On dividing and forking in random structures

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1 Introduction

This is an exposition of a joint work with Akito Tsuboi [4].

Let $m < l < \omega$, and let R be an m-ary relation symbol. By an R-hypergraph, we mean an R-structure where R is symmetric and irreflexive.

For a finite R-hypergraph X, we define $\mathrm{e}(X)$ and $\mathrm{ne}(X)$ as follows:

$$\mathrm{e}(X) = |\{A \in [X]^m : X \models R(A)\}|,$$
$$\mathrm{ne}(X) = |\{B \in [X]^m : X \models \neg R(B)\}|.$$

Suppose X and Y are subsets of an R-hypergraph. We define $\mathrm{e}(X/Y)$ and $\mathrm{ne}(X/Y)$ as follows:

$$\mathrm{e}(X/Y) = |\{A \in [X \cup Y]^m : Y \subset A,\ XY \models R(A)\}|,$$
$$\mathrm{ne}(X/Y) = |\{B \in [X \cup Y]^m : Y \subset A,\ XY \models \neg R(B)\}|.$$

We write $\mathrm{ne}(X/y_1, \ldots, y_k)$ for $\mathrm{ne}(X/\{y_1, \ldots, y_k\})$, when Y is explicitly given. We use $\mathrm{e}(X/y_1, \ldots, y_k)$ similarly.

Let $A \in [X]^m$. We call A an *R-hyperedge* if $X \models R(A)$. We call A an *$\neg R$-hyperedge* if $X \models \neg R(A)$.

Let s be another integer. Let $\mathcal{H}^m_{l,s}$ be the class defined by:

$X \in \mathcal{H}^m_{l,s}$ if and only if whenever $A \subseteq X$ with $|A| = l$ then $\mathrm{e}(A) < \binom{l}{m} - s$.

Note that $X \in \mathcal{H}^m_{l,s}$ if and only if whenever $A \subseteq X$ with $|A| = l$ then $\mathrm{ne}(A) > s$.

Let $\mathcal{K}$ be an infinite class of finite R-structures.

M is a *random* structure for $\mathcal{K}$ if

(1) M is countable,

(2) whenever $A \subset_{finite} M$ then $A \in \mathcal{K}$, and

(3) whenever $A \subset_{finite} M$, $A \subset B \in \mathcal{K}$ and B is finite then there is an L-embedding $f : B \to M$ with $f(x) = x$ for $x \in A$.

A random structure for $\mathcal{K}$ is also known as a Fraïssé limit of $\mathcal{K}$ and a generic structure for $\mathcal{K}$.

If $\mathcal{K}$ has HP, JEP, and AP then there is a random structure for $\mathcal{K}$.

Let A, B, C and D be R-structures. We say that D is a *free amalgam* of B and C over A if $D = B \cup C$, $B \cap C = A$ as the sets of domains, and $R(D) = R(B) \cup R(C)$. We say that a class of finite R-structures $\mathcal{K}$ has the *free amalgamation property* (FAP in short) if whenever D is a free amalgam of B and C over A with A, B, $C \in \mathcal{K}$ then $D \in \mathcal{K}$.

Proposition 1 (K., Tsuboi[4]). *We have the following:*

(1) $\mathcal{H}^m_{l,s}$ *has the free amalgamation property if* $s < \binom{l-2}{m-2}$.

(2) *The amalgamation property fails in* $\mathcal{H}^m_{l,s}$ *if* $s \geq \binom{l-2}{m-2}$.

Let $F^m_{l,s}$ be a random $\mathcal{H}^m_{l,s}$ hypergraph. $F^2_{l,0}$ are known as Henson graphs. Conant proved that forking and dividing are different concepts in the theory of Henson graph $F^2_{l,0}$. While Hrushovski proved that each theory of $F^m_{l,0}$ is a simple theory of SU-rank one if $m \geq 3$. Assuming $m \geq 3$, we found that the theory of $F^m_{l,s}$ is simple theory of SU-rank one if s is small, while dividing and forking are different concepts if s is large. The general proofs are technical. We will explain the idea of the proof with certain values of l, m, and s.

2 Main Theorem

Theorem 2 (K., Tsuboi [4]). *Suppose* $3 \leq m < l$, *and* $s < \binom{l-3}{m-3}$. *Then* $\mathrm{Th}(F^m_{l,s})$ *is simple with* SU*-rank one.*

Theorem 3 (K., Tsuboi [4])**.** *Suppose* $3 \leq m < l$, *and* $\binom{l-3}{m-3} \leq s < \binom{l-2}{m-3}$. *Then dividing and forking are different concepts in* $\mathrm{Th}(F^m_{l,s})$.

Sketch of proof of Theorem 3. We work in a monster elementary extension $\mathcal{M}$ of $F^m_{l,s}$.

Assume $\binom{l-3}{m-3} \leq s < \binom{l-2}{m-2}$. Let $s = \binom{l-3}{m-3} + 3s_0 + r$ with $r = 0$, 1, or 2. With $|A_0| = l - 3$, let $\varphi(x, A_0, b, c)$ be a formula describing the following:

- $\mathrm{ne}(A_0) = 0$ (A_0 is a complete R-hypergraph of size $l - 3$),
- $\mathrm{ne}(A_0/b) = 0$, $\mathrm{ne}(A_0/c) = 0$ (A_0b and A_0c are complete R-hypergraphs of size $l - 2$),
- $\mathrm{ne}(A_0/b, c) = \binom{l-3}{m-2}$ (no R-hyperedges containing b and c),
- $\mathrm{ne}(A_0/x, b, c) = \binom{l-3}{m-3}$ (no R-hyperedges containing b, c, and x),
- $\mathrm{ne}(A_0/x, b) = s_1$ ($s_1 = s_0 + 1$ if $r = 2$, otherwise $s_1 = s_0$),
- $\mathrm{ne}(A_0/x, c) = s_1$,
- $\mathrm{ne}(A_0/x) = r_0$ ($r_0 = 1$ if $r = 1$, otherwise $r_0 = 0$).

Claim A. *There is an indiscernible sequence* $\{(b_i, c_i)\}_{i\in\omega}$ *over* A_0 *with the following properties:*

1. $A_0b_ic_i \cong A_0bc$ *as* R*-structures.*
2. $\mathrm{ne}(A_0/b_i, c_j) = 0$ *if* $i < j < \omega$ *(Full of* R*-hyperedges).*
3. $\mathrm{ne}(A_0/b_i, c_j) = \binom{l-3}{m-2}$ *if* $j \leq i < \omega$ *(No* R*-hyperedges, or full of* $\neg R$*-hyperedges).*

Properties 1–3 above specifies R-hyperedges and $\neg R$-hyperedges on $U = A_0 \cup \{b_i\}_{i<\omega} \cup \{c_i\}_{i<\omega}$. Assume further that U has no more R-hyperedges than those specified by 1–3. Choosing $D \subset U$ with $|D| = l$, we can show that $D \in \mathcal{H}^m_{l,s}$. Therefore, U can be embedded in the monster model $\mathcal{M}$. By the form of the properties 1–3, $\{(b_i, c_i)\}_{i<\omega}$ can be chosen to be an indiscernible sequence over A_0.

Claim B. $\varphi(x, A_0, b, c)$ *divides over* A_0.

Claim C. *Let $\{(b_i, c_i)\}_{i\in\omega}$ be an indiscernible sequence over A_0 with $(b_0, c_0) = (b, c)$. Suppose that $\mathrm{ne}(A_0/b_i, c_j) > s_0$ for all $i \neq j$ with $i, j \in \omega$. Then the set $\{\varphi(x, A_0, b_i, c_i) : i \in \omega\}$ is consistent.*

Claim D. *Let $A^* = (A_0, b_1, b_2 \ldots, b_{n^*})$ be a tuple with $(A_0, b_i, b_j) \cong (A_0, b, c)$ for $1 \leq i < j \leq n^*$, where n^* is a sufficiently large integer. Let $\psi(x, A^*)$ be the formula*

$$\bigvee_{1\leq i<j\leq n^*} \varphi(x, A_0, b_i, b_j).$$

Then $\psi(x, A^*)$ does not divide over A_0.

Supporse $\psi(x, A^*)$ divides over A_0.

Choose an indiscernible sequence $\{(b_{i,1}, \ldots, b_{i,n^*})\}_{i\in\omega}$ over A_0 witnessing the dividing with $(b_{0,1}, \ldots, b_{0,n^*}) = (b_1, \ldots, b_{n^*})$.

Suppose $1 \leq i < j \leq n^*$. By indescernibility, we have the following:

(a) $\mathrm{ne}(A_0/b_{n,i}, b_{n',j}) \leq s_0$ for $n < n' < \omega$, or

(b) $\mathrm{ne}(A_0/b_{n,i}, b_{n',j}) > s_0$ for $n < n' < \omega$.

Also,

(c) $\mathrm{ne}(A_0/b_{n,j}, b_{n',i}) \leq s_0$ for $n < n' < \omega$, or

(d) $\mathrm{ne}(A_0/b_{n,j}, b_{n',i}) > s_0$ for $n < n' < \omega$.

If (b) and (d) hold simultaneously, $\{\varphi(x, A_0, b_{n,i}, b_{n,j}) : n < \omega\}$ is consistent by Claim B. This implies $\{\psi(x, A_0, b_{n,1}, \ldots, b_{n,n^*}) : n < \omega\}$ is consistent, contradicting our assumption.

At least one of the following is true if $1 \leq i < j \leq n^*$:

(i) $\mathrm{ne}(A_0/b_{n,i}, b_{n',j}) \leq s_0$ for $n < n' < \omega$.

(ii) $\mathrm{ne}(A_0/b_{n,j}, b_{n',i}) \leq s_0$ for $n < n' < \omega$.

Taking n^* sufficiently large, we can use Ramsey's Theorem to choose i, j, k satisfying $1 \leq i, j, k \leq n^*$,

$\mathrm{ne}(A_0/b_{0,i}, b_{1,j}) \leq s_0$,

$\mathrm{ne}(A_0/b_{1,j}, b_{2,k}) \leq s_0$, and

$\mathrm{ne}(A_0/b_{0,i}, b_{2,k}) \leq s_0$.

But we have $\mathrm{ne}(A_0 b_{0,i} b_{1,j} b_{2,k}) \leq s$. Since $A_0 b_{0,i} b_{1,j} b_{2,k} \subset \mathcal{M}$, we have $A_0 b_{0,i} b_{1,j} b_{2,k} \in \mathcal{H}^m_{l,s}$. This is a contradiction. □

We describe more detailed proofs of Claims B and C for parameters $m = 4$, $l = 8$, and $s = 11$, 12, 13.

Proof of Claims B, and C with $m = 4$, $l = 8$, and $s = 11$. We have

$$\binom{l-3}{m-3} = \binom{5}{1} = 5, \quad \binom{l-2}{m-2} = \binom{6}{2} = 15, \quad \binom{l-3}{m-2} = \binom{5}{2} = 10,$$

and $s = 11 = 5 + 3 \cdot 2$. A_0 has exactly $l - 3 = 5$ elements. The formula $\varphi(x, A_0, b, c)$ describes the following:

- A_0 is a complete R-hypergraph of size 5, i.e., $\mathrm{ne}(A_0) = 0$.
- $A_0 b$ is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/b) = 0$.
- $A_0 c$ is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/c) = 0$.
- $A_0 x$ is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/x) = 0$.
- There are no R-hyperedges containing both b and c, i.e., the number of $\neg R$-hyperedges containing both b and c is maximal. In other words, $\mathrm{ne}(A_0/b, c) = \binom{l-3}{m-2} = 10$.
- $\mathrm{ne}(A_0/x, b) = 2$.
- $\mathrm{ne}(A_0/x, c) = 2$.
- There are no R-edges containing b, c, and x, i.e., the number of $\neg R$-hyperedges containing x, b and c is maximal. In other words, $\mathrm{ne}(A_0/x, b, c) = \binom{l-3}{m-3} = 5$.

Consider $D = A_0 xbc$ such that $\varphi(x, A_0, b, c)$ holds. We can make such R-hypergraph D. We have $|D| = 8$, and

$$\mathrm{ne}(D) \geq \mathrm{ne}(A_0/bc) + \mathrm{ne}(A_0/xbc) = 10 + 5 > 11 = s.$$

Therefore, $D \in \mathcal{H}^4_{8,11}$ and there is a copy of D in $F^4_{8,11}$.

We can assume that $A_0 bc$ is a substructure of $F^4_{8,11}$.

Consider an indiscernible sequence $\{(b_i, c_i)\}_{i<\omega}$ over A_0 from Claim A. We have the following:

- $\mathrm{ne}(A_0/b_i, c_j) = 0$ if $i < j < \omega$.
- $\mathrm{ne}(A_0/b_i, c_j) = \binom{5}{2} = 10$ if $j \leq i < \omega$.

Suppose that there is an element $d \in \mathcal{M}$ and there are integers i, j with $i < j < \omega$, $\varphi(d, A_0, b_i, c_i)$, and $\varphi(d, A_0, b_j, c_j)$. Consider $D_1 = A_0 d b_i c_j$. We have $|D_1| = 8$ while

$$\begin{aligned} \mathrm{ne}(D_1) = \mathrm{ne}(A_0) &+ \mathrm{ne}(A_0/d, b_i, c_j) \\ &+ \mathrm{ne}(A_0/d, b_i) + \mathrm{ne}(A_0/d, c_j)\mathrm{ne}(A_0/b_i, c_j) \\ &+ \mathrm{ne}(A_0/d) + \mathrm{ne}(A_0/b_i) + \mathrm{ne}(A_0/c_j) \\ = 0 &+ 5 + 2 + 2 + 0 + 0 + 0 + 0 = 9 < 11 = s. \end{aligned}$$

This contradicts with $D_1 \in \mathcal{H}^4_{8,11}$. Therefore, $\varphi(x, A_0, b, c)$ divides over A_0. This finishes a proof of Claim B.

We turn to a proof of Claim C. Let $\{(b_i, c_i)\}_{i \in \omega}$ be an indiscernible sequence over A_0 with $(b_0, c_0) = (b, c)$. Suppose that $\mathrm{ne}(A_0/b_i, c_j) > s_0$ for all $i \neq j$ with $i, j \in \omega$.

Our aim is to prove that $\Sigma(x) = \{\varphi(x, A_0, b_i, c_i) : i < \omega\}$ is consistent.

By strengthning the formula φ, We can assume that $\varphi(x, A_0, b_i, c_i)$ specifies all R-hyperedges and $\neg R$-hyperedges on $A_0 x b_i c_i$.

Put $I = \{b_i\}_{i<\omega} \cup \{c_i\}_{i<\omega}$ and let $U = A_0 \cup I$ be a substructure of $\mathcal{M}$. Consider an extension Ux of R-structure U by adding R-hyperedges specified in $\Sigma(x)$. We add no more R-hyperedges to Ux other than those specified in $\Sigma(x)$. Note that there are no R-hyperedge Y on Ux such that $\{x, d, d'\} \subset Y$ with $d, d' \in I$ and $d \neq d'$.

We show that any $D \subset Ux$ with $|D| = 8$ belongs to $\mathcal{H}^4_{8,11}$. Then x can be embedded to the monster model.

There are several cases to consider.

Case $A_0 \subset D$. Since $|D - A_0| = 3$, one of the following holds in this case.

1. $D = A_0 b_i c_j x$.

2. $D = A_0 b_i b_j x$ with $i < j$.

3. $D = A_0 c_i c_j x$ with $i < j$.

Note that 2 and 3 are essentially the same case.

Suppose $D = A_0 b_i c_j x$. If $i = j$ then $\varphi(x, A_0, b_i, c_i)$ holds in Ux. Hence, $\mathrm{ne}(D) > 11 = s$.

If $i \neq j$ then $\mathrm{ne}(A_0/b_i, c_j) > s_0 = 2$ by the assumption. Also, $\mathrm{ne}(A_0/b_i, c_j, x) = \binom{5}{1} = 5$ by the definition of the structure Ux. Hence,

$$\begin{aligned}\mathrm{ne}(D) &\geq \mathrm{ne}(A_0/b_i, c_j, x) + \mathrm{ne}(A_0/b_i, x) + \mathrm{ne}(A_0/c_i, x) + \mathrm{ne}(A_0/b_i, c_j) \\ &> 5 + 2 + 2 + 2 = 11 = s.\end{aligned}$$

Now, suppose $D = A_0 b_i b_j x$ with $i \neq j$.

First, we claim that $\mathrm{ne}(A_0/b_i, b_j) > s_0 = 2$ for any $i < j$. Note that $\mathrm{ne}(A_0/b_i, b_j) = \mathrm{ne}(A_0/b_0, b_1)$ by indiscernibility. So, otherwise, we have $\mathrm{ne}(A_0/b_i, b_j) \leq 2$ for any $i < j$. Consider $D_0 = A_0 b_0 b_1 b_2$. We have $\mathrm{ne}(A_0) = 0$, $\mathrm{ne}(A_0/b_i) = 0$ for any i. Hence,

$$\begin{aligned}\mathrm{ne}(D_0) &= \mathrm{ne}(A_0/b_0, b_1, b_2) + \mathrm{ne}(A_0/b_0, b_1) + \mathrm{ne}(A_0/b_1, b_2) + \mathrm{ne}(A_0/b_0, b_2) \\ &\leq 5 + 2 + 2 + 2 = 11 = s.\end{aligned}$$

But since $D_0 \subset \mathcal{M}$, D_0 should be a member of $\mathcal{H}^m_{l,s}$. A contradiction.

Note that $\mathrm{ne}(A_0/b_i, b_j, x) = 5$. So, we have

$$\begin{aligned}\mathrm{ne}(D) &\geq \mathrm{ne}(A_0/b_i, b_j, x) + \mathrm{ne}(A_0/b_i, x) + \mathrm{ne}(A_0/b_j, x) + \mathrm{ne}(A_0/b_i, b_j) \\ &> 5 + 2 + 2 + 2 = 11 = s.\end{aligned}$$

Case $A_0 - D$ is non-empty. Choose $a \in A_0 - D$. If $x \notin D$ then $D \subset \mathcal{M}$. Then $D \in \mathcal{H}^m_{l,s}$. So, we can assume that $x \in D$. Consider a map $\sigma : D \to \mathcal{M}$ such that $\sigma(x) = a$, and σ is an identity map on $D - \{x\}$. σ is clearly an injective map. We claim that σ is an R-homomorphism. That is, any R-hyperedge on D is mapped to an R-hyperedge on $\sigma(D)$. Let Y be an R-hyperedge on D. If $x \notin Y$, then $\sigma Y = Y$ and there is nothing to prove. If $x \in Y$, $Y = A_1 \cup \{x\}$ with $A_1 \subset A_0$ or $Y = A_2 \cup \{x, d\}$ with $A_2 \subset A_0$ and $d \in I$. In either cases, Y is mapped to an R-hyperedge in $\mathcal{M}$. We have $\mathrm{e}(D) \leq \mathrm{e}(\sigma(D))$. Therefore, $\mathrm{ne}(\sigma(D)) \leq \mathrm{ne}(D)$. Since $\sigma(D) \subset \mathcal{M}$, we have $s < \mathrm{ne}(\sigma(D))$. Therefore, $s < \mathrm{ne}(D)$ and hence $D \in \mathcal{H}^m_{l,s}$. □

Proof of Claims B, and C with $m = 4$, $l = 8$, and $s = 12$. We have

$$\binom{l-3}{m-3} = \binom{5}{1} = 5, \quad \binom{l-2}{m-2} = \binom{6}{2} = 15, \quad \binom{l-3}{m-2} = \binom{5}{2} = 10,$$

and $s = 11 = 5 + 3 \cdot 2 + 1$.

A_0 has exactly $l - 3 = 5$ elements. The formula $\varphi(x, A_0, b, c)$ describes the following:

- A_0 is a complete R-hypergraph of size 5, i.e., $\mathrm{ne}(A_0) = 0$.
- $A_0 b$ is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/b) = 0$.
- $A_0 c$ is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/c) = 0$.
- $\mathrm{ne}(A_0/x) = 1$.
- $\mathrm{ne}(A_0/b, c) = \binom{l-3}{m-2} = 10$.
- $\mathrm{ne}(A_0/x, b) = 2$.
- $\mathrm{ne}(A_0/x, c) = 2$.
- $\mathrm{ne}(A_0/x, b, c) = \binom{l-3}{m-3} = 5$.

Consider $D = A_0xbc$ such that $\varphi(x, A_0, b, c)$ holds. We can make such R-hypergraph D. We have $|D| = 8$, and

$$\mathrm{ne}(D) \geq \mathrm{ne}(A_0/bc) + \mathrm{ne}(A_0/xbc) = 10 + 5 > 12 = s.$$

Therefore, $D \in \mathcal{H}^4_{8,12}$ and there is a copy of D in $F^4_{8,12}$.

We can assume that A_0bc is a substructure of $F^4_{8,12}$.

Consider an indiscernible sequence $\{(b_i, c_i)\}_{i<\omega}$ over A_0 from Claim A. We have the following:

- $\mathrm{ne}(A_0/b_i, c_j) = 0$ if $i < j < \omega$.
- $\mathrm{ne}(A_0/b_i, c_j) = \binom{5}{2} = 10$ if $j \leq i < \omega$.

Suppose that there is an element $d \in \mathcal{M}$ and there are integers i, j with $i < j < \omega$, $\varphi(d, A_0, b_i, c_i)$, and $\varphi(d, A_0, b_j, c_j)$. Consider $D_1 = A_0 d b_i c_j$. We have $|D_1| = 8$ while

$$\begin{aligned}\mathrm{ne}(D_1) &= \mathrm{ne}(A_0) + \mathrm{ne}(A_0/d, b_i, c_j) \\ &\qquad + \mathrm{ne}(A_0/d, b_i) + \mathrm{ne}(A_0/d, c_j) + \mathrm{ne}(A_0/b_i, c_j) \\ &\qquad + \mathrm{ne}(A_0/d) + \mathrm{ne}(A_0/b_i) + \mathrm{ne}(A_0/c_j) \\ &= 0 + 5 + 2 + 2 + 0 + 1 + 0 + 0 = 10 < 12 = s.\end{aligned}$$

This contradicts with $D_1 \in \mathcal{H}^4_{8,12}$. Therefore, $\varphi(x, A_0, b, c)$ divides over A_0. This finishes a proof of Claim B.

We turn to a proof of Claim C. Let $\{(b_i, c_i)\}_{i\in\omega}$ be an indiscernible sequence over A_0 with $(b_0, c_0) = (b, c)$. Suppose that $\mathrm{ne}(A_0/b_i, c_j) > s_0$ for all $i \neq j$ with $i, j \in \omega$.

Our aim is to prove that $\Sigma(x) = \{\varphi(x, A_0, b_i, c_i) : i < \omega\}$ is consistent.

By strengthning the formula φ, We can assume that $\varphi(x, A_0, b_i, c_i)$ specifies all R-hyperedges and $\neg R$-hyperedges on $A_0 x b_i c_i$.

Put $I = \{b_i\}_{i<\omega} \cup \{c_i\}_{i<\omega}$ and let $U = A_0 \cup I$ be a substructure of $\mathcal{M}$. Consider an extension Ux of R-structure U by adding R-hyperedges specified in $\Sigma(x)$. We add no more R-hyperedges to Ux other than those specified in $\Sigma(x)$. Note that there are no R-hyperedge Y on Ux such that $\{x, d, d'\} \subset Y$ with $d, d' \in I$ and $d \neq d'$.

We show that any $D \subset Ux$ with $|D| = 8$ belongs to $\mathcal{H}^4_{8,12}$. Then x can be embedded to the monster model.

There are several cases to consider.

Case $A_0 \subset D$. Since $|D - A_0| = 3$, one of the following holds in this case.

1. $D = A_0 b_i c_j x$.
2. $D = A_0 b_i b_j x$ with $i < j$.
3. $D = A_0 c_i c_j x$ with $i < j$.

Note that 2 and 3 are essentially the same case.

Suppose $D = A_0 b_i c_j x$. If $i = j$ then $\varphi(x, A_0, b_i, c_i)$ holds in Ux. Hence, $\mathrm{ne}(D) > 12 = s$.

If $i \neq j$ then $\mathrm{ne}(A_0/b_i, c_j) > s_0 = 2$ by the assumption. Also, $\mathrm{ne}(A_0/b_i, c_j, x) = \binom{5}{1} = 5$ by the definition of the structure Ux. Hence,

$$\begin{aligned}\mathrm{ne}(D) &\geq \mathrm{ne}(A_0/b_i, c_j, x) + \mathrm{ne}(A_0/b_i, x) + \mathrm{ne}(A_0/c_i, x) + \mathrm{ne}(A_0/b_i, c_j)\\ &\quad + \mathrm{ne}(A_0/x)\\ &> 5 + 2 + 2 + 2 + 1 = 12 = s.\end{aligned}$$

Now, suppose $D = A_0 b_i b_j x$ with $i \neq j$.

First, we claim that $\mathrm{ne}(A_0/b_i, b_j) > s_0 = 2$ for any $i < j$. Note that $\mathrm{ne}(A_0/b_i, b_j) = \mathrm{ne}(A_0/b_0, b_1)$ by indiscernibility. So, otherwise, we have $\mathrm{ne}(A_0/b_i, b_j) \leq 2$ for any $i < j$. Consider $D_0 = A_0 b_0 b_1 b_2$. We have

$\mathrm{ne}(A_0) = 0$, $\mathrm{ne}(A_0/b_i) = 0$ for any i. Hence,

$$\begin{aligned}\mathrm{ne}(D_0) &= \mathrm{ne}(A_0/b_0, b_1, b_2) + \mathrm{ne}(A_0/b_0, b_1) + \mathrm{ne}(A_0/b_1, b_2) + \mathrm{ne}(A_0/b_0, b_2)\\ &\leq 5 + 2 + 2 + 2 = 11 < 12 = s.\end{aligned}$$

But since $D_0 \subset \mathcal{M}$, D_0 should be a member of $\mathcal{H}^m_{l,s}$. A contradiction.

Note that $\mathrm{ne}(A_0/b_i, b_j, x) = 5$. So, we have

$$\begin{aligned}\mathrm{ne}(D) \geq\ &\mathrm{ne}(A_0/b_i, b_j, x) + \mathrm{ne}(A_0/b_i, x) + \mathrm{ne}(A_0/b_j, x) + \mathrm{ne}(A_0/b_i, b_j)\\ &+ \mathrm{ne}(A_0/x)\\ >\ &5 + 2 + 2 + 2 + 1 = 12 = s.\end{aligned}$$

Case $A_0 - D$ is non-empty. The proof is the same as the previous one. □

Proof of Claims B, and C with $m = 4$, $l = 8$, and $s = 13$. We have

$$\binom{l-3}{m-3} = \binom{5}{1} = 5, \quad \binom{l-2}{m-2} = \binom{6}{2} = 15, \quad \binom{l-3}{m-2} = \binom{5}{2} = 10,$$

and $s = 11 = 5 + 3 \cdot 2 + 2$.

A_0 has exactly $l - 3 = 5$ elements. The formula $\varphi(x, A_0, b, c)$ describes the following:

- A_0 is a complete R-hypergraph of size 5, i.e., $\mathrm{ne}(A_0) = 0$.
- A_0b is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/b) = 0$.
- A_0c is a complete R-hypergraph, i.e., $\mathrm{ne}(A_0/c) = 0$.
- $\mathrm{ne}(A_0/x) = 0$.
- $\mathrm{ne}(A_0/b, c) = \binom{l-3}{m-2} = 10$.
- $\mathrm{ne}(A_0/x, b) = 3$.
- $\mathrm{ne}(A_0/x, c) = 3$.
- $\mathrm{ne}(A_0/x, b, c) = \binom{l-3}{m-3} = 5$.

Consider $D = A_0xbc$ such that $\varphi(x, A_0, b, c)$ holds. We can make such R-hypergraph D. We have $|D| = 8$, and

$$\mathrm{ne}(D) \geq \mathrm{ne}(A_0/bc) + \mathrm{ne}(A_0/xbc) = 10 + 5 > 13 = s.$$

Therefore, $D \in \mathcal{H}^4_{8,13}$ and there is a copy of D in $F^4_{8,13}$.

We can assume that A_0bc is a substructure of $F^4_{8,13}$.

Consider an indiscernible sequence $\{(b_i, c_i)\}_{i<\omega}$ over A_0 from Claim A. We have the following:

- $\mathrm{ne}(A_0/b_i, c_j) = 0$ if $i < j < \omega$.
- $\mathrm{ne}(A_0/b_i, c_j) = \binom{5}{2} = 10$ if $j \leq i < \omega$.

Suppose that there is an element $d \in \mathcal{M}$ and there are integers i, j with $i < j < \omega$, $\varphi(d, A_0, b_i, c_i)$, and $\varphi(d, A_0, b_j, c_j)$. Consider $D_1 = A_0db_ic_j$. We have $|D_1| = 8$ while

$$\begin{aligned}
\mathrm{ne}(D_1) &= \mathrm{ne}(A_0) + \mathrm{ne}(A_0/d, b_i, c_j) \\
&\qquad + \mathrm{ne}(A_0/d, b_i) + \mathrm{ne}(A_0/d, c_j) + \mathrm{ne}(A_0/b_i, c_j) \\
&\qquad + \mathrm{ne}(A_0/d) + \mathrm{ne}(A_0/b_i) + \mathrm{ne}(A_0/c_j) \\
&= 0 + 5 + 3 + 3 + 0 + 0 + 0 + 0 = 11 < 13 = s.
\end{aligned}$$

This contradicts with $D_1 \in \mathcal{H}^4_{8,13}$. Therefore, $\varphi(x, A_0, b, c)$ divides over A_0. This finishes a proof of Claim B.

We turn to a proof of Claim C. Let $\{(b_i, c_i)\}_{i\in\omega}$ be an indiscernible sequence over A_0 with $(b_0, c_0) = (b, c)$. Suppose that $\mathrm{ne}(A_0/b_i, c_j) > s_0$ for all $i \neq j$ with $i, j \in \omega$.

Our aim is to prove that $\Sigma(x) = \{\varphi(x, A_0, b_i, c_i) : i < \omega\}$ is consistent.

By strengthning the formula φ, We can assume that $\varphi(x, A_0, b_i, c_i)$ specifies all R-hyperedges and $\neg R$-hyperedges on $A_0xb_ic_i$.

Put $I = \{b_i\}_{i<\omega} \cup \{c_i\}_{i<\omega}$ and let $U = A_0 \cup I$ be a substructure of $\mathcal{M}$. Consider an extension Ux of R-structure U by adding R-hyperedges specified in $\Sigma(x)$. We add no more R-hyperedges to Ux other than those specified in $\Sigma(x)$. Note that there are no R-hyperedge Y on Ux such that $\{x, d, d'\} \subset Y$ with $d, d' \in I$ and $d \neq d'$.

We show that any $D \subset Ux$ with $|D| = 8$ belongs to $\mathcal{H}^4_{8,13}$. Then x can be embedded to the monster model.

There are several cases to consider.

Case $A_0 \subset D$. Since $|D - A_0| = 3$, One of the following holds in this case.

1. $D = A_0 b_i c_j x$.

2. $D = A_0 b_i b_j x$ with $i < j$.

3. $D = A_0 c_i c_j x$ with $i < j$.

Note that 2 and 3 are essentially the same case.

Suppose $D = A_0 b_i c_j x$. If $i = j$ then $\varphi(x, A_0, b_i, c_i)$ holds in Ux. Hence, $\mathrm{ne}(D) > 13 = s$.

If $i \neq j$ then $\mathrm{ne}(A_0/b_i, c_j) > s_0 = 2$ by the assumption. Also, $\mathrm{ne}(A_0/b_i, c_j, x) = \binom{5}{1} = 5$ by the definition of the structure Ux. Hence,

$$\begin{aligned}\mathrm{ne}(D) &\geq \mathrm{ne}(A_0/b_i, c_j, x) + \mathrm{ne}(A_0/b_i, x) + \mathrm{ne}(A_0/c_i, x) + \mathrm{ne}(A_0/b_i, c_j) \\ &> 5 + 3 + 3 + 2 = 13 = s.\end{aligned}$$

Now, suppose $D = A_0 b_i b_j x$ with $i \neq j$.

First, we claim that $\mathrm{ne}(A_0/b_i, b_j) > s_0 = 2$ for any $i < j$. Note that $\mathrm{ne}(A_0/b_i, b_j) = \mathrm{ne}(A_0/b_0, b_1)$ by indiscernibility. So, otherwise, we have $\mathrm{ne}(A_0/b_i, b_j) \leq 2$ for any $i < j$. Consider $D_0 = A_0 b_0 b_1 b_2$. We have $\mathrm{ne}(A_0) = 0$, $\mathrm{ne}(A_0/b_i) = 0$ for any i. Hence,

$$\begin{aligned}\mathrm{ne}(D_0) &= \mathrm{ne}(A_0/b_0, b_1, b_2) + \mathrm{ne}(A_0/b_0, b_1) + \mathrm{ne}(A_0/b_1, b_2) + \mathrm{ne}(A_0/b_0, b_2) \\ &\leq 5 + 3 + 3 + 2 = 13 = s.\end{aligned}$$

But since $D_0 \subset \mathcal{M}$, D_0 should be a member of $\mathcal{H}^m_{l,s}$. A contradiction.

Note that $\mathrm{ne}(A_0/b_i, b_j, x) = 5$. So, we have

$$\begin{aligned}\mathrm{ne}(D) &\geq \mathrm{ne}(A_0/b_i, b_j, x) + \mathrm{ne}(A_0/b_i, x) + \mathrm{ne}(A_0/b_j, x) + \mathrm{ne}(A_0/b_i, b_j) \\ &> 5 + 3 + 3 + 2 = 13 = s.\end{aligned}$$

Case $A_0 - D$ is non-empty. The proof is the same as the previous one. □

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