Ehrenfeucht theories approximated by countably categorical theories

Akito Tsuboi
University of Tsukuba*

March 24, 2025

1 Preliminaries

Throughout, L denotes a countable language, and T is a complete theory formulated in L. L-formulas are denoted by φ, ψ, and so on. Variables are denoted by x, y, z and so on, and finite tuples of variables are denoted by $\bar{x}, \bar{y}, \bar{z}$ and so on. $I(\omega, T)$ denotes the number of countable models of T modulo isomorphism. $S(T)$ is the set of all complete types in T (over the empty set). ω is the set of natural numbers.

The most famous example of a theory with $1 < I(\omega, T) < \omega$ will be the following:

Example 1 (Ehrenfeucht)**.** Let $L = \{<, 0, 1, 2, \dots\}$, and let T be the theory of the L-structure $(\mathbb{Q}, 0, 1, 2, \dots)$. Then, $I(\omega, T) = 3$, i.e., there are exactly three countable models of T: M_0, the standard model; M_1, the countable nonstandard model whose nonstandard part has a minimum element; and M_2, the countable nonstandard model whose nonstandard part has no minimum element.

In this paper, a theory with $1 < I(\omega, T) < \omega$ is called an Ehrenfeucth theory. Lachlan's conjecture states that there is no stable Ehrenfeucht theory T. It was shown in [3] that by mimicking the method in Example 1, we cannot obtain a stable Ehrenfeucht theory. Indeed, it was shown that if $\{T_n : n \in \omega\}$ is an increasing sequence of $\aleph_0$-categorical stable theories, then $T = \bigcup_{n\in\omega} T_n$ is non-Ehrenfeucht.

Definition 2. A type $p(\bar{x}) \in S(T)$ is called powerful, if whenever $M \models T$ realizes p, then M realizes all types in $S(T)$.

Below, we summarize the necessary facts for this paper, providing a brief explanation for each as to why it holds.

Fact 3. *Let T be an Ehrenfeucht theory.*

1. *T is small, i.e., $S(T)$ is countable. Suppose, for a contradiction, that $S(T)$ is uncountable. Choose any $p_0 \in S(T)$ and a countable model M_0 realizing p_0. Since $S(T)$ is uncountable, there is $p_1 \in S(T)$ and a countable model M_1 realizing both p_0 and p_1. Continuing this process, we get $p_0, \ldots, p_n \in S(T)$ and countable models $M_0, \ldots, M_n$ such that M_n realizes $p_0, \ldots, p_n$ but does not realize p_{n+1}. Then, the M_n are mutually non-isomorphic, leading to a contradiction.*

2. *A powerful type exists. A similar argument as above shows that if there is no powerful type, then there would be infinitely many non-isomorphic countable models.*

Now we work in a countably saturated model $\mathcal{M}$ of T, where T is Ehrenfeucht. For $\bar{a}$ and A in $\mathcal{M}$, the type of $\bar{a}$ over A is denoted by $\mathrm{tp}(\bar{a}/A)$.

Fact 4.

1. *For all finite set $A \subset \mathcal{M}$, there is a prime and atomic model over A. This follows from the fact that $S(T)$ is countable.*

2. *Let $p(\bar{x}) \in S(T)$ be a powerful type. There exist realizations $\bar{a}$ and $\bar{a}'$ of p such that (i) $\mathrm{tp}(\bar{a}'/\bar{a})$ is an isolated type, while (ii) $\mathrm{tp}(\bar{a}/\bar{a}')$ is not. This can be shown as follows: Let $\bar{a}$ be a realization of p. Choose a realization $\bar{b}$ of p such that $\mathrm{tp}(\bar{b}/\bar{a})$ is non-isolated. Let $q(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{a}, \bar{b})$. Since p is powerful, we can choose a realization $(\bar{a}', \bar{b}')$ of q from a prime model $M_{\bar{a}}$ over $\bar{a}$. Then, it is easy to see that $(\bar{a}, \bar{a}')$ satisfies the conditions (i) and (ii).*

2 Main Results

Continuing from the previous section, we work in $\mathcal{M}$.

Definition 5. We say $p(\bar{x}) = \mathrm{tp}(\bar{a}/\bar{b})$ is semi-isolated, if there is a formula $\varphi(\bar{x}, \bar{b}) \in p$ such that $\forall \bar{x}(\varphi(\bar{x}, \bar{b}) \to \psi(\bar{x}))$ is true for all $\psi \in \mathrm{tp}(\bar{a})$. If this situation holds, we also say $\varphi(\bar{x}, \bar{b})$ generates $p(\bar{x})$, and write $\varphi(\bar{x}, \bar{b}) \models p(\bar{x})$.

It is clear that an isolated type is semi-isolated, but the converse is not true in general. However, if $\mathrm{tp}(\bar{b}/\bar{a})$ is isolated, then the following conditions are equivalent:

1. $\mathrm{tp}(\bar{a}/\bar{b})$ is isolated;
2. $\mathrm{tp}(\bar{a}/\bar{b})$ is semi-isolated.

This can be explained as follows. Since $\mathrm{tp}(\bar{b}/\bar{a})$ is isolated, there is a formula $\varphi(\bar{x}, \bar{a})$ isolating the type.

Theorem 6. *Let $\{T_n\}_{n\in\omega}$ be an increasing sequence of $\aleph_0$-categorical theories. Suppose that $T = \bigcup_{n\in\omega} T_n$ is an Ehrenfeucht theory. Then, there exist a definable order $<$ and elements e_n ($n \in \omega$) in T^{eq} such that (i) $e_n < e_{n+1}$ for all $n \in \omega$ and (ii) $<$ is a dense order without endpoints.*

Proof. We fix a powerful type $p \in S(T)$. For simplicity in notation, we assume that p is a 1-type, i.e., $p = p(x)$ where x is a single variable. Using Fact 4.2, choose realizations a and a' of p such that $\mathrm{tp}(a'/a)$ is isolated and $\mathrm{tp}(a/a')$ is not isolated. Let $r(x, y) = \mathrm{tp}(a', a)$ and choose a formula $\psi(x, y) \in r(x, y)$ witnessing that $\mathrm{tp}(a'/a)$ is isolated. Then, $\psi(x, y)$ satisfies the following property:

(*) For all d realizing p, $\psi(x, d)$ generates $p(x)$, and $\psi(d, y)$ does not generate $p(y)$.

Observe that the formulas satisfying (*) are closed under a finite disjunction. Choose $m \in \omega$ such that all symbols in $\psi(x, y)$ belong to $L(T_m)$, which is the language of T_m. Since T_m is $\aleph_0$-categorical, there are only a finite number of formulas satisfying (*). Thus, we can assume $\psi(x, y) \in r(x, y)$ is the weakest formula possessing the property (*).

Claim A. *For $n \in \omega$, let $\psi^n(x, y)$ be the formula $\exists z_1, \ldots, z_n[\psi(x, z_1) \wedge \bigwedge_{i=1,\ldots,n-1} \psi(z_i, z_{i+1}) \wedge \psi(z_n, y)]$. Then, each formula $\psi^n(x, y)$ satisfies the condition (*) above.*

We assume $n = 2$ for simplicity since other cases are treated similarly. It is clear that $\psi^2(x, d) \models p(x)$ holds for all d realizing p. Suppose, for a contradiction, that $\psi^2(d, y) \models p(y)$. Choose d_1 and d_2 such that both (d, d_1) and (d_1, d_2) realize $r(x, y)$. Since $\psi(x, y)$ belongs to r, $\psi^2(d, d_2)$ holds. So, $\mathrm{tp}(d_2/d)$ is semi-isolated. As $\mathrm{tp}(d_1/d_2)$ is also semi-isolated, we must have that $\mathrm{tp}(d_1/d)$ is semi-isolated. By the remark just after Definition 5, $\mathrm{tp}(d_1/d)$ must be isolated. A contradiction. (End of Proof of Claim A)

Claim B. *Let (d, d_1) and (d_1, d_2) realize $r(x, y)$. Then, $\psi(\mathcal{M}, d)$ is a proper subset of $\psi(\mathcal{M}, d_2)$.*

By Claim A, $\psi^3(x, y)$ satisfies (*), so $\psi^3(x, d)$ proves $\psi(x, d)$. Also, by the choice of d, d_1, d_2, we see that $\psi(x, d_2)$ proves $\psi^3(x, d)$. Thus, $\psi(\mathcal{M}, d_2)$ is a subset of $\psi(\mathcal{M}, d)$. If the inclusion is not proper, since $\psi(d_1, d_2)$ holds, we must have $\psi(d_1, d)$. This is impossible, since $\mathrm{tp}(d_1/d)$ is not semi-isolated. (End of Proof of Claim B)

Using Claim B, we can choose a sequence $\{a_n\}_{n\in\omega}$ of realizations of p such that $\{\psi(\mathcal{M}, a_n)\}_{n\in\omega}$ forms a strictly increasing sequence of uniformly defined definable sets. Thus, T has the strict order property.

Now we define formulas:

$$\begin{aligned} &y \succeq z \equiv \forall x(\psi(x, y) \to \psi(x, z)),\\ &y \sim z \equiv y \succeq z \succeq y,\\ &y \succ z \equiv y \succeq z \land \neg z \succeq y. \end{aligned}$$

Clearly, $\succ$ defines a pre-order without endpoints.

Claim C. *There is a dense pre-order without end points.*

For all $n \in \omega$, let $y \succ^n z$ be the formula $\exists x_1, \dots, x_n(y \succ x_1 \succ \cdots \succ x_n \succ z)$. These formulas are all in $L(T_m)$. Since T_m is $\aleph_0$-categorical, there is $k \in \omega$ such that $\succ^n$ are equivalent for all numbers $n \geq k$. Then, $\succ^k$ defines a dense pre-order. Indeed, if $b \succ^k d$, then $b \succ^{2k+1} d$. So, there exists e with $b \succ^k e \succ^k d$. (End of Proof of Claim C)

In the quotient structure by $\sim$, the order induced by $\succ$ satisfies our requirement. $\square$

Question 7. The language of Example 1 is infinite. However, there exist Ehrenfeucht theories whose languages are finite. For instance, consider the language $L = \{\preceq, \sim\}$ and the L-theory T axiomatized by the following sentences:

1. $\preceq$ is a dense linear preorder without endpoints;
2. $x \sim y$ if and only if $x \preceq y \preceq x$;
3. For every positive natural number n, there exists a unique $\sim$-equivalence class, denoted c_n, containing exactly n elements. Moreover, these equivalence classes form a strictly increasing chain: $c_1 \prec c_2 \prec \dots$.

Can we construct a theorem demonstrating that mimicking this example does not yield a stable Ehrenfeucht theory?

References

[1] Benda, M., Remark on countable models, Fund. Math., 81 (1974), 107-119.

[2] Shelah, S. (1990). Classification Theory: and the Number of Non-Isomorphic Models. Elsevier.

[3] Tsuboi, A., On Theories Having a Finite Number of Nonisomorphic Countable Models, JSL Vol. 50, No. 3 (Sep., 1985), pp. 806-808

[4] Pillay, A. (1983). An Introduction to Stability Theory. Clarendon Press, Oxford University Press.

[5] Kim, B. On the number of countable models of a countable super simple theory. Journal of the London Mathematical Society. 1999;60(3):641-645.

[6] Hrushovski E. Finitely based theories. Journal of Symbolic Logic. 1989;54(1):221-225.

Professor Emeritus at the University of Tsukuba,
E-mail address: tsuboi@math.tsukuba.ac.jp