A note on stability of generic structures

Koichiro Ikeda *
Faculty of Business Administration, Hosei University

In this short note, we want to prove that under some assumption the theory of a generic structure is stable (Proposition 0.6). This result is a preparation for showing an existence of stable theories having a type with the tree property.

Assumption 0.1 (The language L) Let $R(x, y)$ be a binary relation which is irreflexive and symmetric. For each $i \in \omega$, let $U_i(x)$ be a unary relation with $\models \forall x(U_i(x) \to U_{i+1}(x))$, and $V(x, y)$ a binary relation. Let $L = \{R, U_0, U_1, \cdots, V\}$. For any $a \in A$, we define the weight of a, $w(a)$, as follows: For each $n \in \omega$, $w(a) = n$ if $\models U_n(a) \wedge \neg U_{n+1}(a)$, and $w(a) = \omega$ if $\models \neg U_n(a)$ for any $n \in \omega$.

$\{R\}$-structures are considered as (undirected simple) graphs. Let $A, B, C, \cdots$ be finite graphs. For α with $0 < \alpha < 1$, we define a predimension of A by $\delta_\alpha(A) = |A| - \alpha|R^A|$. We often abbreviate $\delta_\alpha(*)$ to $\delta(*)$. Let denote $\delta(B/A) = \delta(B \cup A) - \delta(A)$. For finite graphs $A \subset B$, A is said to be closed in B (in symbol $A \leq B$), if $\delta(X/A) \geq 0$ for any $X \subset B - A$. Let denote the closure of A in B by $\mathrm{cl}_B(A) = \bigcap\{C : A \subset C \leq B\}$.

For A, B with $A = B \cap C$, B and C is said to be free over A (in symbol $B \perp_A C$), if $R^{B \cup C} = R^B \cup R^C$. Moreover, the free amalgam B and C over A is defined by $B \oplus_A C = (B \cup C, R^B \cup R^C)$.

Let $\mathcal{K}$ be a class of finite L-structures closed under substructures. Then $\mathcal{K}$ is said to have the free amalgamation property, if whenever $A \leq B, C \in \mathcal{K}$ then $B \oplus_A C \in \mathcal{K}$. A countable L-structure M is said to be $\mathcal{K}$-generic, if it satisfies the following conditions.

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1. $A \subset_{\text{fin}} M$ implies $A \in \mathcal{K}$,
2. if $A \leq B \in \mathcal{K}$ and $A \leq M$, then there exists some $B' \cong_A B$ with $B' \leq M$,
3. if $A \subset_{\text{fin}} M$, then $\text{cl}_M(A)$ is finite.

Let $\mathcal{K}$ be a class of finite L-structures which is closed under substructures and free amalgam. $\mathcal{K}$ is said to have a control function, if there exists an unbounded function $f : \omega \to \mathbb{R}$ such that $\delta(A) \geq f(|A|)$ for any $A \in \mathcal{K}$. For $A, B \in \mathcal{K}$, $A \cong^n B$ means that A is isomorphic to B in $L_n = \{R, U_0, \cdots, U_n, V\}$.

Assumption 0.2 (The class $\mathcal{K}$) Suppose that $\mathcal{K}$ satisfies the following:

1. $\mathcal{K}$ is closed under free amalgam,
2. $\mathcal{K}$ has a control function,
3. For any A, B with $A \leq B \in \mathcal{K}$ there is $m = m(A, B) \in \omega$ such that if $n \geq m$ and $A' \cong^n A$ then there is $B' \in \mathcal{K}$ with $A'B' \cong^n AB$.

By Assumption 0.2.1, there exists the $\mathcal{K}$-generic structure M.

Lemma 0.3 The $\mathcal{K}$-generic M is saturated.

Proof. Let $\mathcal{M}$ be a big model of $\text{Th}(M)$. By Assumption 0.2.2, $\mathcal{K}$ has a control function f. So, for finite A, $A \leq \mathcal{M}$ can be expressed by a first order formula. Take any A, B with $A \leq B \in \mathcal{K}$. By Assumption 0.2.3, take $m = m(A, B)$.
Claim: $\mathcal{M} \models \forall X(X \cong^n A \wedge X \leq \mathcal{M}) \to \exists Y(XY \cong^n AB \wedge Y \leq \mathcal{M}))$ for any $n \geq m$.
Proof. It is enough to show that the above statement is true in M. Take any $A' \leq M$ with $A' \cong^n A$. By Assumption 0.2.3, there is $B' \in \mathcal{K}$ with $A'B' \cong^n AB$. By genericty, there is $B'' \leq M$ with $B'' \cong_{A'} B'$.

We want to show that M is saturated. Take any finite $A \subset M$ and any $p \in S(A)$. We can assume that $A \leq M$. Take any $e \models p$ in $\mathcal{M}$ and let $B = \text{cl}(eA)$. By genericity, we can take some $B' \leq M$ with $B' \cong_A B$. Next, take any $b' \in \mathcal{M} - B'$ and let $B_1' = \text{cl}(b'B')$. By the claim, there is $B_1 \leq \mathcal{M}$ with $B_1 B \cong B_1' B'$. Iterationg this process, we have $\text{tp}(B'/A) = \text{tp}(B/A)$. Hence we can take an realization $b' \models p$ in M.

For $C \subset \mathcal{M}$, let $S_V(C)$ denote the set of maximal consistent set of the form $V(x, c)$ or $\neg V(x, c)$ where $c \in C$. Let $\text{tp}_V(B/C)$ be an element of $S_V(C)$ whose realization is B.

Remark 0.4 Let $\mathrm{tp}(B_1/A) = \mathrm{tp}(B_2/A)$ and $B_i \perp_A C, B_i C \leq \mathcal{M}$ for each $i = 1, 2$. Then $\mathrm{tp}_V(B_1/C) = \mathrm{tp}_V(B_2/C)$ implies $\mathrm{tp}(B_1/C) = \mathrm{tp}(B_2/C)$.

Proof. Since $\mathrm{tp}(B_1/A) = \mathrm{tp}(B_2/A)$ and $B_i \perp_A C$, we have $B_1 \cong_C B_2$ in $L - \{V\}$. Since $\mathrm{tp}_V(B_1/C) = \mathrm{tp}_V(B_2/C)$, we have $B_1 \cong_C B_2$ in L. By $B_1C, B_2C \leq \mathcal{M}$, we have $\mathrm{tp}(B_1/C) = \mathrm{tp}(B_2/C)$.

For finite $A, B \subset \mathcal{M}$, let $d(A) = \delta(\mathrm{cl}(A))$. Let denote $d(A/B) = d(AB) - d(B)$. For possibly infinite B, we define $d(A/B) = \inf\{d(A/B_0) : B_0 \subset_{\mathrm{fin}} B\}$. Then it is known the following.

Fact 0.5 ([1, 4]) Let $B, C \leq \mathcal{M}$ with $A = B \cap C$ and $B = \mathrm{cl}(bA)$ for some $b \in B$. Then $d(B/C) = d(B/A)$ if and only if $B \perp_A C$ and $BC \leq \mathcal{M}$.

$V(x, y)$ is said to have the order property, if there are $a_0, a_1, \cdots, b_0, b_1, \cdots$ such that for all i, j, $\models V(a_i, b_j)$ iff $i < j$. It is known that $V(x, y)$ has the order property if and only if $|S_V(A)| \leq |A|$ for any $A \subset \mathcal{M}$.

Proposition 0.6 Let L be a language satisfying Assumption 0.1, and $\mathcal{K}$ a class of finite L-structures satisfying Assumption 0.2. If $V(x, y)$ does not have the order property, then the $\mathcal{K}$-generic structure is stable.

Proof. Take any $\lambda = 2^{\aleph_0}$ and $N \prec \mathcal{M}$ with $|N| = \lambda$. We want to count the number of types over N. Take any $e \in \mathcal{M} - N$. Then there is a countable subset $A \subset N$ with $d(e/N) = d(e/A)$. We can assume that $\mathrm{cl}(eA) \cap N = A$. Let $B = \mathrm{cl}(eA)$. Then $d(B/N) = d(B/A)$. Since $V(x, y)$ does not have the order property, $|S_V(N)| \leq |N| = \lambda$. By Remark 0.4 and Fact 0.5, we have $|S(N)| \leq |\mathcal{P}_{\leq\omega}(N)| \times |S_V(N)| \times$(the number of all types over a countable set)$\leq \lambda^{\aleph_0} \times \lambda \times 2^{\aleph_0} = \lambda$.

Remark 0.7 Let T be a stable theory in a countable language. Then $p \in S(T)$ is said to be powerful, if any model realizing p realizes all type $q \in S(T)$. The following two notions can be found in [2]: $p \in S(T)$ is said to have the tree property, if there are $a, b, c \models p$ such that $b \downarrow c$ and $\mathrm{tp}(bc/a)$ is isolated. $p \in S(T)$ is said to have infinite weight, if there are $a, b_1, b_2, \cdots \models p$ with $\downarrow \{b_1, b_2, \cdots\}$ and $a \not\downarrow b_i$ for each $i \in \omega$. It is known that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ holds, where

1. $1 < I(\aleph_0, T) < \aleph_0$.

2. T has a non-isolated powerful type.
3. T has a type with the tree property.
4. T has a type of infinite weight.

The Lachlan conjecture says that there is no stable theory T with $1 < I(\aleph_0, T) < \aleph_0$. By the above statement, if there would be a stable theory T with $1 < I(\aleph_0, T) < \aleph_0$, then T must have a type of infinite weight. In fact, Herwig has proved that there is a generic structure whose theory is stable with a type of infinite weight([2]). Our final goal is to show that there is a stable generic structure having a type with the tree property.

Question 0.8 Is there a stable generic structure whose theory has a non-isolated powerful type?

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Faculty of Business Administration
Hosei University
2-17-1 Fujimi, Chiyoda-ku
Tokyo 102-8160
JAPAN
E-mail address: ikeda@hosei.ac.jp