HYPERIMAGINARIES AND RELATIVIZED LASCAR GROUPS, REVISITED

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Abstract. We aim to relativize the (model-theoretic) notion of strong types to a solution set of a partial type, and then characterize them using hyperimaginaries. We also present some unresolved natural questions appeared during our research.

The purpose of this article is to give a brief overview of [LL24]. Any reader interested in the technical details can consult [LL24]. In fact, there, we have worked in the more generalized context, over an arbitrary hyperimaginary (not just over $\emptyset$). This causes more technical difficulties, and sometimes, even an additional assumption is necessary. All of these issues when we work over a hyperimaginary are well explained in [LL24]. The references in this article may not be very accurate and even omit them sometimes; we apologize for this. More accurate references may be found on [LL24].

1. Preliminaries

Throughout, we will work in a sufficiently saturated and strongly homogeneous (monster) model $\mathfrak{C}$.

Definition 1.1. (1) Any $a \in \mathfrak{C}$ is called a **(real) element** of $\mathfrak{C}$.
(2) An equivalence class of an $\emptyset$-definable equivalence relation E is called an **imaginary**.
(3) An equivalence class of an $\emptyset$-type-definable equivalence relation E is called a **hyperimaginary**.

Note that any real tuple is an imaginary ($a = a_=$), and any imaginary is a hyperimaginary.

Example 1.2. There are natural examples of hyperimaginaries. Let $M = (S^1, C, \{g_{\frac{1}{n}} : n \geq 1\})$, where C is a ternary relation and each $g_{\frac{1}{n}}$ is a unary function symbol:
(1) S^1 is a unit circle on the real plane,
(2) $M \models C(a, b, c)$ if and only if a, b and c are in clockwise-order, and
(3) $g_{\frac{1}{n}}(a)$ = rotation of a by $\frac{2\pi}{n}$-radians clockwise.

$$E(x, y) := \bigwedge_{1<n} C(x, y, g_{\frac{1}{n}}(x)) \vee C(y, x, g_{\frac{1}{n}}(y)).$$

We can make the following observations:

- $E(x, y)$ is an $\emptyset$-type-definable equivalence relation.
- Let $\mathfrak{C} \models \mathrm{Th}(M)$ be a monster model. For any $a \in \mathfrak{C}$, a_E is a hyperimaginary, which collects all elements 'infinitesimally close' to or having 'distance 0' from a.
- $|\{f(a_E) : f \in \mathrm{Aut}(\mathfrak{C})\}|$ is infinite, but bounded by $2^{\aleph_0}$. That is, $a_E \in \mathrm{bdd}(\emptyset)$ (we will define $\mathrm{bdd}(\emptyset)$ soon).

Note that we can do similar work in an expansion of T_{DLO}, but no boundedness there.

Now we extend the usual definable, algebraic closures in terms of real elements into the context of hyperimaginaries.

Definition 1.3. (1) The definable closure over $\emptyset$, $\mathrm{dcl}(\emptyset)$ is the set of hyperimaginaries $\boldsymbol{e}$ such that $|\{f(\boldsymbol{e}) : f \in \mathrm{Aut}(\mathfrak{C})\}| = 1$. If $\boldsymbol{e} \in \mathrm{dcl}(\emptyset)$, then we say $\boldsymbol{e}$ is definable over $\emptyset$.
(2) The algebraic closure over $\emptyset$, $\mathrm{acl}(\emptyset)$ is the set of hyperimaginaries $\boldsymbol{e}$ such that $|\{f(\boldsymbol{e}) : f \in \mathrm{Aut}(\mathfrak{C})\}| < \omega$ If $\boldsymbol{e} \in \mathrm{acl}(\emptyset)$, then we say $\boldsymbol{e}$ is definable over $\emptyset$.
(3) The bounded closure over $\emptyset$, $\mathrm{bdd}(\emptyset)$ is the set of hyperimaginaries $\boldsymbol{e}$ such that $|\{f(\boldsymbol{e}) : f \in \mathrm{Aut}(\mathfrak{C})\}|$ is small (that is, less than the degree of saturation and strong homogeneity of the monster model). If $\boldsymbol{e} \in \mathrm{bdd}(\emptyset)$, then we say $\boldsymbol{e}$ is definable over $\emptyset$.

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From the definition, it can be checked that $\mathrm{dcl}(\emptyset) \cap \mathfrak{C}$ is the usual definable closure of $\emptyset$. Note that for imaginaries, being bounded is the same as being algebraic; $\mathrm{bdd}(\emptyset) \cap \mathfrak{C}$ is the usual algebraic closure of $\emptyset$.

Recall that given (small) $A \subseteq \mathfrak{C}$, $\mathrm{Aut}_A(\mathfrak{C})$ is an automorphism group of $\mathfrak{C}$, which fixes A pointwise. Regarding a hyperimaginary as a 'single object', it is natural to define the automorphism group fixing a hyperimaginary in the following way.

Definition 1.4. Let $\boldsymbol{e}$ be a hyperimaginary. $\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C}) = \{f \in \mathrm{Aut}(\mathfrak{C}) : f(\boldsymbol{e}) = \boldsymbol{e} \text{ (setwise)}\}$.

We can also naturally define a collection of automorphisms fixing a set of hyperimaginaries. Note that a sequence of hyperimaginaries is interdefinable with a single hyperimaginary, so Definition 1.4 indeed extends the definition of $\mathrm{Aut}_A(\mathfrak{C})$.

Note that $\mathrm{dcl}(\emptyset)$, $\mathrm{acl}(\emptyset)$, and $\mathrm{bdd}(\emptyset)$ are not necessarily small. But for each of them, we can find a small set of hyperimaginaries interdefianble with the corresponding one, so we will pretend that they are small sets. This fact can be found on [C11] or [K14].

The definable closure over $\emptyset$ is just interdefinable with $\emptyset$, hence it will be not very interesting to study definable closures. But for algebraic closures and bounded closures, it is well-knwon that there are nice characterizations:

Theorem 1.5. *(1) The following are equivalent.*

(a) There is $f \in \mathrm{Aut}_{\mathrm{acl}(\emptyset)}(\mathfrak{C})$ such that $f(a) = b$.

(b) $a \equiv^{\mathrm{S}} b$, i.e. for any $\emptyset$-definable finite equivalence relation E, $E(a, b)$. We say that a and b have the same Shelah-strong type.

(2) The following are equivalent.

(a) There is $f \in \mathrm{Aut}_{\mathrm{bdd}(\emptyset)}(\mathfrak{C})$ such that $f(a) = b$.

(b) $a \equiv^{\mathrm{KP}} b$, i.e. for any $\emptyset$-type-definable bounded equivalence relation E, $E(a, b)$. We say that a and b have the same KP-strong type (KP stands for Kim-Pillay).

Generalizing type-definability to invariance, we have the following definition.

Definition 1.6. $a \equiv^{\mathrm{L}} b$ if for any $\emptyset$-invariant bounded equivalence relation E, $E(a, b)$. We say that a and b have the same Lascar-strong type.

Similar as $\equiv^{\mathrm{S}}$ and $\equiv^{\mathrm{KP}}$, there is $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}) \leq \mathrm{Aut}(\mathfrak{C})$ such that $a \equiv^{\mathrm{L}} b$ if and only if there is $f \in \mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ such that $f(a) = b$. Explicitly, $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ is a the subgroup of Aut)$\mathfrak{C}$), generated by

$$\{f \in \mathrm{Aut}(\mathfrak{C}) : \text{ there is } M \models T \text{ such that } f \text{ fixes } M \text{ pointwise}\}.$$

2. Relativization to the solution set of a partial type

Now we define the notion of main interest. Let Σ be any (partial) type over $\emptyset$, possibly with infinitely many variables.

Definition 2.1. $\mathrm{Autf}_{\mathrm{L}}(\Sigma) =$

$$\{\sigma \in \mathrm{Aut}(\mathfrak{C}) \upharpoonright \Sigma(\mathfrak{C}) : \text{for any (small) cardinal } \lambda, \text{ for any tuple } a = (a_i)_{i<\lambda} \text{ where each } a_i \models \Sigma(x_i),\ a \equiv^{\mathrm{L}} \sigma(a)\}.$$

$\mathrm{Autf}_{\mathrm{S}}(\Sigma)$ and $\mathrm{Autf}_{\mathrm{KP}}(\Sigma)$ can be defined similarly.

The next proposition says that in Definition 2.1, we only need to consider countable tuples.

Proposition 2.2. $\mathrm{Autf}_{\mathrm{L}}(\Sigma) =$

$$\{\sigma \in \mathrm{Aut}(\mathfrak{C}) \upharpoonright \Sigma(\mathfrak{C}) : \textit{for any countable tuple } a = (a_i)_{i<\omega} \textit{ where each } a_i \models \Sigma(x_i),\ a \equiv^{\mathrm{L}} \sigma(a)\}.$$

Note that this proposition is nontrivial even when $\Sigma = \{x = x\}$.

Sketch of the proof. Suppose that for any corresponding countable subtuples of a and b, they have the same Lascar strong type. $a \equiv^{\mathrm{L}} b$ if and only if the "Lascar distance", $d(a, b)$ between a and b, $d(a, b)$, is finite. Then by induction and the fact that $d(a, b) \leq k$ is type-definable, by compactness, it can be proved that for any subtuples a_0 and b_0 of a and b, $d(a_0, b_0) \leq k$ for some uniform $k < \omega$. Then again by compactness, $d(a, b) \leq k$. □

By definition, for any $M \models T$ and $f \in \mathrm{Aut}_M(\mathfrak{C})$, $a \equiv^{\mathrm{L}} f(a)$ for any tuple a in $\mathfrak{C}$. Similarly, it is natural to ask whether there is small b in $\Sigma(\mathfrak{C})$ such that if $f \in \mathrm{Aut}_b(\mathfrak{C})$, then for any tuple a in $\Sigma(\mathfrak{C})$, $a \equiv^{\mathrm{L}} f(a)$. Relaxing even further, we can ask whether there is some small tuple b in $\Sigma(\mathfrak{C})$ such that if $b \equiv^{\mathrm{L}} f(b)$, then for any tuple a in $\Sigma(\mathfrak{C})$, $a \equiv^{\mathrm{L}} f(a)$.

Lemma 2.3. *A small tuple b of realizations of Σ is called a* **Lascar tuple** *(in Σ) if*

$$\mathrm{Autf}_{\mathrm{L}}(\Sigma) = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) \restriction \Sigma(\mathfrak{C}) : b \equiv^{\mathrm{L}} \sigma(b)\}.$$

For any Σ, there is a Lascar tuple b in Σ.

Sketch of the proof.

$$\{c/\equiv^{\mathrm{L}}: \ c \text{ is a (at most) countable tuple of realizations of } \Sigma\}$$

is a small set. Take exactly one representative for each $c/\equiv^{\mathrm{L}}$, and make a sequence which enumerates all of them. Then it is a Lascar tuple. □

Recall the following classical notion and a fact.

Definition 2.4. $\mathrm{Gal}_{\mathrm{L}}(T) := \mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ is the **Lascar group** of T.

Proposition 2.5. $\nu : S_M(M) \to \mathrm{Gal}_{\mathrm{L}}(T)$, $\mathrm{tp}(f(M)/M) \mapsto f/\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ *is a (well-defined) surjective map, and* $\mathrm{Gal}_{\mathrm{L}}(T)$ *is a quasi-compact topological group with the quotient topology induced by ν.*

In $\Sigma(\mathfrak{C})$, we can do the same thing with a Lascar tuple b, instead of a model M. The proof of Corollary 2.7 is straightforward; note that there is a natural quotient map from $\mathrm{Gal}_{\mathrm{L}}(\mathfrak{C})$ to $\mathrm{Gal}_{\mathrm{L}}(\Sigma)$.

Definition 2.6. $\mathrm{Gal}_{\mathrm{L}}(\Sigma) = \mathrm{Aut}(\mathfrak{C}) \restriction \Sigma(\mathfrak{C})/\mathrm{Autf}_{\mathrm{L}}(\Sigma)$, the **Lascar group relativized to** Σ.

Corollary 2.7. $\nu_b : S_b(b) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma)$, $\mathrm{tp}(f(b)/b) \mapsto f \cdot \mathrm{Autf}_{\mathrm{L}}(\Sigma)$ *is well-defined and makes* $\mathrm{Gal}_{\mathrm{L}}(\Sigma)$ *into a quasi-compact topological group.*

We now recall one of the most important property of the Lascar group, and then compare with the (newly proved) relativized one. Let $\pi : \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_{\mathrm{L}}(T)$ be the natural projection map and $\pi_\Sigma : \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma)$ be the natural projection map.

Proposition 2.8. *The following are equivalent.*

(1) $H \leq \mathrm{Gal}_{\mathrm{L}}(T)$ *is closed.*
(2) $\pi^{-1}[H] = \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$ *for some hyperimaginary $\boldsymbol{e}$ bounded over $\emptyset$.*

Proposition 2.9. *The following are equivalent.*

(1) $H \leq \mathrm{Gal}_{\mathrm{L}}(\Sigma)$ *is closed.*
(2) $\pi_\Sigma^{-1}[H] = \mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})$ *for some hyperimaginary $\boldsymbol{e}$ bounded over $\emptyset$, and one of the representatives of $\boldsymbol{e}$ is a tuple in $\Sigma(\mathfrak{C})$.*

Sketch of the proof of Proposition 2.9. (1) ⇒ (2): $\nu_b^{-1}[H]$ is closed in $S_b(b)$, thus $\{h(b) : h \in H\}$ is type-definable over b. Also, $\mathrm{Aut}_b(\mathfrak{C}) \leq \pi_\Sigma^{-1}[H]$. Then using some facts (please refer to [LL24]), $\pi_\Sigma^{-1}[H] = \mathrm{Aut}_{b_E}(\mathfrak{C})$ for some $\emptyset$-type-definable equivalence relation E. It can be checked that b_E is bounded over $\emptyset$.

(2) ⇒ (1): Say $\boldsymbol{e} = a_E$. We may assume that a is contained in a Lascar tuple b. Then (because b is a Lascar tuple), $\nu_b^{-1}[H] = \{p(x') \in S_b(b) : E(x, a) \subseteq p(x')\}$, where $|x'| = |b|$ and $x \subseteq x'$. Thus H is closed in $\mathrm{Gal}_{\mathrm{L}}(\Sigma)$. □

By the previous proposition, the following definition seems to be a good choice for the 'relativized' bounded closure. We also can define $\mathrm{dcl}(\emptyset) \cap \Sigma$ and $\mathrm{acl}(\emptyset) \cap \Sigma$ similarly.

Definition 2.10. $\mathrm{bdd}(\emptyset) \cap \Sigma$ is the set of all hyperimaginaries bounded over $\emptyset$, where one of the representatives is a tuple in $\Sigma(\mathfrak{C})$.

Again, we recall a fundamental fact on the Lascar group, and compare with the new, relativized one.

Proposition 2.11. *Let* $\pi : \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_{\mathrm{L}}(T)$ *be the natural projection.*

(1) The closure of $\{\mathrm{id}\}$ *in* $\mathrm{Gal}_{\mathrm{L}}(T)$ *is* $\pi[\mathrm{Aut}_{\mathrm{bdd}(\emptyset)}(\mathfrak{C})]$.
(2) The connected component containing $\{\mathrm{id}\}$ *in* $\mathrm{Gal}_{\mathrm{L}}(T)$ *is* $\pi[\mathrm{Aut}_{\mathrm{acl}(\emptyset)}(\mathfrak{C})]$.

Proposition 2.12. *Let Σ be a partial type over $\emptyset$ and* $\pi_\Sigma : \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_{\mathrm{L}}(\Sigma)$ *the natural projection.*

(1) The closure of $\{\mathrm{id}\}$ *in* $\mathrm{Gal}_{\mathrm{L}}(\Sigma)$ *is* $\pi_\Sigma[\mathrm{Aut}_{\mathrm{bdd}(\emptyset)\cap\Sigma}(\mathfrak{C})]$.

(2) The connected component containing $\{\mathrm{id}\}$ *in* $\mathrm{Gal_L}(\Sigma)$ *is* $\pi_\Sigma[\mathrm{Aut}_{\mathrm{acl}(\emptyset)\cap\Sigma}(\mathfrak{C})]$.

Sketch of the proof of Proposition 2.12. (1) Since $\mathrm{Gal_L}(\Sigma)$ is a topological group, the closure of $\{\mathrm{id}\}$ is the intersection of all closed subgroups. But each of the closed subgroups is of the form $\pi_\Sigma[\mathrm{Aut}_{\boldsymbol{e}}(\mathfrak{C})]$, where $\boldsymbol{e} \in \mathrm{bdd}(\emptyset) \restriction \Sigma$.

(2) Again, since $\mathrm{Gal_L}(\Sigma)$ is a topological group, The connected component containing $\{\mathrm{id}\}$ is the intersection of all closed subgroups of finite indices. Thus $\mathrm{bdd}(\emptyset) \restriction \Sigma$ is replaced with $\mathrm{acl}(\emptyset) \restriction \Sigma$. □

Finally, we can compare the following two theorems.

Theorem 2.13. *(1) The following are equivalent.*
(a) There is $f \in \mathrm{Aut}_{\mathrm{acl}(\emptyset)}(\mathfrak{C})$ *such that* $f(a) = b$.
(b) $a \equiv^{\mathrm{S}} b$, *i.e. for any* $\emptyset$*-definable finite equivalence relation* E, $E(a, b)$.
(2) The following are equivalent.
(a) There is $f \in \mathrm{Aut}_{\mathrm{bdd}(\emptyset)}(\mathfrak{C})$ *such that* $f(a) = b$.
(b) $a \equiv^{\mathrm{KP}} b$, *i.e. for any* $\emptyset$*-type-definable bounded equivalence relation* E, $E(a, b)$.

Theorem 2.14. *Let* Σ *be a partial type over* $\emptyset$. *For any tuples* a *and* b *in* $\Sigma(\mathfrak{C})$,
(1) $a \equiv^{\mathrm{S}} b$ *if and only if there is* $f \in \mathrm{Aut}_{\mathrm{acl}(\emptyset)\cap\Sigma}(\mathfrak{C})$ *such that* $f(a) = b$.
(2) $a \equiv^{\mathrm{KP}} b$ *if and only if there is* $f \in \mathrm{Aut}_{\mathrm{bdd}(\emptyset)\cap\Sigma}(\mathfrak{C})$ *such that* $f(a) = b$.

Note that $a \equiv^{\mathrm{S}} b$ or $a \equiv^{\mathrm{KP}} b$ implies that $\mathrm{tp}(a) = \mathrm{tp}(b)$, hence we can get the following corollary easily by letting $\Sigma = \mathrm{tp}(a) = \mathrm{tp}(b)$.

Corollary 2.15. *For any tuples* a *and* b *in* $\mathfrak{C}$,
(1) $a \equiv^{\mathrm{S}} b$ *if and only if* $\mathrm{tp}(a) = \mathrm{tp}(b)$ *and there is* $f \in \mathrm{Aut}_{\mathrm{acl}(\emptyset)\cap\mathrm{tp}(a)}(\mathfrak{C})$ *such that* $f(a) = b$.
(2) $a \equiv^{\mathrm{KP}} b$ *if and only if* $\mathrm{tp}(a) = \mathrm{tp}(b)$ *and there is* $f \in \mathrm{Aut}_{\mathrm{bdd}(\emptyset)\cap\mathrm{tp}(a)}(\mathfrak{C})$ *such that* $f(a) = b$.

Above corollary and the following question is not mentioned in [LL24].

Question 2.16. (1) (Informal) Will Corollary 2.15 help or give more information in proving (or disproving) that $\equiv^{\mathrm{S}}$ and $\equiv^{\mathrm{KP}}$ are the same in simple theories?

(2) (Question of unexpected difficulty) $\{f \restriction \Sigma(\mathfrak{C}) : f \in \mathrm{Autf_L}(\mathfrak{C})\} = \mathrm{Autf_L}(\Sigma)$?

Note that one direction, $\subseteq$ is trivial. This question can be restated in the following way: If f fixes all Lascar strong types in $\Sigma(\mathfrak{C})$, then is there g that fixes all Lascar strong types in $\mathfrak{C}$ and coincides with f in $\Sigma(\mathfrak{C})$?

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