A QUESTION ON SEMI-RETRACTIONS AND PRE-ADJUNCTIONS

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Abstract. We survey recent progress on open questions that we stated in [4] as well as related work. We suggest some new directions for research in this area.

1. Introduction

This note explores recent progress related to open questions stated in our joint paper [4]. In the latter paper, we investigated the notion of *semi-retraction* (defined in [17]) from multiple perspectives. The second author was motivated by the application of semi-retractions to generalized indiscernible sequences (see Definition 2.23 of [4]). The *Ramsey property* (see Definition 2.4) for the age of a locally finite ordered structure $\mathcal{I}$ is equivalent to a useful property called the *modeling property* for I-indexed indiscernible sequences (see Theorem 2.30 of [4] for the most recently refined statement of this equivalence). The first author was motivated by the goal of formulating the interaction between semi-retractions and the Ramsey property in a way that is less bound to first-order syntax, but is truer to the perspective of topological dynamics.

Semi-retractions have a direct connection to generalized indiscernible sequences (as stated in Theorem 2.33 of [4]), but a more indirect connection to the Ramsey property, as stated in Theorem 2.9 and Corollary 2.10. In [4], we were motivated by the question of which minimal conditions are required for a semi-retraction to transfer the Ramsey property between the ages of two first-order structures. We were also interested in finding multiple examples of semi-retractions, to better understand their function in explaining how the Ramsey property for the age of one structure requires the Ramsey property for the age of another structure.

This note is organized as follows. In Section 2, we give a minimal list of conventions, definitions, and relevant background from [4] needed to explain the open questions. We also describe related work that has come to our attention in Remark 2.3. In Section 3, we embark on a discussion of the challenges around Question 3.2 from our previous paper, which question concerns the existence of semi-retractions between generic ordered expansions of the random graph and the countable atomless Boolean algebra. We state Question 3.3 in response to these challenges. In Section 4, we discuss Question 4.5 from our previous paper, which concerned the relationship between our approach using semi-retractions to studying transfer of the Ramsey property and the approach using pre-adjunctions described in [13]. We highlight progress in [2] on the approach using pre-adjunctions, and pose Question 5.1 as a concrete test case for Question 4.5, which question remains open.

2. Preliminaries

Here we will give the definitions needed to precisely state our questions. For the basics of formulas, structures, and Fraïssé theory the reader is referred to [12, 9].

In this paper, the signature of a structure may be infinite and may contain function symbols. The *age* of a structure $\mathcal{M}$ consists of all finitely-generated substructures of $\mathcal{M}$. If all elements in the age of $\mathcal{M}$ are finite structures, we say that $\mathcal{M}$ is *locally finite*. We say that a structure

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$\mathcal{A}$ is *rigid* if the only automorphism of $\mathcal{A}$ is the identity map. Given a signature L and two L-structures A, A', by $A \cong A'$ we mean that A and A' are isomorphic as L-structures.

Given a function $f : X^s \to Y$ and $\bar{\imath}' := (i_k : k < s) \in X^s$, we define $f(\bar{\imath}') = (f(i_k) : k < s)$. For same-length tuples $\overline{a}, \overline{a}'$ from $\mathcal{A}$, we will use $\overline{a} \sim_{\mathcal{A}} \overline{a}'$ to mean that $\overline{a}$ and $\overline{a}'$ satisfy exactly the same quantifier-free formulas in $\mathcal{A}$.

Here we give our basic definitions around semi-retractions.

Definition 2.1 ([17])**.** Given any structures $\mathcal{A}, \mathcal{B}$, not necessarily in the same signature, we say that an injection $h : \mathcal{A} \to \mathcal{B}$ is

(i) *quantifier-free type-respecting (qftp-respecting)* if for all finite, same-length tuples $\bar{\imath}, \bar{\jmath}$ from $\mathcal{A}$,
$$\bar{\imath} \sim_{\mathcal{A}} \bar{\jmath} \Rightarrow h(\bar{\imath}) \sim_{\mathcal{B}} h(\bar{\jmath}),$$
(ii) *quantifier-free type-preserving (qftp-preserving)* if $\mathcal{A}, \mathcal{B}$ are structures in the same signature and $\mathrm{qftp}^{\mathcal{A}}(\bar{\imath}) = \mathrm{qftp}^{\mathcal{B}}(h(\bar{\imath}))$ (thus, it is also qftp-respecting).

When these maps compose in a certain way, we get a semi-retraction.

Definition 2.2 ([17])**.** Let $\mathcal{A}$, $\mathcal{B}$ be any structures. We say that $\mathcal{A}$ is a *semi-retract of $\mathcal{B}$ (via (g, f))* if

(1) there exist qftp-respecting injections $\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{A}$,
(2) such that $\mathcal{A} \xrightarrow{fg} \mathcal{A}$ is an embedding (equivalently, is qftp-preserving).

We refer to the pair (g, f) as a *semi-retraction between $\mathcal{A}$ and $\mathcal{B}$*.

Remark 2.3. Kamsma uses qftp-respecting maps in Lemma 3.15 (iii) of [10], where he defines a construction somewhat weaker than a semi-retraction to transfer facts about indiscernibility and EM-types between indexing structures.

Pinsker brought it to the attention of the authors that a qftp-respecting map as defined above is (in the case that both structures have quantifier elimination) exactly what is defined as a "canonical function" in [6]. In the latter paper, the authors are studying the constraint satisfaction problem for an arbitrary countably infinite relational structure.

Thus, we find that the notion of qftp-respecting maps and semi-retractions is quite natural, and applicable to a variety of contexts.

We first state the structural definition of the Ramsey property, as it has been canonized in works such as [14, 11]. It is convenient to define $\binom{C}{B}$ to be the set of all substructures $B' \subseteq C$, such that $B' \cong B$. Given two L-structures A and B, we denote the L-embeddings from A into B by $\mathrm{Emb}(A, B)$.

Definition 2.4. Let $\mathcal{K}$ be a class of finitely-generated L-structures, for some signature L. We say that $\mathcal{K}$ has *the Ramsey property* (RP) if for all $A, B \in \mathcal{K}$ and for all integers $r \geq 2$, there exists $C \in \mathcal{K}$ such that for any coloring $c : \binom{\mathcal{M}}{A} \to r$, there is $B' \in \binom{\mathcal{M}}{B}$ and $\ell \in r$ such that $c(A') = \ell$ for every $A' \in \binom{B'}{A}$.

Example 2.5. The following classes have RP:

(1) All finite linear orders in $L = \{<\}$ ([16]).
(2) All finite simple graphs with no loops with an ordering on the vertices in $L = \{R, <\}$ ([1],[15]).
(3) Finite Boolean algebras in $L = \{\vee, \wedge, \neg, \mathbf{0}, \mathbf{1}\}$ ([7]). This is what is called the *Finite Dual Ramsey Theorem.*
(4) Finite vector spaces over a fixed finite field $\mathbb{F}$ in $L = \{+, 0, \{\cdot\alpha : \alpha \in \mathbb{F}\}\}$ ([8]). This is what is called the *Graham-Leeb-Rothschild Ramsey property for finite vector spaces over a fixed finite field.*

A more refined version of the Ramsey property is stated in terms of finite Ramsey degrees, and the concept of Ramsey degrees for embeddings is natural for a more category theoretic approach (as well as for applications in topological dynamics, see [11, 3]).

Definition 2.6. Let $\mathcal{K}$ be a class of L-structures and let $A \in \mathcal{K}$. We say that A has *finite Ramsey degree for embeddings in* $\mathcal{K}$ if there is an integer $d \geq 1$ such that for every $B \in \mathcal{K}$ and every $r \geq 2$ there is $C \in \mathcal{K}$ such that for every coloring $c : \mathrm{Emb}(A, C) \to \{0, 1, \ldots, r-1\}$, there is $h \in \mathrm{Emb}(B, C)$ such that c on $h \circ \mathrm{Emb}(A, B)$ takes at most d colors.

If for all $A \in \mathcal{K}$, A has finite Ramsey degree for embeddings in $\mathcal{K}$ witnessed by $d = 1$, then we say that $\mathcal{K}$ has the *Ramsey property for embeddings.*

Remark 2.7. If every structure in $\mathcal{K}$ is rigid, $\mathcal{K}$ has the Ramsey property for embeddings if and only if $\mathcal{K}$ has the Ramsey property.

Definition 2.8. We say that $\mathcal{A}$ *has RP* if age($\mathcal{A}$) has RP.

Here we detail the results from [4] that will be relevant for the next two sections of the paper.

Theorem 2.9 (Theorem 5.1 of [4])**.** *Suppose that $\mathcal{A}$ is a locally finite semi-retract of $\mathcal{B}$ via (g, f). Let $A \in$ age($\mathcal{A}$). Suppose that $\langle g(A)\rangle_{\mathcal{B}}$ has Ramsey degree d for embeddings in age($\mathcal{B}$). Then A has Ramsey degree $\leq d$ for embeddings in age($\mathcal{A}$).*

Corollary 2.10 (Corollary 5.5 of [4])**.** *Let $\mathcal{A}, \mathcal{B}$ be structures and suppose (g, f) is a semi-retraction between $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\mathcal{A}$ is locally finite, and age($\mathcal{B}$) consists of rigid structures. If $\mathcal{B}$ has RP, then $\mathcal{A}$ has RP.*

Though it will not be used in this note, it is especially pleasant to observe that, for relational structures, many assumptions may be dropped. This indicates the considerable work involved to move from conclusions about the generators of a finitely-generated substructure to conclusions about the entire finitely-generated substructure.

Theorem 2.11 (Theorem 5.9 of [4])**.** *Let $\mathcal{A}, \mathcal{B}$ be structures each in relational signatures and suppose that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$, and let $A \in$ age($\mathcal{A}$). If the induced substructure on $g(A)$ has Ramsey degree d in age($\mathcal{B}$), then A has Ramsey degree bounded by d in $\mathcal{A}$.*

3. Question about Graphs and Boolean Algebras

Let $\mathcal{R}$ denote the *(countably infinite) random graph* (the Fraïssé limit of finite graphs) and let $\mathcal{R}^<$ denote the *random ordered graph* (the Fraïssé limit of finite graphs with a linear ordering on vertices). Let $\mathcal{B}_{ba}$ denote the *countable atomless Boolean algebra* (the Fraïssé limit of finite Boolean algebras).

If A is a finite Boolean algebra, a linear order $<_A$ on A is *natural* if it is the antilexicographic order on A induced by some linear (arbitrary) order on the atoms of A (see [11]). The *countable atomless Boolean algebra with a generic normal order* $(\mathcal{B}_{ba}, \prec)$ is the Fraïssé limit of the class of finite Boolean algebras with natural linear orders. It is straightforward to verify that $\mathcal{R}$ is a reduct of $\mathcal{R}^<$ and $\mathcal{B}_{ba}$ is a reduct of $(\mathcal{B}_{ba}, \prec)$, for example using Proposition 5.2 of [11].

In our joint paper, we successfully proved the following result for graphs and Boolean algebras, which we call the "unordered case".

Theorem 3.1 (Theorem 4.1 of [4])**.** *The countable random graph $\mathcal{R}$ is a semi-retract of the countable atomless Boolean algebra $\mathcal{B}_{ba}$.*

In trying to adapt our technique for the "unordered case" to the "ordered case" (the random ordered graph and the countable atomless Boolean algebra with a generic normal order) we hit certain roadblocks. We will describe the limitations of our argument in case our example is instructive. In order to make the first map $g : \mathcal{R} \to \mathcal{B}_{ba}$ qftp-respecting (following the technique of Mašulović in [13]) we enumerate the vertices of $\mathcal{R}$ as $(v_i : i \in \omega)$ (in effect, imposing a well-ordering on the vertices of $\mathcal{R}$), where $b_i := g(v_i)$ are the joins of specific subsets of an antichain

$\{a_i^j : i, j \in \omega\}$ in $\mathcal{B}_{ba}$, visualized as $a_0^0 \prec a_0^1 \prec \ldots \prec a_1^0 \prec a_1^1 \prec \ldots$. Moreover, the b_i's are chosen so that the quantifier-free type of $(b_{i_0}, \ldots, b_{i_{n-1}})$ in $\mathcal{B}_{ba}$ is a function of the quantifier-free type of $(v_{i_0}, \ldots, v_{i_{n-1}})$ in $\mathcal{R}$. If we adjust g to be order-preserving, e.g. to order $g(v_2) \prec g(v_1) \prec g(v_3)$ in $(\mathcal{B}_{ba}, \prec)$ according to the order $v_2 < v_1 < v_3$ in $\mathcal{R}^<$, we find that for vertices $\{v_i, v_j, v_k, v_l\}$ forming a copy of K_4 in $\mathcal{R}^<$, the truth of the order relation $g(v_i) \wedge g(v_j) < g(v_k) \wedge g(v_l)$ is not a function of the $<$-type of (v_i, v_j, v_k, v_l) in $\mathcal{R}^<$, (though it is, unhelpfully, a function of the ordering on indices, e.g. $i < j < k < l$) if we maintain our method of coding the edge relation in $\mathcal{R}$ as meets in $\mathcal{B}_{ba}$.

Given the roadblocks that we encountered, we left the ordered case as an open question.

Question 3.2 (Question 4.3 of [4]). Is the random ordered graph a semi-retract of the countable atomless Boolean algebra with a generic normal order ?

Here is a question about semi-retractions and ordered expansions more generally.

Question 3.3. Is there a general technique that allows a semi-retraction between two structures $\mathcal{A}$ and $\mathcal{B}$ to be "lifted" to a semi-retraction of "generic" ordered expansions $\mathcal{A}^<$ and $\mathcal{B}^<$ (in the case that both expansions have RP, or both fail to have RP)?

4. Question about Pre-adjunctions and Progress

Semi-retractions arise naturally as pointwise maps on the universes of structures, however the mechanism to transfer the Ramsey property operates at a more abstract level. To investigate this thesis, we compared semi-retractions to pre-adjunctions, which were shown by Mašulović in [13] to transfer the Ramsey property.

For a category $\mathbf{C}$, we denote by $\mathrm{Obj}(\mathbf{C})$ its objects and by $\hom_{\mathbf{C}}(A, B)$ the collection of morphisms between objects $A, B \in \mathrm{Obj}(\mathbf{C})$.

Definition 4.1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathrm{Obj}(\mathbf{D}) \to \mathrm{Obj}(\mathbf{C})$ and $G : \mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{D})$ be maps on objects. We say that (F, G) is a *pre-adjunction* if for every $A \in \mathrm{Obj}(\mathbf{D})$ and $C \in \mathrm{Obj}(\mathbf{C})$ we have a map

$$\Phi_{A,C} : \hom_{\mathbf{C}}(F(A), C) \to \hom_{\mathbf{D}}(A, G(C)),$$

such that

$$\forall A, B \in \mathrm{Obj}(\mathbf{D})\ \forall C \in \mathrm{Obj}(\mathbf{C})\ \forall v \in \hom_{\mathbf{D}}(A, B)\ \forall \psi \in \hom_{\mathbf{C}}(F(B), C)$$
$$\exists w \in \hom_{\mathbf{C}}(F(A), F(B)) \text{ such that } \Phi_{A,C}(\psi \circ w) = \Phi_{B,C}(\psi) \circ v.$$

This is a version of Mašulović's result restricted to our setting.

Theorem 4.2 (Theorem 3.2 of [13]). *Let $\mathbf{C}$ and $\mathbf{D}$ be categories of finite structures with embeddings as morphisms. Assume that $F : Obj(\mathbf{D}) \rightleftarrows Obj(\mathbf{C}) : G$ is a pre-adjunction and that $\mathbf{C}$ has the Ramsey property for embeddings. Then $\mathbf{D}$ has the Ramsey property for embeddings.*

In our paper, we grappled with the connection between pre-adjunctions and semi-retractions in the following two results.

Theorem 4.3 (Theorem 6.4 of [4]). *Let $\mathcal{A}$ and $\mathcal{B}$ be locally finite and let (g, f) be a semi-retraction between $\mathcal{A}$ and $\mathcal{B}$. Then there is a pre-adjunction between age($\mathcal{A}$) and age($\mathcal{B}$) with embeddings as morphisms.*

Theorem 4.4 (Theorem 6.5 of [4]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are countable structures in relational signatures. Let $\mathbf{D}$ be age($\mathcal{A}$) with embeddings as morphisms and let $\mathbf{C}$ be age($\mathcal{B}$) with embeddings as morphisms.*

Suppose that (F, G) is a pre-adjunction between $\mathbf{D}$ and $\mathbf{C}$ and assume that F and G preserve cardinality. Then there is a semi-retraction between $\mathcal{A}$ and a structure $\mathcal{B}'$ whose age is contained in $\mathbf{C}$.

In our paper, we presented this second question, which remains open.

Question 4.5 (Question 6.6 of [4])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be (locally finite) structures and suppose that there is a pre-adjunction between age($\mathcal{A}$) and age($\mathcal{B}$) with embeddings as morphisms. Under which conditions is there a semi-retraction between $\mathcal{A}$ and $\mathcal{B}$?

Remark 4.6. It is particularly challenging to answer when a transfer of RP from a class $\mathcal{K}_1$ to a class $\mathcal{K}_0$ is witnessed by a semi-retraction, since the constraint for choosing infinite representatives $\mathcal{A}$ and $\mathcal{B}$ is merely that age($\mathcal{A}$) $= \mathcal{K}_0$ and age($\mathcal{B}$) $= \mathcal{K}_1$. Proposition 3.10 of [4] illustrates an example in which the structure $\mathcal{A}$ is not chosen to be the Fraïssé limit of age($\mathcal{A}$).

At first, the authors tasked themselves with finding as many examples as possible where a semi-retraction exists to support the fact that one Ramsey property is considered conventionally to be "stronger" than another (or to yield the other). This project is quite similar to what is announced by Mašulović and Dasilva Barbosa in [2], where they pursue pre-adjunctions as being explanatory in this manner. To the extent that both projects are successful, there is some evidence that semi-retractions and pre-adjunctions are capturing the same phenomena, even as we have not yet proved them to be equivalent in all cases.

5. Conclusion

Recent progress makes Question 4.5 more intriguing than ever. Bartošová pointed out to Mašulović the fact (which is cited in the Introduction of [2]) that the arguments in [5] implicitly show that there is a pre-adjunction between the class of finite vector spaces with certain embeddings and the class of finite naturally ordered Boolean algebras with embeddings. This witnesses that the Dual Ramsey Theorem is equally strong as the Graham-Leeb-Rothschild Ramsey property for finite vector spaces over a fixed finite field (in the sense of [2]), as the former is easily implied by the latter for the choice of the field $\{0, 1\}$. This supports the thesis in [2] that the Dual Ramsey Theorem is the strongest among structural Ramsey theorems. In connection with what we have just stated about vector spaces, we end with a special case of Question 4.5.

Question 5.1. Are there infinite structures $\mathcal{A}$ and $\mathcal{B}$ such that age($\mathcal{A}$) is the class of all finite dimensional vector spaces over some fixed finite field $\mathbb{F}$ and age($\mathcal{B}$) $=$ age($\mathcal{B}_{ba}$), such that $\mathcal{A}$ is a semi-retract of $\mathcal{B}$? Moreover, for appropriate ordered expansions $\mathcal{A}^<$ of A and $\mathcal{B}_{ba}^<$ of $\mathcal{B}_{ba}$, is $\mathcal{A}^<$ a semi-retract of $\mathcal{B}_{ba}^<$?

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