On monotonicity theorems and dp-rank

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1 Introduction

The monotonicity theorem for o-minimal structures is a remarkable result providing a basis on which wide and deep studies on them are carried out. It is well-known that a weakened version of monotonicity theorem holds for weakly o-minimal theories. Then one question naturally arises, that is, does it hold also for dp-minimal ordered abelian groups? Although partial results have been given, the whole problem is still open. At RIMS Model Theory Workshop 2021, Goodrick[3] gave an expansion of an ordered abelian group not admitting the kind of monotonicity and raised a question whether or not it is dp-minimal. We gave an answer to this question, i.e., its dp-rank is 2.

In this report, we are going to give a brief summary of the background of the question and a part of the proof of our result showing that the dp-rank is 2 or more.

2 O-minimality and the monotonicity theorem

Among orders, we consider only dense linear ones without endpoints in this article.

Definition 2.1. A structure equipped with an order $\mathcal{M} = (M; <, \dots)$ is said to be *o-minimal* if any unary definable subset $X \subseteq M$ can be partitioned into finitely many points and intervals.

Example 2.2. The real field $(\mathbb{R}; <, +, \cdot)$ is easily seen to be o-minimal by Tarski's celebrated quantifier elimination. The proof can be found in most introductory textbooks on model theory.

Another example is $\mathbb{R}_{\exp}$, the real field augmented by the exponential function. It is a highly important result by Wilkie[7].

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The monotonicity theorem is a fundamental result on o-minimal structures. It yields some important theorems such as the cell decomposition theorem, which is a powerful tool for studies of o-minimal structures.

Theorem 2.3 (The monotonicity theorem[5]). *Let $\mathcal{M}$ be an o-minimal structure and $f\colon X \to M$ a unary function definable in $\mathcal{M}$. Then, X can be partitioned into finitely many points and intervals so that, for each interval I, $f|_I$ is continuous and either strictly increasing, strictly decreasing, or constant.*

For further details, see van den Dries' textbook[6].

Due to the importance of the monotonicity theorem, it is natural to ask whether or not we can give a similar result for wider classes of ordered structures. Actually, a local version of monotonicity can be shown for weakly o-minimal theories.

Definition 2.4. An ordered structure $\mathcal{M} = (M; <, \dots)$ is *weakly o-minimal* if any unary definable subset $X \subseteq M$ can be partitioned into finitely many points and convex sets.

A complete theory is said to be weakly o-minimal if all of its models are so.

Example 2.5. $(\mathbb{Q}; <, +, (-\pi, \pi))$, the ordered group of rational numbers with a distincted subset $(-\pi, \pi)$, is known to be weakly o-minimal. However, it is not o-minimal since the convex set $(-\pi, \pi)$ does not have its endpoints in the universe $\mathbb{Q}$.

It is also known that $(\mathbb{R}^{\mathrm{alg}}; <, +, \cdot, (-\pi, \pi))$ is weakly o-minimal where $\mathbb{R}^{\mathrm{alg}}$ is the set of real algebraic numbers.

Although the monotonicity theorem itself is not true, its weakened form still holds for weakly o-minimal theories.

Definition 2.6. A unary function $f\colon X \to M$ is *locally increasing* if for any point $a \in X$, f is strictly increasing on some sufficiently small neighborhood of a.

Theorem 2.7 ([4]). *Suppose that $\mathcal{M}$ is a structure with weakly o-minimal theory and $f\colon X \to M$ a unary function definable in $\mathcal{M}$. Then, X can be partitioned into finitely many points and convex sets so that, for each convex sets C, $f|_C$ is continuous and either locally increasing, locally decreasing, or locally constant.*

Hence, weakly o-minimal theories have the property which should be called local monotonicity.

3 Dp-minimality and local monotonicity conjecture

Dose local monotonicity hold for a wider class? As to this question, it is conjectured that ordered abelian groups satisfying a stability-theoretic property called dp-minimality also have local monotonicity.

Definition 3.1. Let $\mathcal{M}$ be a structure (possibly without an order) and κ a cardinal. We say that $\mathcal{M}$ has dp-rank κ or more if there exist a family of formulas $(\varphi_\alpha(x, y_\alpha))_{\alpha\in\kappa}$ where $|x| = 1$ and a family of tuples $(b_{\alpha,i})_{\alpha\in\kappa,i<\omega}$ from a monster model such that, for any function $\eta\colon \kappa \to \omega$, one can take a_η from a monster model which satisfies

$$\models \varphi_\alpha(a_\eta, b_{\alpha,i}) \iff \eta(\alpha) = i \quad \text{for all } \alpha < \kappa \text{ and } i < \omega.$$

Example 3.2. Weakly o-minimal theories are known to be dp-minimal. See [1] for details. Other examples include strongly minimal theories, C-minimal theories, and P-minimal theories.

Question 3.3. Let $\mathcal{M} = (M; <, +, \dots)$ be a dp-minimal expansion of an ordered abelian group. Is the following statement true?

> Suppose that $f\colon X \to M$ is a unary function definable in $\mathcal{M}$. Then, X can be partitioned into finitely many definable subsets $X_1, \dots, X_m$ so that, for each X_i, $f|_{X_i}$ is continuous and either locally increasing, locally decreasing, or locally constant.

This problem is still open. See Goodrick's article[3] for details. In there, He gave an ordered abelian group augmented by a function without local monotonicity and posed a question whether it is dp-minimal or not. If so, this structure would be a counterexample to the problem above. To see the details, we will describe the ordered abelian group first.

Definition 3.4. We denote by $\mathbb{R}((t))$ the set of formal Laurent series over $\mathbb{R}$, that is,

$$\mathbb{R}((t)) = \left\{ \sum_{i=m}^{\infty} a_i t^i \mid m \in \mathbb{Z}, a_i \in \mathbb{R} \right\}.$$

The addition and the multiplication is defined in the same mannar as done in polynomial rings.

The order is given in the following way: $\sum_{i=m}^{\infty} a_i t^i < \sum_{i=m}^{\infty} b_i t^i$ if and only if there is an index k such that $a_k < b_k$ and $a_i = b_i$ for all $i < k$.

One can see that $\mathbb{R}((t))$ is an ordered field with the addition, the multiplication and the order defined above. In addition, we introduce a function on it which does not have local monotonicity (the definition of the function is slightly different from that given by Goodrick(ibid.) but we believe this difference is not essential.).

Definition 3.5. Let $f\colon \mathbb{R}((t)) \to \mathbb{R}((t))$ be a function defined as:

$$f\left(\sum_{i=m}^{\infty} a_i t^i\right) = \sum_{i=m}^{\infty} (-1)^i a_i t^i.$$

It is clear that f is a group homomorphism.

Proposition 3.6. *For any $a \in \mathbb{R}((t))$, f is not monotone on any interval including a.*

Proof. Take an arbitrarily small positive element ϵ from $\mathbb{R}((t))$ and let $\epsilon = \sum_{i=m}^{\infty} e_i t^i$ where $e_m > 0$. One can suppose that $m > 0$. Then, we have

$$|t^{2m}| < |\epsilon|, \qquad |t^{2m+1}| < |\epsilon|$$

and

$$f(a + t^{2m}) - f(a) = f(t^{2m}) = t^{2m} > 0,$$
$$f(a + t^{2m+1}) - f(a) = f(t^{2m+1}) = -t^{2m-1} < 0,$$

which prove the claim. ■

Here is the question Goodrick gave;

Question 3.7. Is a structure $(\mathbb{R}((t)); <, +, f)$ dp-minimal?

If it holds, then the structure gives a counterexample to Question 3.3.
Our result is:

Theorem 3.8. $(\mathbb{R}((t)); <, +, f)$ *has dp-rank* 2.

In this report, we will give only the proof that shows it has dp-rank 2 or more as the rest requires a much longer proof. First, we define two projection functions.

Definition 3.9. $\pi_0, \pi_1 \colon \mathbb{R}((t)) \to \mathbb{R}((t))$ are functions defined in the following way:

$$\pi_0\left(\sum_i^{\infty} a_i t^i\right) = \sum_{i \in 2\mathbb{Z}}^{\infty} a_i t^i$$
$$\pi_1\left(\sum_i^{\infty} a_i t^i\right) = \sum_{i \in 2\mathbb{Z}+1}^{\infty} a_i t^i$$

It is easily seen that, for $a \in \mathbb{R}((t))$,

$$\pi_0(a) = \frac{a + f(a)}{2}, \qquad \pi_1(a) = \frac{a - f(a)}{2}.$$

Since the additive group $\mathbb{R}((t))$ is divisible, it is meaningful and definable dividing by 2. Hence, π_0 and π_1 are definable in the structure $(\mathbb{R}((t)); <, +, f)$.

Proposition 3.10. *The dp-rank of* $(\mathbb{R}((t)); <, +, f)$ *is* 2 *or more.*

Proof. Define two formulas

$$\varphi_0(x, y_0, y_1) \equiv y_0 < \pi_0(x) \le y_1, \qquad \varphi_1(x, y_0, y_1) \equiv y_0 < \pi_1(x) \le y_1.$$

For any $\eta \colon 2 \to \omega$, we let a_η be $t^{2\eta(0)} + t^{2\eta(1)+1}$. Then, we have

$$\mathbb{R}((t)) \models \varphi_\alpha(a_\eta, t^{2i+2}, t^{2i}) \iff \eta(\alpha) = i$$

for all $\alpha < 2$ and $i < \omega$. ■

References

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