Some characterization of locally o-minimal structures

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概要

abstract Locally o-minimal structures are some local adaptation from o-minimal structures. However, there are examples of them whose theories have TP_2. In this note, we characterize definably complete locally o-minimal structures under some additional conditions.

1. Introduction

We recall some definitions at first.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is *o-minimal* if every definable subset of M^1 is a finite union of points and intervals.

M is *locally o-minimal* if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is *uniformly locally o-minimal* if for any formula $\varphi(x, \bar{y})$ over $\emptyset$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$ where $\varphi(M, \bar{b})$ is the realization set of $\varphi(x, \bar{b})$ in M.

M is *strongly locally o-minimal* if for any $a \in M$, there is an open interval $I \ni a$ such that whenever X is a definable subset of M^1, then $I \cap X$ is a finite union of points and intervals.

M is *definably complete* if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Example 2. [1], [2]

$(\mathbb{R}, +, <, \mathbb{Z})$ where $\mathbb{Z}$ is the interpretation of a unary predicate, and $(\mathbb{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

And we recall some fundamental facts.

Fact 3. [1], [2]

Definably complete local o-minimality is preserved under elementary equivalence.

Proposition 4. [2]

Let M be a uniformly locally o-minimal structure. Suppose that M is ω-saturated. Then M is strongly locally o-minimal.

Here we recall types of locally o-minimal structures for the argument below.

Definition 5. Let M be a densely linearly ordered structure and $p(x) \in S_1(M)$.

We say that $p(x)$ is *cut over M* if for any $a \in M$, if $a < x \in p(x)$, then there is $b \in M$ such that $a < b < x \in p(x)$, and similarly if $x < a \in p(x)$, then there is $c \in M$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1(M)$ is *noncut over M* if $q(x)$ is not a cut type.

Here we consider nonisolated types only.

Cut types are called *irrational cuts*, and noncut types are called *rational cuts* by some people.

There are several kinds of noncut types.

Definition 6. Let M be locally o-minimal and $p(x) \in S_1(M)$ be noncut.

There are four kinds of noncut types.

$p(x) \vdash \{m < x < a \ : \ m < a \in M\}$ for some fixed a,

or $\{a < x < m \ : \ a < m \in M\}$ for some fixed a.

Here we call these types a *bounded $left$* noncut type and a bounded *$right$* noncut one.

$p(x) \vdash \{m < x \ : \ m \in M\}$ or $\{x < m \ : \ m \in M\}$.

We call these types *unbounded* noncut types.

It is known that o-minimal structures have a small amount of definable sets. However, locally o-minimal structures have definable complexity to some extent.

Definition 7. An *independence pattern* of length κ is a sequence of pairs $(\phi^\alpha(\bar{x}, \bar{y}), k^\alpha)_{\alpha<\kappa}$ of formulas such that there exists an array $\langle \bar{b}_i^\alpha : \alpha < \kappa, i < \lambda \rangle$ for some $\lambda \geq \omega$ such that :

· for each $\alpha < \kappa$, the set $\{\phi^\alpha(\bar{x}, \bar{b}_i^\alpha) : i < \lambda\}$ is k^α-inconsistent, and

· for all $\eta \in \lambda^\kappa$, the set $\{\phi^\alpha(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$ is consistent.

A thoery T is *inp-minimal* if there is no inp-pattern of length two in a single variable x.

A theory T has *the tree property of the second kind* (TP_2) if there is an inp-pattern of size ω for which the formulas $\phi^\alpha(\bar{x}, \bar{y})$ in the definition above are all equal to some $\phi(\bar{x}, \bar{y})$.

Fact 8. [12], [13]

There are locally o-minimal structures whose theories have TP_2.

For example, some modified simple product by M.Fujita, and some ultraproduct of o-minimal *structures by A.Tsuboi and the author.*

These examples indicate that it is difficult to characterize locally o-minimal structures by using the definition only. Thus we try to characterize them by stability theoretic properties under some additional conditions.

2. Characterization of locally o-minimal structures under some conditions.

In this section, we try to characterize definably complete locally o-minimal structures in relation to dp-rank. In particular, we investigate locally o-minimal structures whose theories are dp-minimal at first.

There are many research papers about dp-minimal structures. Especially, we refer to the argument by P.Simon here.

Definition 9. Let $p(\bar{x})$ be a partial type over a set $A \subset \mathcal{U}$ where $\mathcal{U}$ is the monster model.
We define the *dp-rank* of $p(\bar{x})$ as follows :
Let μ be a cardinal.
We say that $p(\bar{x})$ has dp-rank $< \mu$ if given any realization $\bar{a}$ of p and any family $(I_t : t < \mu)$ of mutually A-indiscernible sequences, at least one of them is indiscernible over $A\bar{a}$.
Dp-minimal theories are theories in which all 1-types have dp-rank 1 (dp-rk$(x = x) = 1$).

Fact 10. *e.g.* [4]
A theory T is dp-minimal
if and only if
T is NIP and inp-minimal.

We argue about the definability of types first.

Fact 11. *Let M be locally o-minimal and $p(x) \in S_1(M)$ be bounded noncut.*
Then $p(x)$ is definable.
For instance, let $p(x) \vdash \{m < x < a : m < a \in M\}$ for some fixed a.
For any L-formula $\phi(x, \bar{y})$,
$$d\phi(\bar{y}) := \exists z \forall x (z < a \wedge z < x < a \longrightarrow \phi(x, \bar{y})).$$

In o-minimal structures, unbounded noncut types are also definable. But they are not definable in locally o-minimal structures generally.

Proposition 12. *Let M be definably complete locally o-minimal and $Th(M)$ be dp-minimal.*
And let $p(x) \in S_1(M)$ be unbounded noncut.

Then $p(x)$ is definable.

Proof ;

Let $p(x) \vdash \{m < x \,:\, m \in M\}$. And let $E(\bar{y}, \bar{z})$ be the equivalence relation defined on tuples by $E(\bar{b}, \bar{b}')$ if and only if $\exists z \forall x\, (\, x > z \longrightarrow (\, \phi(x, \bar{b}) \longleftrightarrow \phi(x, \bar{b}')\,)\,)$.

Let $\bar{b}$ and $\bar{b}'$ have the same type over M_0 for some model M_0. We denote the formula $\phi(x, \bar{b}) \triangle \phi(x, \bar{b}') := (\phi(\bar{x}, \bar{b}) \wedge \neg\phi(\bar{x}, \bar{b}')\,) \vee (\,\neg\phi(\bar{x}, \bar{b}) \wedge \phi(\bar{x}, \bar{b}'))$. Now as $\bar{b}$ and $\bar{b}'$ have the same Lascar strong type over M_0, there is an indiscernible sequence $\{\, \bar{b}_i \,:\, i < \omega \,\}$ over M_0 such that $\bar{b}_0 = \bar{b}$ and $\bar{b}_1 = \bar{b}'$.

Consider the set of formulas $\Phi(x) := \{\, \phi(x, \bar{b}_{2i}) \triangle \phi(x, \bar{b}_{2i+1}) \,:\, i < \omega \,\}$. If $\Phi(x)$ is consistent, then the alternation rank of $\phi(x, \bar{y})$ is infinite. Thus it contradicts that $Th(M)$ is NIP. So the formula $\phi(x, \bar{b}) \triangle \phi(x, \bar{b}')$ divides over M_0. Thus the set of formulas $\Phi(x)$ is k-inconsistent for some $k < \omega$.

If the realization set $\phi(M, \bar{b}) \triangle \phi(M, \bar{b}')$ is cofinal in M, then for any $i < \omega$, the set $\phi_i(M) := \phi(M, \bar{b}_{2i}) \triangle \phi(M, \bar{b}_{2i+1})$ is cofinal. Thus we can take infinitely many intervals which have realizations of ϕ_i for any $i < \omega$. Then this contradicts the inp-minimality.

Thus this set $\phi(M, \bar{b}) \triangle \phi(M, \bar{b}')$ cannot be cofinal, so $\bar{b}$ and $\bar{b}'$ are E-equivalent. Then $E(\bar{y}, \bar{z})$ is a bounded equivalence relation. As E is definable, E has finitely many classes. By Compactness theorem, E has n_ϕ classes for some $n_\phi < \omega$.

Let $\psi(x, \bar{y}) \in L$-formula. And let $\{\, \bar{m}_i \,:\, i < n_\psi \,\} \subset M$ be the set of representatives of the E-classes. We can take $d\psi(\bar{y}) := \exists z \forall x(\, x > z \longrightarrow \bigvee_{i<n_\psi} (\psi(x, \bar{y}) \longleftrightarrow \psi(x, \bar{m}_i))$. ∎

And I show the next theorem.

Theorem 13. *Let M be a definably complete locally o-minimal and $Th(M)$ be dp-minimal. Then M is uniformly locally o-minimal.*

Sketch of proof ;

Let $d \in M$ and we consider the bounded left noncut type $p(x) \in S_1(M)$ of d, so $p(x)$ is definable over M. And we take a Morley sequence $\bar{b} := \{b_i \,:\, i < \omega\}$ of $p(x)$ such that $b_i \models p(x) \,|\, M\{b_j : j < i\}$ where $p(x) \,|\, M\{b_j : j < i\}$ is definable over M, so it is M-invariant.

We choose an element $b_i \in \bar{b}$ $(\, i \neq 0\,)$ and we denote $b_i := a$ and $\bar{b} := \{b_j : i \neq j, j < \omega\}$ in the following. So $\bar{b}$ is indiscernible over M and is not indiscernible over Ma.

Let $\phi(x, \bar{y}) \in L$-formula with $|\,x\,| = 1$. Now we consider $q(\bar{y}) \in S_k(M\bar{b})$ which is finitely satisfiable in M and its global extension $q'(\bar{y}) \in S_k(\mathcal{U})$ which is also finitely satisfiable in M. And let $q'(\bar{y}) \vdash \phi^l(a, \bar{y})$ where $l = 0, 1$, that is, $\phi^0(a, \bar{y}) := \neg\phi(a, \bar{y})$ and $\phi^1(a, \bar{y}) := \phi(a, \bar{y})$.

If there is $\bar{c} \in \mathcal{U}$ such that $q(\bar{y}) \vdash \neg\phi^l(a, \bar{y})$ and $\models q(\bar{c})$, then we take a Morley sequence I of $q'(\bar{y})$ over everything and let $\bar{c}' := \bar{c} + I$. Thus as $\mathrm{tp}(\bar{c}'/\mathcal{U})$ is finitely satisfiable in M, $\bar{b}$

and $\bar{c}'$ are mutually indiscernible over M and neither $\bar{b}$ nor $\bar{c}'$ is indiscernible over Ma. This contradicts the dp-minimality of T.

So by the compactness theorem, there are $\theta_q(\bar{y}) \in q(\bar{y})$ and $\psi_q(x) \in \mathrm{tp}(a/M\bar{b})$ such that $\models \theta_q(\bar{y}) \wedge \psi_q(x) \longrightarrow \phi^l(x,\bar{y})$. Let $\theta_q(\bar{y}) := \theta_q(\bar{y}, \bar{b}_0\bar{b}_1\bar{m}_y)$ and $\psi(x) := \psi(x, \bar{b}_0\bar{b}_1\bar{m}_x)$ where $\bar{b}_0 := b_0^0 < b_0^1 < \cdots < b_0^{k'-1}$ and $\bar{b}_1 := b_1^0 < b_1^1 < \cdots < b_1^{l'-1}$, and $\bar{b}_0 < a = b_i < \bar{b}_1$ in the original $\bar{b}$ and $\bar{m}_x, \bar{m}_y \in M$.

As the original $\bar{b}$ is a Morley sequence of extensions of types by the defining schema, we can check the next claim.

Claim 1. $\mathrm{tp}(\bar{b}_1 a/\bar{b}_0 M)$ *is an heir of* $\mathrm{tp}(\bar{b}_1 a/M)$.

As $\mathrm{tp}(\bar{b}_1 a/\bar{b}_0 M)$ is an heir of $\mathrm{tp}(\bar{b}_1 a/M)$ and $q(\bar{y})$ are finitely satisfiable in M, for any $\bar{m} \in M$ with $\models \theta_q(\bar{m}, \bar{b}_0\bar{b}_1\bar{m}_y)$, there is $\bar{m}_0 \in M$ such that :

$\models \forall x \forall \bar{y}\, \{\theta_q(\bar{y}, \bar{m}_0\bar{b}_1\bar{m}_y) \wedge \psi_q(x, \bar{m}_0\bar{b}_1\bar{m}_x) \longrightarrow \theta^l(x,\bar{y})\,\} \wedge \{\theta_q(\bar{m}, \bar{m}_0\bar{b}_1\bar{m}_y) \wedge \psi_q(a, \bar{m}_0\bar{b}_1\bar{m}_x) \longrightarrow \theta^l(a,\bar{m})\,\}$, *that is,* $\cdots\cdots(*)$

$\forall x \forall \bar{y}\, \{\theta_q(\bar{y}, \bar{m}_0\bar{z}_1\bar{m}_y) \wedge \psi_q(x, \bar{m}_0\bar{z}_1\bar{m}_x) \longrightarrow \theta^l(x,\bar{y})\,\} \wedge \{\theta_q(\bar{m}, \bar{m}_0\bar{z}_1\bar{m}_y) \wedge \psi_q(x, \bar{m}_0\bar{z}_1\bar{m}_x) \longrightarrow \theta^l(x,\bar{m})\,\} \in \mathrm{tp}(\bar{b}_1 a/M)$.

Now we denote $\tilde{\theta}_q^{\bar{m}}(\bar{y}) := \theta_q(\bar{y}, \bar{m}_0\bar{b}_1\bar{m}_y)$ and $\tilde{\psi}_q^{\bar{m}}(x) := \psi_q(x, \bar{m}_0\bar{b}_1\bar{m}_x)$.

Let $S \subset S_{\bar{y}}(M\bar{b})$ be the set of types finitely satisfiable in M. It is closed set, thus compact and contains all types realized in M. We can extract from the family $\{\,\tilde{\theta}_q^{\bar{m}}(\bar{y}) : q(\bar{y}) \in S\,\}$ a finite subcover $\{\,\tilde{\theta}_q^{\bar{m}}(\bar{y}) : q(\bar{y}) \in S^*\,\}$. For $l = 0, 1$, let $S_l^* = \{\,q(\bar{y}) \in S^* : (*)$ holds for $l\,\}$.And we define $\tilde{\theta}_l(\bar{y}) = \bigvee_{q \in S_l^*} \tilde{\theta}_q^{\bar{m}}(\bar{y})$ and $\tilde{\psi}(x) = \bigwedge_{q \in S^*} \tilde{\psi}_q^{\bar{m}}(x)$.

We have that $\tilde{\theta}_0(\bar{y})$ and $\tilde{\theta}_1(\bar{y})$ cover S, in particular, $\tilde{\theta}_0(M) \cup \tilde{\theta}_1(M) = M^{|y|}$. Also for $l = 0, 1$, $\tilde{\psi}(x) \in \mathrm{tp}(a/M\bar{b})$ and $\mathcal{U} \models \forall \bar{y} \forall x\, \{\,\tilde{\theta}_l(\bar{y}) \wedge \tilde{\psi}(x) \longrightarrow \phi^l(x,\bar{y})\,\}$. And let the parameters of $\tilde{\theta}_l(\bar{y})$ and $\tilde{\psi}(x)$ outside M be $\bar{b}_1'$ where $a < \bar{b}_1'$ in the original $\bar{b}$.

By the same argument above, $\mathrm{tp}(\bar{b}_1'/a\,M)$ is an heir of $\mathrm{tp}(\bar{b}_1'/M)$, so $\tilde{\psi}(x)$ is satisfiable in M. As $\bar{b}_1' a$ is a Morley sequence by definability, the next claim is checked.

Claim 2. There is $m_l \in M$ such that for any $m \in M$ with $m_l < m < d$, $\models \tilde{\phi}(m)$.

Thus we can choose a point $m_l \in M$ such that for any $\bar{m}' \in M$, either for any point m with $m_l < m < d$, $\models \neg\phi(m, \bar{m}')$ or for any point m with $m_l < m < d$, $\models \phi(m, \bar{m}')$.

If we consider the bounded right noncut type of d over M, then we can take the formulas which work like $\tilde{\psi}(x)$ and $\tilde{\theta}_l(\bar{y})$ by the same way for the formula $\phi(x,\bar{y})$.

Then M is a unifromly locally o-minimal structure. ∎

Remark 14. *P.Simon proved the next Lemma.*

In the proof above, $p(x) = \mathrm{tp}(a/M)$ is definable over M, but $\mathrm{tp}(a/\bar{b}M)$ is not definable over M.

Definition 15. *Let $p(\bar{x})$ and $q(\bar{y})$ be global invariant types.*

$p(\bar{x}) \otimes q(\bar{y})$ *denotes* $\mathrm{tp}(\bar{a}\bar{b}/\mathcal{U})$ *where* $\bar{b} \models q$ *and* $\bar{a} \models p \,|\, \mathcal{U}\bar{b}$.

We say that $p(\bar{x})$ *and* $q(\bar{y})$ *commute if* $p(\bar{x}) \otimes q(\bar{y}) = q(\bar{y}) \otimes p(\bar{x})$, *and we say that* $p(\bar{x})$ *and* $q(\bar{y})$ *commute over* M *if* $p(\bar{x}) \otimes q(\bar{y}) \restriction M = q(\bar{y}) \otimes p(\bar{x}) \restriction M$.

Lemma 16. [5] *Let T be any theory.*

An M-invariant type $p(\bar{x})$ is definable if and only if for every M-finitely satisfiable type $q(\bar{y})$, $p(\bar{x}) \otimes q(\bar{y}) \restriction M = q(\bar{y}) \otimes p(\bar{x}) \restriction M$.

3. Some characterization about invariant extensions of types

There is a result about invariant extensions of types in dp-minimal theories by P.Simon.

Theorem 17. [5] *Let T be any theory. And let $p(\bar{x})$ be a global M-invariant type of dp-rank$= 1$.*

Then $p(\bar{x})$ is either definable over M or finitely satisfiable in M.

I try to characterize invariant extensions of types in locally o-minimal structures. Here I consider this problem for 1-variable types under the condition that T is NIP only.

Fact 18. *Let T be definably complete locally o-minimal and NIP. And let $M \prec \mathcal{U} \models T$.*

If $p(x) \in S_1(\mathcal{U})$ is M-invariant and both $p(x)$ and $p(x) \restriction M$ are complete by the order formulas,

then $p(x)$ is either definable over M or finitely satisfiable in M.

Even if either $p(x)$ or $p(x) \restriction M$ is not complete by the order formulas, there are cases in which $p(x)$ is definable over M or finitely satisfiable in M. Here I can not explain in detail.

4. Further problems

I continue to characterize locally o-minimal structures whose theories are dp-minimal, or finite dp-rank. After that I consider this problem under weaker conditions, for example, the theory is NIP and some additional conditions.

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