Supplementing dimension function in d-minimal structures

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概要

The author recently proved the existence of definable quotient of a definable equivalence relation and definable $\mathcal{C}^r$ group structure on a definable topological group in d-minimal structures. They are generalization of the results on definably complete locally o-minimal structures. In these studies, the author developed measures which supplement dimension function. We expect that these measures are useful for other studies on d-minimal structures. We introduce these measures in this paper.

1 Introduction

The author has recently tried to generalize the geometric assertions for o-minimal structures and definably complete locally o-minimal structures to d-minimal structures. For instance, the existence of definable quotient was first studied by Brumfel [1] in the semialgebraic category and it was extended to o-minimality by van den Dries in [3]. Scheiderer showed Brumfel's result holds under a relaxed assumption in semialgebraic category [18], and the author and Kawakami extended Scheiderer's result to definably complete o-minimal case in [10]. Finally, the author generalized their result to the d-minimal case in [7].

In Pillay's seminal paper [16], he proved that every definable group has a 'natural'

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definable topology under which group operations are continuous. His result was generalized to more general structures satisfying several technical conditions on dimension function by Wencel [20]. It is not a hard task to show that dimension function satisfies Wencel's requirements in definably complete locally o-minimal structures using the dimension formulas in [6, 11, 9]. The author proved an assertion similar but not identical to Wencel's result in d-minimal expansions of ordered fields [8]. He showed that a definable topological group has a definable $\mathcal{C}^r$ structure under which group operations are of class $\mathcal{C}^r$.

In both studies [7, 8], the author needed to develop new measures which supplement dimension function, though we could complete the proofs using dimension function alone in previous studies. This is dues to the absence of 'continuity property' (and 'strong frontier formula') of dimension function in d-minimal structures though continuity property is enjoyed both in o-minimal structures and definably complete locally o-minimal structures. The author expects that these two measures are also useful in other studies on d-minimal structures. The purpose of this proceeding is to introduce these two new measures, that is, extended rank and partition degree.

In this note, we only consider expansions of dense linear orders without endpoints. Under this setting, we may assume that the underlying space, say F, is a topological space equipped with the order topology. Since any definable set is a subset of F^n for some positive integer n, every definable set is equipped with the topology induced from the product topology on F^n. Unless we explicitly give a topology on a definable set, we assume that the definable set is equipped with the induced topology.

2 Dimension formulas

This section is a preliminary section. We first recall several basic definitions.

Definition 2.1 ([3, 19, 13, 5])**.** An expansion of a dense linear order without endpoints $\mathcal{F} = (F, <, \dots)$ is *o-minimal* if every definable subset of F is a finite union of points and open intervals.

The expansion $\mathcal{F}$ is *locally o-minimal* if, for every definable subset X of F and for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is the union of finitely many points and open intervals.

The expansion $\mathcal{F}$ is *definably complete* if any definable subset X of F has the supremum and infimum in $M \cup \{\pm\infty\}$.

The structure $\mathcal{F}$ is *d-minimal* if it is definably complete, and every definable subset X of F is the union of an open set and finitely many discrete sets, where the number of discrete sets is bounded by the number which does not depend on the parameters of definition of X.

An o-minimal structure is always a definably complete locally o-minimal structure, and a definably complete locally o-minimal structure is always d-minimal. A standard textbook for o-minimal structure is [3]. The papers [12, 17] also treat o-minimality. See [19, 4, 6, 11] for the basic properties of sets definable in locally o-minimal structures. The author recommends [5, 14] for d-minimal structures.

We next recall the definition of dimension.

Definition 2.2 (Dimension)**.** Consider an expansion of a dense linear order without endpoints $\mathcal{F} = (F, <, \ldots)$. Let X be a nonempty definable subset of F^n. Recall that F^0 is a singleton with the trivial topology. The dimension of X is the maximal nonnegative integer d such that $\pi(X)$ has a nonempty interior for some coordinate projection $\pi : F^n \to F^d$. We set $\dim(X) = -\infty$ when X is an empty set.

In o-minimal structures, definably complete locally o-minimal structures and d-minimal expansions of an ordered field, dimension function enjoys the following properties called van den Dries's requirements named after van den Dries's paper [2].

Definition 2.3. Consider a structure $\mathcal{F} = (F, \ldots)$. Let $\mathcal{D}$ be the set of all definable sets and $\mathbb{Z}_{\geq 0} := \{n \in \mathbb{Z} \mid n \geq 0\}$. A map $\dim : \mathcal{D} \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ *satisfies van den Dries's requirements* if the following conditions are satisfied:

(1) $\dim(S) = -\infty \Leftrightarrow S = \emptyset$; $\dim(\{x\}) = 0$ for all $x \in F$ and $\dim F = 1$.
(2) $\dim(S_1 \cup S_2) = \max\{\dim S_1, \dim S_2\}$.
(3) $\dim S^\sigma = \dim S$ for any definable set $S \subseteq F^n$ and any permutation σ of $\{1, \ldots, n\}$. $S^\sigma = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in F^n \mid (x_1, \ldots, x_n) \in S\}$.
(4) Let T be a definable subset of F^{n+1} and $T_x = \{y \in F \mid (x, y) \in T\}$ for any $x \in F^n$. $T(i) := \{x \in F^n \mid \dim(T_x) = i\}$ are definable and $\dim(\{(x, y) \in T \mid x \in T(i)\}) = \dim T(i) + i$.

When the dimension function satisfies van den Dries's requirement, the following formulas hold [2].

(1) Let $f : X \to F^n$ be a definable map. We have

$$\dim(f(X)) \leq \dim X.$$

(2) **(Addition formula)** Let $\varphi : X \to Y$ be a definable surjective map whose fibers are equi-dimensional. We have

$$\dim X = \dim Y + \dim \varphi^{-1}(y)$$

for all $y \in Y$.

We know that dimension function satisfies the following continuity property and strong frontier formula in definably complete locally o-minimal structures.

(1) **(Continuity property)** Let $f : X \to M^n$ be a definable map. Set $\mathcal{D}(f) := \{x \in X \mid f \text{ is discontinuous at } x\}$. The following inequality holds:

$$\dim(\mathcal{D}(f)) < \dim X$$

(2) **(Strong frontier formula)** Let ∂X be the frontier of a definable set X. We have

$$\dim(\partial X) < \dim X.$$

The continuity property (1) implies the inequality in (2). Consider the definable function which is one on X and zero on ∂X for instance. Neither properties (1) nor (2) are enjoyed in d-minimal structures. The dimension function behaves slightly wilder in d-minimal structures than definably complete locally o-minimal structures. In the application studies of definably complete locally o-minimal structures [20, 10], we use both van den Dries's requirements and continuity property. Recall that $\mathcal{D}$ is the set of all definable sets. We needed to develop some linearly ordered set $\mathcal{E}$ and some measures $m : \mathcal{D} \to \mathcal{E} \cup \{-\infty\}$ which satisfies at least one of the following conditions in order to conduct application studies in d-minimal structures under a similar strategy.

(A) Let $f : X \to F^n$ be a definable map. The following inequality holds:

$$m(\mathcal{D}(f)) < m(X). \tag{1}$$

(B) We have

$$m(\partial X) < m(X). \tag{2}$$

We introduce extended rank in Section 3 and partition degree in Section 4. Extended rank satisfies the inequality (2) and the pair of dimension and partition degree satisfies the inequality (1).

3 Extended rank

We introduce the notion of extended rank in this section. The facts introduced in this section are found in [7] with proofs. In order to give the definition of extended rank, we first recall Cantor-Bendixson rank.

Definition 3.1. We denote the set of isolated points in a topological space S by $\mathrm{iso}(S)$. We set $\mathrm{lpt}(S) := S \setminus \mathrm{iso}(S)$. In other word, a point $x \in S$ belongs to $\mathrm{lpt}(S)$ if and only if $x \in \mathrm{cl}_S(S \setminus \{x\})$.

Let X be a nonempty closed subset of a topological space S. We set $X[i]$ as follows for every nonnegative integer i:

$$X[0] = X$$
$$X[m] = \mathrm{lpt}(X[m-1])$$

We say that $\mathrm{rank}(X) = m$ if $X[m] = \emptyset$ and $X[m-1] \neq \emptyset$. We say that $\mathrm{rank}\, X = \infty$ when $X[m] \neq \emptyset$ for every natural number m.

Lemma 3.2. *Let $\mathcal{F} = (F, <, \ldots)$ be an expansion of a dense linear order without endpoints. For a definable closed subset A of F with empty interior,* $\mathrm{rank}(A) = k$ *if and only if k is the least number of discrete sets whose union is A.*

Recall that every definable subset of F with empty interior is a finite union of discrete set in d-minimal structures. This lemma indicates that the Cantor-Bendixson rank is a good supplement for definable subsets of F of dimension zero.

We need to extend this notion to handle definable sets of positive dimension. The following extended rank is defined so as to handle such definable sets.

Definition 3.3 (Extended rank)**.** Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$. Let $\Pi(n, d)$ be the set of coordinate projections of F^n onto

F^d. Recall that F^0 is a singleton. We consider that $\Pi(n, 0)$ is a singleton whose element is a trivial map onto F^0. Since $\Pi(n, d)$ is a finite set, we can define a linear order on it. We denote it by $<_{\Pi(n,d)}$. Let $\mathcal{E}_n$ be the set of triples (d, π, r) such that d is a nonnegative integer not larger than n, $\pi \in \Pi(n, d)$ and r is a positive integer. The linear order $<_{\mathcal{E}_n}$ on $\mathcal{E}_n$ is the lexicographic order. We abbreviate the subscript $\mathcal{E}_n$ of $<_{\mathcal{E}_n}$ in the rest of the paper, but it will not confuse readers. Let X be a nonempty bounded definable subset of F^n. The triple (d, π, r) is the *extended rank* of X and denoted by $\mathrm{eRank}_n(X)$ if it is an element of $\mathcal{E}_n$ satisfying the following conditions:

- $d = \dim X$;
- the projection π is a largest element in $\Pi(n, d)$ such that $\pi(X)$ has a nonempty interior;
- r is a largest positive integer such that there exists a definable open subset U of F^d contained in $\pi(X)$ such that the set $\pi^{-1}(x) \cap X$ is of dimension zero and the equality $\mathrm{rank}(\pi^{-1}(x) \cap X) = r$ holds for each $x \in U$.

Note that such a positive integer r exists by [5, Lemma 5.10]. We set $\mathrm{eRank}_n(\emptyset) = -\infty$ and define that $-\infty$ is smaller than any element in $\mathcal{E}_n$.

Let us consider the case in which X is an unbounded definable subset of F^n. Let $\varphi : F \to (-1, 1)$ be the definable homeomorphism given by $\varphi(x) = \frac{x}{\sqrt{1+x^2}}$. We define $\varphi_n : F^n \to (-1, 1)^n$ by $\varphi_n(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$. We set $\mathrm{eRank}_n(X) = \mathrm{eRank}_n(\varphi_n(X))$. We confirmed that the equation $\mathrm{eRank}_n(X) = \mathrm{eRank}_n(\varphi_n(X))$ holds when X is bounded.

In [7], the author first collects the desired properties of a new measure which are required to generalize the proof of [10] as follows: We then proved that the extended rank defined above satisfies the requirements.

Definition 3.4. Let $\mathcal{F} = (F, <, \ldots)$ be an expansion of a dense linear order without endpoints. Let $\mathrm{Def}(F^n)$ be the set of definable subsets of F^n. The structure $\mathcal{F}$ has a *good extended rank function* if, for each nonnegative integer n, there exists a map $\mathrm{eRank}_n : \mathrm{Def}(F^n) \to \mathcal{E}_n \cup \{-\infty\}$ satisfying the following conditions:

(a) The target space $\mathcal{E}_n$ is a linearly ordered set having a smallest element and $-\infty$ is smaller than any element in $\mathcal{E}_n$.

(b) $\mathrm{eRank}_n(X) = -\infty \Leftrightarrow X = \emptyset$.
(c) There is no infinite decreasing sequence $e_1 > e_2 > \cdots$ of elements in $\mathcal{E}_n$.
(d) $\mathrm{eRank}_n(A \cup B) = \max\{\mathrm{eRank}_n(A), \mathrm{eRank}_n(B)\}$
(e) Let $X, Y \in \mathrm{Def}(F^n)$ with $X \subseteq Y$. If $\mathrm{eRank}_n(X) = \mathrm{eRank}_n(Y)$, there exists a definable subset U of X such that U is open in Y and $\mathrm{eRank}_n(X \setminus U) < \mathrm{eRank}_n(Y)$. ($X$ coincides with Y except 'thin' subset.)
(f) A nonempty subset X of F^n belonging to $\mathrm{Def}(F^n)$ is discrete whenever $\mathrm{eRank}_n(X)$ is the smallest element in $\mathcal{E}_n$.

We simply denote $\mathrm{eRank}_n(X)$ by $\mathrm{eRank}(X)$ when n is clear from the context.

The following lemma says that the extended rank satisfies inequality (2).

Lemma 3.5. *Let $S \subseteq X$ be definable subsets of F^n. Then, the inequality $\mathrm{eRank}\, \partial_X S < \mathrm{eRank}\, S$ holds.*

Extended rank has several drawbacks. First of all, the definition of extended rank is ugly. The second weak point is that two definable sets in a common ambient space F^n are only comparable. Finally, in the case where X is a definably bijective to Y, we may have $\mathrm{eRank}_n(X) \neq \mathrm{eRank}_n(Y)$ though we have $\dim X = \dim Y$.

4 Partition degree

We want to introduce the notion of partition degree in this section. The facts given here are found in [8] with proofs. We first give its definition.

Definition 4.1. Let r be a positive integer. Consider a definably complete expansion of an ordered field. Let F be its universe. The *partition degree*, denoted by $\mathrm{p.deg}(X)$, of a nonempty definable subset X of F^n of dimension d is the minimum nonnegative integer m such that X is partitioned into $m+2$ definable subsets $X_{-1}, X_0, \dots, X_m$ satisfying the following conditions:

(1) $\dim X_{-1} < d$:
(2) the definable set X_i is a nonempty definable $\mathcal{C}^r$ submanifold of dimension d and it is open in $\bigcup_{j=-1}^{i} X_j$ for each $0 \leq i \leq m$.

Note that we do not require that X_{-1} is nonempty. The sequence $X_{-1}, X_0, \dots, X_m$ of definable subsets of X satisfying the above conditions is called an *r-partition sequence*.

The natural question is whether $\mathrm{p.deg}(X)$ is well-defined. In other word, does such a minimum integer m satisfying the conditions in the definition exist and does m depend on r even when such m exists? The following two assertions provide affirmative answers to this questions.

Proposition 4.2. *The partition degree* $\mathrm{p.deg}(X)$ *is independent of the choice of r.*

Theorem 4.3. *Consider a d-minimal expansion of an ordered field. We have* $\mathrm{p.deg}(X) < \infty$ *for every nonempty definable set X.*

The following theorem illustrates uniformness of partition degree.

Theorem 4.4. *Consider a d-minimal expansion of an ordered field. Let $\pi : F^{m+n} \to F^m$ be the coordinate projection onto the first m coordinate. Let X be a definable subset of F^{m+n}. There exists positive integer q such that* $\mathrm{p.deg}(X \cap \pi^{-1}(x)) \leq q$ *for every $x \in \pi(X)$.*

We next introduce a standard procedure for taking a shortest r-partition sequence. We begin with the introduction of a preliminary notion. We do not give a definition of definable $\mathcal{C}^r$ submanifolds of F^n because they are defined straightforwardly.

Definition 4.5. Let X be a definable subset of F^n of dimension d. We say that a point x in F^n is *r-regular* in X if $x \in X$ and there exists a definable $\mathcal{C}^r$ diffeomorphism $\varphi : U \to V$ from a definable open neighborhood U of x in X which is simultaneously a definable $\mathcal{C}^r$ submanifold onto a definable open subset V of F^d.

We set $\mathrm{Reg}_r(X)$ as follows:

$$\mathrm{Reg}_r(X) = \{r\text{-regular points in } X\}.$$

We get the following proposition:

Proposition 4.6. *A definable subset X of F^n of dimension d is a definable $\mathcal{C}^r$ submanifold of dimension d if and only if every point in X is r-regular in X.*

The following proposition provides a standard procedure to generate an r-partition sequence of minimum length:

Proposition 4.7. *Consider a d-minimal expansion of an ordered field. We define* $X\langle i\rangle$ *as follows for each* $i \geq -1$*:*

- $X\langle -1\rangle = X$*;*
- $X\langle i\rangle = X\langle i-1\rangle \setminus \operatorname{Reg}_r(X\langle i-1\rangle)$*.*

There exists a nonnegative integer m *such that* $\dim X\langle m\rangle < \dim X$ *and the equality* $\operatorname{p.deg}(X) = m$ *holds. In addition,* $(X\langle m\rangle, \operatorname{Reg}_r^m(X), \ldots, \operatorname{Reg}_r^0(X))$ *is an* r*-partition sequence of* X*, where* $\operatorname{Reg}_r^i(X) := \operatorname{Reg}_r(X\langle i-1\rangle)$ *for* $0 \leq i \leq m$*.*

A significant feature of partition degree is that it is preserved under definable homeomorphism.

Theorem 4.8. *Let* X *and* Y *be definable sets which are definably homeomorphic to each other. Then the equality* $\operatorname{p.deg}(X) = \operatorname{p.deg}(Y)$ *holds.*

It is easy to show that partition degree is not preserved under definable bijection.

Proposition 4.9. *Let* X *be a nonempty definable subset of* F^n*. There exist a definable subset* Y *of* F^{n+1} *of* $\operatorname{p.deg}(Y) = 0$ *and a definable bijection* $\varphi : X \to Y$*.*

Using the standard r-partition sequence of minimum length, we can prove the following:

Theorem 4.10. *Let* X *and* Y *be definable sets. Then the equality* $\operatorname{p.deg}(X \times Y) = \operatorname{p.deg}(X) + \operatorname{p.deg}(Y)$ *holds.*

The following proposition asserts that the pair of dimension function and partition degree satisfies a similar inequality to the inequality (1).

Proposition 4.11. *Let* X *be a definable set and* $f_1, \ldots, f_k : X \to F$ *be definable functions. There exists a definable open subset* U *of* X *such that at least one of the inequalities*

$$\dim X \setminus U < \dim X \text{ and } \operatorname{p.deg} X \setminus U < \operatorname{p.deg} X$$

holds, U *is a definable* $\mathcal{C}^r$ *submanifold and the function* f_i *restricted to* U *is of class* $\mathcal{C}^r$ *for each* $1 \leq i \leq k$*.*

We expect that every definable function defined on a definable set X is either continuous or of class $\mathcal{C}^r$ on a definable subset Y of X which is 'almost the same' as X. For instance, the notion of d-largeness is introduced as such a notion in [20]. The notion of d-largeness possesses the expected property in definably complete locally o-minimal structures, but it does not necessarily have when the structure is d-minimal. We introduce the notion of hugeness and r-largeness as alternatives of d-largeness.

Definition 4.12. Consider an expansion of a dense linear order without endpoints. Let X and Y be definable sets with $X \subseteq Y$. We say that X is *lean* in Y if and only if $\dim(\operatorname{int}_Y X) < \dim Y$. We call that X is *huge* in Y if $Y \setminus X$ is lean in Y.

Let r be a positive integer. Consider a definably complete expansion of an ordered field. Let X and Y be definable sets with $X \subseteq Y$. We say that X is *r-large* in Y if $\operatorname{int}_Y(X \cap \operatorname{Reg}_r(Y)) \neq \emptyset$.

We have the following equivalence in d-minimal expansion of an ordered field.

Proposition 4.13. *Consider a d-minimal expansion of an ordered field. Let X and Y be definable sets with $X \subseteq Y$. Let r be a positive integer. The following are equivalent:*

(1) The definable set X is lean in Y;
(2) The definable set X is not r-large in Y;
(3) $\dim(X \cap \operatorname{Reg}_r(Y)) < \dim Y$.

In addition, conditions (1) through (3) hold when $\dim X < \dim Y$.

The notion of d-largeness coincides with that of hugeness in definably complete locally o-minimal structures. In the last of this paper, we introduce the following proposition: This seems to be technical, but it is very useful. In fact, it is one of key lemmas in [8].

Proposition 4.14. *Consider a d-minimal expansion of an ordered field $\mathcal{F} = (F, <, +, \cdot, 0, 1, \ldots)$. Let $\pi : F^n \to F^m$ be a coordinate projection. Let X and Z be definable subsets of F^n such that $Z \subseteq X$ and* $\dim X = \dim Z$. *Assume that* $\dim(X \cap \pi^{-1}(x))$ *is independent of $x \in \pi(X)$. Set*

$$W := \{x \in \pi(X) \mid \pi^{-1}(x) \cap Z \textit{ is not lean in } \pi^{-1}(x) \cap X\}.$$

If Z is lean in X, then W is lean in $\pi(X)$.

5 Concluding remarks

The important features of dimension function in definably complete locally o-minimal structures are as follows: These features are used here and there in geometric studies of definably complete locally o-minimal structures. However, the author currently fails to find a reasonable measure m satisfying all the following conditions in d-minimal structures:

- $m(A \cup B) = \max\{m(A), m(B)\}$;
- Let $f : X \to F$ be a definable map. Then we have $m(\mathcal{D}(f)) < m(X)$;
- $m(\partial X) < m(X)$;
- The value of m is preserved under definable bijections (or more strongly, $m(f(X)) \leq m(X)$ for a definable map f defined on X).
- a reasonable addition formula

Two measures, extended rank and partition degree, introduced in this note satisfy a part of the above conditions, but do not satisfy all. The author thinks that the geometric study of d-minimal structures will be more boosted if a measure satisfying all the above conditions is found.

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