) \to F^n$ such that $M \cap U = \tau(\pi(U))$ and $\pi \circ \tau$ is the identity map defined on $\pi(U)$. Then U_π is a $\mathcal{D}^r$ multi-valued graph with respect to π and $\{U_\pi\}_{\pi \in \Pi_{n,d}}$ is a definable open cover of M.*

Proof. It is obvious that U_π is a $\mathcal{D}^r$ multi-valued graph with respect to π. It is also obvious that U_π is open in M. The family $\{U_\pi\}_{\pi \in \Pi_{n,d}}$ is a definable open cover of M by [4, Lemma 3.5, Corollary 3.8]. □

Proposition 2.7 (Definable Sard)**.** *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a $\mathcal{D}^1$ submanifold of F^m of dimension d and $f = (f_1, \dots, f_n) : M \to F^n$ be a $\mathcal{D}^1$ map. The set of critical values of f is definable and of dimension smaller than n.*

Proof. The proposition is trivial when $d < n$ by [3, Lemma 4.5]. We consider the case in which $d \geq n$ in the rest of proof.

We denote the set of critical values of f by Σ_f. The $\mathcal{D}^1$ manifold M is covered by finitely many $\mathcal{D}^1$ multi-valued graphs $U_1, \dots, U_k$ by Lemma 2.6. The equality $\Sigma_f = \bigcup_{i=1}^k \Sigma_{f|_{U_i}}$ obviously holds, where $f|_{U_I}$ is the restriction of f to U_i. The set Σ_f is definable if $\Sigma_{f|_{U_i}}$ is definable for every $1 \leq i \leq k$. We have $\dim \Sigma_f = \max\{\dim \Sigma_{f|_{U_i}} \mid 1 \leq i \leq k\}$ by [3, Lemma 4.5]. Therefore, we may assume that M is a $\mathcal{D}^1$ multi-valued graph with respect to a coordinate projection π. We may further assume that π is the projection onto the first d coordinates by permuting the coordinates if necessary.

By the definition of $\mathcal{D}^1$ multi-valued graphs, for any $a \in M$, there exists a nonempty open box B such that $M \cap B$ is the graph of $\mathcal{D}^1$ map defined on $\pi(B)$. In particular, the restriction of π to $M \cap B$ is a $\mathcal{D}^1$ diffeomorphism onto $\pi(B)$. The global coordinate functions $x_1, \dots, x_d$ are local coordinates of M at a. Let $Df : M \to F^{d \times n}$ be the map giving the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})_{1 \leq i \leq n, 1 \leq j \leq d}$. Set $\Gamma_f = \{x \in M \mid \operatorname{rank}(Df(x)) < n\}$. The set Γ_f is definable. The set Σ_f is also definable because the equality $\Sigma_f = f(\Lambda_f)$ holds, where Λ_f denotes the set of critical points of f.

Assume for contradiction that $\dim \Sigma_f = n$. The definable set Σ_f contains a nonempty open box U. We can take a definable map $g : U \to \Sigma_f$ such that $f \circ g$ is an identity map on U. We may assume that g is of class $\mathcal{C}^1$ by [3, Lemma 3.14] by shrinking U if necessary. By differentiation, the matrix$Df(g(x)) \cdot Dg(x)$ is the identity matrix of size n. It implies that $Df(g(x))$ has rank at least n, which contradicts the definition of Γ_f. ☐

Lemma 2.8. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let π be the coordinate projection of F^n onto the first d coordinates. Let U be a $\mathcal{D}^2$ multi-valued graph with respect to π and $f : U \to F$ be a $\mathcal{D}^2$ map. We can find $a_1, \dots, a_d \in F$ such that the definable function $\Phi : U \to F$ given by $\Phi(x_1, \dots, x_n) = f(x_1, \dots, x_n) - \sum_{i=1}^d a_i x_i$ is a Morse function and $|a_1|, \dots, |a_d|$ are sufficiently small.*

Proof. Observe that the global coordinate functions $x_1, \dots, x_d$ in F^n is a local coordinate function of U by the definition of multi-valued graphs as proven in the proof of Proposition 2.7. Consider the map $H : U \to F^d$ given by $H(x) = (\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x))$. Observe that $p_0 \in U$ is a critical point of H if and only if $\det(H_f)(p_0) = \det(H_\Phi)(p_0) = 0$, where H_f and H_Φ are Hessians of f and Φ, respectively. Let $(a_1, \dots, a_d)$ be the point in $F^d \setminus \Sigma_H$, where Σ_H denotes the set of critical values of H. We can choose such $(a_1, \dots, a_d)$ so that $|a_1|, \dots, |a_d|$ are sufficiently small because Σ_H has an empty interior by Proposition 2.7. It is easy to check that Φ is a Morse function. We omit the details. ☐

Lemma 2.9. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a $\mathcal{D}^r$ submanifold of F^m with $0 \leq r < \infty$. Given a* *definable closed subset X of M, there exists a D^r function $f : M \to F$ whose zero set is X.*

Proof. Consider the closure $\mathrm{cl}(X)$ of X in F^m. There exists a $\mathcal{D}^r$ function $G : F^m \to F$ with $G^{-1}(0) = \mathrm{cl}(X)$ by [10]. The restriction of G to M satisfies the requirement. □

Lemma 2.10. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a definable $\mathcal{D}^r$ submanifold of F^m with $0 \leq r < \infty$. Let X and Y be closed definable subsets of M with $X \cap Y = \emptyset$. Then, there exists a $\mathcal{D}^r$ function $f : M \to [0,1]$ with $f^{-1}(0) = X$ and $f^{-1}(1) = Y$.*

Proof. There exist $\mathcal{D}^r$ functions $g, h : M \to F$ with $g^{-1}(0) = X$ and $h^{-1}(0) = Y$ by Lemma 2.9. The function $f : M \to [0,1]$ defined by $f(x) = \frac{g(x)^2}{g(x)^2 + h(x)^2}$ satisfies the requirement. □

Lemma 2.11. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a $\mathcal{D}^r$ submanifold with $0 \leq r < \infty$. Let C and U be closed and open definable subsets of M, respectively. Assume that C is contained in U. Then, there exists an open definable subset V of M with $C \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.*

Proof. There is a definable continuous function $h : M \to [0,1]$ with $h^{-1}(0) = C$ and $h^{-1}(1) = M \setminus U$ by Lemma 2.10. The set $V = \{x \in M \,;\, h(x) < \frac{1}{2}\}$ satisfies the requirement. □

Lemma 2.12 (Fine definable open covering)**.** *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a $\mathcal{D}^r$ submanifold with $0 \leq r < \infty$. Let $\{U_i\}_{i=1}^q$ be a finite definable open covering of M. For each $1 \leq i \leq q$, there exists a definable open subset V_i of M satisfying the following conditions:*

(a) the closure $\mathrm{cl}(V_i)$ in M is contained in U_i for each $1 \leq i \leq q$, and
(b) the collection $\{V_i\}_{i=1}^q$ is again a finite definable open covering of M.

Proof. We inductively construct V_i so that $\mathrm{cl}(V_i) \subset U_i$ and $\{V_i\}_{i=1}^{k-1} \cup \{U_i\}_{i=k}^q$ is a finite definable open covering of M. We fix a positive integer k with $k \leq q$. Set $C_k = M \setminus (\bigcup_{i=1}^{k-1} V_i \cup \bigcup_{i=k+1}^q U_i)$. The set C_k is a definable closed subset of M contained in U_k. There exists a definable open subset V_k of M with $C_k \subseteq V_k \subseteq \mathrm{cl}(V_k) \subseteq U_k$ by Lemma 2.11. It is obvious that $\{V_i\}_{i=1}^k \cup \{U_i\}_{i=k+1}^q$ is a finite definable open covering of M. □

Definition 2.13. Let $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$ be a d-minimal expansion of an ordered field. Let M be a $\mathcal{D}^r$ submanifold of F^n and $\mathcal{D}^r(M)$ be the set of $\mathcal{D}^r$ functions. The space $\mathcal{D}^r(M)$ equips the topology defined in [2].

Theorem 2.14. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \dots)$. Let M be a definably compact $\mathcal{D}^2$ submanifold of F^n. The set of all definable Morse functions on M is open and dense in $\mathcal{D}^2(M)$.*

Proof. We first prepare several notations and define several sets and maps for later use. Set $d = \dim M$. The $\mathcal{D}^2$ submanifold M is covered by finitely many $\mathcal{D}^2$ multi-valued graphs $U_1, \dots, U_k$ by Lemma 2.6. Let U_i be a $\mathcal{D}^2$ multi-valued graph with respect to a coordinate projection $\pi_i : F^n \to F^d$ for $1 \leq i \leq k$. Let $\{V_i\}_{i=1}^k$ be a fine definable open covering of $\{U_i\}_{i=1}^k$ given in Lemma 2.12. Set $C_i = \mathrm{cl}(V_i)$.

Observe that $C_i \subseteq U_i$ and $M = \bigcup_{i=1}^k C_i$. The definable set C_i is a closed subset of M. It deduces that C_i is definably compact. By Lemma 2.10, we can take a definable function $\lambda_i : M \to [0,1]$ such that $\lambda_i^{-1}(0) = M \setminus U_i$ and $\lambda_i^{-1}(1) = C_i$.

Let D_i be the $\mathcal{D}^1$ vector field such that $D_i(x)$ is the projection image of $\partial/\partial x_i$ onto the tangent space T_xM of M at $x \in M$. We can find a subset I_i of $\{1,\ldots,n\}$ of cardinality d such that π_i is the projection onto the d coordinates $(x_j \mid j \in I_i)$. The j-th smallest element in I_i is denoted by $\sigma_i(j)$. By the definition of multi-valued graphs, T_xM is spanned by $(D_j(x) \mid j \in I_i)$ for every $x \in U_i$ and for every $1 \leq i \leq k$.

Let $\mathcal{U}$ be the set of all definable Morse functions on M. We first show that $\mathcal{U}$ is open. We prove a stronger claim for later use. For every nonempty subset I of $\{1,\ldots,k\}$, we set

$$\mathcal{U}_I := \{f \in \mathcal{D}^2(M) \mid f \text{ has no degenerate critical points on } C_i \text{ for each } i \in I\}.$$

We prove that $\mathcal{U}_I$ is open. Take an arbitrary definable Morse function $h : M \to F$. We set $h_j^i = D_{\sigma_i(j)}h$ for $1 \leq j \leq d$ and $i \in I$. We set $H_i = \det(D_{\sigma_i(j_1)}D_{\sigma_i(j_2)}h)_{1\leq j_1,j_2\leq d}$. They are definable continuous functions defined on M. Since h has no degenerate critical points on C_i for $i \in I$ and the coordinates $x_{\sigma_i(1)},\ldots,x_{\sigma_i(d)}$ give a local coordinate of M at $x \in U_i$, the function $G_i := \sum_{j=1}^d |h_j^i| + |H_i|$ is positive on U_i for $i \in I$. In particular, we can find a positive $K_i \in F$ such that $G_i > K_i$ on the definably compact definable set C_i for $i \in I$. We can take $L_i > 0$ so that $|D_{\sigma_i(j_1)}D_{\sigma_i(j_2)}h| < L_i$ on C_i. Take a sufficiently small $\varepsilon > 0$ so that $d!((L_i+\varepsilon)^d - L_i^d) + d\varepsilon < K_i$ for every $i \in I$. Consider the open set

$$\begin{aligned}\mathcal{V}_{h,\varepsilon} = \{g \in \mathcal{D}^2(M) \mid\ & |g-h| < \varepsilon, |D_j(g-h)| < \varepsilon\ (1 \leq j \leq d),\\ & |D_{j_1}D_{j_2}(g-h)| < \varepsilon\ (1 \leq j_1, j_2 \leq d)\}\end{aligned}$$

in $\mathcal{D}^2(M)$. We can verify that $\sum_{j=1}^d |D_{\sigma_i(j)}g| + |\det(D_{\sigma_i(j_1)}D_{\sigma_i(j_2)}g)_{1\leq j_1,j_2\leq d}| > 0$ on C_i for every $i \in I$ and $g \in \mathcal{V}_{h,\varepsilon}$. It deduces that $\mathcal{V}_{h,\varepsilon} \subseteq \mathcal{U}_I$ and $\mathcal{U}_I$ is open.

We next show that $\mathcal{U}$ is dense in $\mathcal{D}^2(M)$. We first take arbitrary $h \in \mathcal{D}^2(M)$. We define an open set $\mathcal{V}_{h,\varepsilon}$ for positive $\varepsilon \in F$ in the same manner as above. Since M is definably compact, every positive definable continuous function is bounded from below by a positive constant. This deduces that $\{\mathcal{V}_{h,\varepsilon}\}_{\varepsilon>0}$ is a basis of open neighborhoods of h in $\mathcal{D}^2(M)$.

Fix an arbitrary positive $\varepsilon \in F$ and set $\varepsilon' = \varepsilon/k$. We have only to construct a function $g \in \mathcal{U} \cap \mathcal{V}_{h,\varepsilon}$ so as to show that $\mathcal{U}$ is dense in $\mathcal{D}^2(M)$. Set $\mathcal{U}_i := \mathcal{U}_{\{1,\ldots,i\}}$ for $1 \leq i \leq k$ for simplicity. We construct $g_i \in \mathcal{U}_i \cap \mathcal{V}_{h,i\varepsilon'}$. It is obvious that $g := g_k \in \mathcal{U} \cap \mathcal{V}_{h,\varepsilon}$.

We construct g_i by induction on i. We may assume that π_i is the projection onto the first d coordinates by permuting the coordinates if necessary. We first consider the case in which $i = 1$. We can find $a_1,\ldots,a_d$ such that $|a_j| < \varepsilon'$ for $1 \leq j \leq d$ and $g_1(x) = h(x) + \sum_{i=1}^d a_i x_i$ is a Morse function by Lemma 2.8. It is obvious that $g_1 \in \mathcal{U}_1 \cap \mathcal{V}_{h,\varepsilon'}$.

We next consider the case in which $i > 1$. We can find $g_{i-1} \in \mathcal{U}_{i-1} \cap \mathcal{V}_{h,(i-1)\varepsilon'}$ by induction hypothesis. We construct $g_i \in \mathcal{U}_i \cap \mathcal{V}_{g_{i-1},\varepsilon'}$. Such a g_i obviously belongs to $\mathcal{U}_i \cap \mathcal{V}_{h,i\varepsilon'}$. We have already shown that $\mathcal{U}_{i-1}$ is open. Therefore, we can find $\delta > 0$ such that $\mathcal{V}_{g_{i-1},\delta} \subseteq \mathcal{U}_{i-1} \cap \mathcal{V}_{g_{i-1},\varepsilon'}$ because $\{\mathcal{V}_{g_{i-1},\varepsilon''}\}_{\varepsilon''>0}$ is a basis of open neighborhoods of g_{i-1} in $\mathcal{D}^2(M)$.

Set $g_i := g_{i-1}+\lambda_i\cdot(\sum_{i=1}^d a_ix_i)$ for $a_1,\ldots,a_d \in F$. We want to choose $a_1,\ldots,a_d \in F$ satisfying the following conditions:

(1) $g'_i := g_{i-1} + \sum_{i=1}^d a_ix_i$ has no degenerate critical points in U_i.
(2) $g_i \in \mathcal{V}_{g_{i-1},\delta}$.

We check that g_i belong to $U_i \cap \mathcal{V}_{g_{i-1},\varepsilon'}$ when $a_1,\ldots,a_d$ satisfy the above conditions (1) and (2). It is obvious that $g_i \in V_{g_{i-1},\varepsilon'}$ by the inclusion $\mathcal{V}_{g_{i-1},\delta} \subseteq \mathcal{V}_{g_{i-1},\varepsilon'}$. The inclusion $\mathcal{V}_{g_{i-1},\delta} \subseteq \mathcal{U}_{i-1}$ implies that g_i has no degenerate critical points on C_j for $1 \leq j \leq i-1$. Since λ_i is identically one on C_i, we have $g_i = g'_i$ on C_i. Condition (1) implies that g_i has no degenerate critical points in C_i. We have shown that g_i has no degenerate critical points on C_j for $1 \leq j \leq i$, and this means $g_i \in \mathcal{U}_i$.

The remaining task is to find $a_1,\ldots,a_d \in F$ so that conditions (1) and (2) are satisfied. The following inequalities are satisfied:

$$|g_i - g_{i-1}| \leq \sum_{l=1}^d |a_l||\lambda_i x_l| < K\cdot\sum_{j=1}^k |a_j|$$

$$|D_j(g_i - g_{i-1})| \leq \sum_{l=1}^d |a_l||D_j(\lambda_i x_l)| < K\cdot\sum_{j=1}^k |a_j|$$

$$|D_{j_1}D_{j_2}(g_i - g_{i-1})| \leq \sum_{l=1}^d |a_l||D_{j_1}D_{j_2}(\lambda_i x_l)| < K\cdot\sum_{j=1}^k |a_j|$$

Finitely many definable continuous functions $|\lambda_ix_l|$, $|D_j(\lambda_ix_l)|$ and $|D_{j_1}D_{j_2}(\lambda_ix_l)|$ defined on M appear in the above calculation. Since M is definably compact, we can find $0 < K \in F$ such that these functions are bounded above by K in M. We used this fact in the calculation. We can find $(a_1,\ldots,a_d)$ so that $K\cdot\sum_{j=1}^k |a_j| < \delta$ and g'_i has no degenerate critical points in U_i by Lemma 2.8. This $(a_1,\ldots,a_d)$ satisfies conditions (1) and (2). $\square$

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