A Proof of the Equivalence of Forking and Dividing with Indiscernible Trees

Koitaro Nakaura*and Akito Tsuboi†

1 Introduction

In this paper, we consider the indiscernibility of trees, and by using this, we see another proof of the equivalence of forking and dividing over models in NTP_2 theories.

The notion of indiscernible sequences is a useful tool for simplifying complex arguments in model theory. In classification theory, we often consider a tree of tuples with certain property, and hence a construction of an indiscernible tree can be valuable, which is a tree whose well-shaped subsequences are indiscernible.

The existence of an indiscernible sequence is usually demonstrated through a simple compactness argument using the finite version of Ramsey's theorem. However, the existence proof of an indiscernible tree has required a stronger set-theoretic theorem, such as the Erdös-Rado theorem ([1], [2]). In this study, we present a theorem proving the existence of an indiscernible tree using only the finite Ramsey theorem.

In addition, we see that this construction method can be applied to prove the fact that forking and dividing are equivalent over models in NTP_2 theories due to Chernikov and Kaplan([3]).

2 Preliminaries

First, we recall some definitions. We work in a monster model $\mathbb{M}$ of a complete L-theory T. Let us consider the set of finite sequences $\omega^{<\omega}$ as a tree.

Definition 2.1. Let $\varphi(x, y)$ be a formula.

1. We say $\varphi(x, y)$ has the *tree property* (TP) if there exist a tree of tuples $(a_\nu)_{\nu \in \omega^{<\omega}}$ and $k < \omega$ such that
   (a) for all $\eta \in \omega^\omega$, $\{\varphi(x, a_{\eta \upharpoonright i}) \mid i < \omega\}$ is consistent, and

*The University of Tokyo
†Professor Emeritus, University of Tsukuba

(b) for all $\nu \in \omega^{<\omega}$, $\{\varphi(x, a_{\nu^\frown\langle j\rangle}) \mid j < \omega\}$ is k-inconsistent.

2. We say $\varphi(x, y)$ has the *tree property of the second kind* (TP_2) if there exist an array of tuples $(a_{i,j})_{i,j<\omega}$ and $k < \omega$ such that

   (a) for all $f\colon \omega \to \omega$, $\{\varphi(x, a_{i,f(i)}) \mid i < \omega\}$ is consistent, and

   (b) for all $i < \omega$, $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is k-inconsistent.

3. We say a theory T has TP (resp. TP_2) if there exists a formula which has TP (resp. TP_2). A theory without TP (resp. TP_2) is called an NTP (resp. NTP_2) theory. An NTP theory is also called a *simple* theory.

By definition, simple theories are NTP_2.

Definition 2.2. Let b be a tuple and A a subset of $\mathbb{M}$. A formula $\varphi(x, b)$ *divides* over A if there is some A-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ such that $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent. A formula $\varphi(x)$ *forks* over A if $\varphi(x)$ implies a disjunction $\bigvee_{j<n} \psi_j(x)$ of formulas $\psi_j(x)$ which divides over A.

By definition, if a formula $\varphi(x)$ divides over A, then $\varphi(x)$ forks over A. However, the converse does not hold in general.

The rest of this section is devoted to considering the indiscernibility on sequences and arrays. We define a good property of sets of formulas called "the subsequence property," in order to construct a required indiscernible sequence.

Let X and Y be disjoint sets of variables and $\Gamma = \Gamma(X, Y)$ a set of formulas whose free variables are among $X \cup Y$. Then, $\exists Y \Gamma(X, Y)$ denotes the set of all formulas of the form

$$\exists \bar{y} \bigwedge_{i<n} \theta_i(\bar{x}, \bar{y}),$$

where $\bar{x} \in X$, $\bar{y} \in Y$ and all $\theta_i(\bar{x}, \bar{y})$ are from Γ.

Notice that a formula in $\exists Y \Gamma(X, Y)$ has only free variables from X.

Definition 2.3. Let $\Gamma(X, Y, Z)$ be a set of formulas whose free variables are among $X \cup Y \cup Z$, where $X = \{x_i\}_{i<\omega}$. We say Γ has the *subsequence property* with respect to X over Y, if there exist sets $\{a_i\}_{i<\omega}$ and B such that the following condition is satisfied:

- For all strictly increasing functions $f : \omega \to \omega$, the set $\exists Z \Gamma(X, Y, Z)$ is realized by the sequence $\{a_{f(i)}\}_{i<\omega}$ for X and B for Y.

The following lemma is proved by an easy argument using Ramsey's theorem.

Lemma 2.4. *Suppose that $\Gamma(\{x_i\}_{i<\omega}, Y, Z)$ has the subsequence property with respect to $\{x_i\}_{i<\omega}$ over Y. Then, there exists a realization $(\{a_i\}_{i<\omega}, B, C)$ of Γ such that the sequence $\{a_i\}_{i<\omega}$ is indiscernible over B.*

We first use this lemma for considering the indiscernibility on arrays.

Definition 2.5. Let $(I_t)_{t\in X}$ be a family of sequences of tuples from a model M and A a subset of M. We say that the sequences $(I_t)_{t\in X}$ are *mutually indiscernible* over A if for any $t \in X$, the sequence I_t is indiscernible over $A \cup I_{\neq t}$, where $I_{\neq t} = \{a \in I_s \mid s \neq t\}$.

We define the similar property on arrays to the subsequence property on suquences.

Definition 2.6. Let α be an ordinal and $\Gamma((X_i)_{i<\alpha}, Y, Z)$ a set of formulas whose free variables are among $\bigcup_{i<\alpha} X_i \cup Y \cup Z$, where $X_i = \{x_{ij}\}_{j<\omega}$. We say Γ has the *multiple subsequence property* with respect to $(X_i)_{i<\alpha}$ over Y, if there exist $(B_i)_{i<\alpha}$ with $B_i = \{b_{ij}\}_{j<\omega}$ and C such that the following condition is satisfied:

- For any tuple of strictly increasing functions $(f_i \colon \omega \to \omega)_{i<\alpha}$, $\exists Z \Gamma$ is realized by $\{b_{i,f_i(j)}\}_{j<\omega}$ for X_i and C for Y.

Lemma 2.7. *Suppose that $\Gamma((X_i)_{i<\alpha}, Y, Z)$ has the multiple subsequence property with respect to $(X_i)_{i<\alpha}$ over Y, where $X_i = \{x_{ij}\}_{j<\omega}$. Then, there exists a realization $((B_i)_{i<\alpha}, C, D)$ of Γ such that the sets B_i ($i < \alpha$) are mutually indiscernible over C.*

Proof. Let $\Gamma^*((X_i)_{i<\alpha}, Y)$ denote the set

$$\bigcup_{(f_i)_{i<\alpha}} (\exists Z)\Gamma((\{x_{i,f_i(j)}\}_{j<\omega})_{i<\alpha}, Y, Z),$$

where $(f_i \colon \omega \to \omega)_{i<\alpha}$ ranges over the set of all α-tuples of strictly increasing functions. Note that Γ has the multiple subsequence property (with respect to $(X_i)_{i<\alpha}$ over Y) if and only if Γ^* is consistent. Since $\Gamma^*((X_i)_{i<\alpha}, Y)$ has the subsequence property with respect to $X_0 = \{x_{0,j}\}_{j<\omega}$ over $X_{\neq 0} \cup Y$, there exists some realization $((B_i)_{i<\alpha}, C)$ of $\Gamma*$ such that B_0 is indiscernible over $\bigcup_{i\neq 0} B_i \cup C$. This means that the set

$$\Gamma^* \cup \text{`}X_0 \text{ is indiscernible over } \bigcup_{i\neq 0} X_i \cup Y\text{'}$$

is consistent. Notice that this set also has the multiple subsequence property (with respect to $(X_i)_{i<\alpha}$ over Y). Thus, by considering $X_1, X_2, \dots$ in order, we see that the following set is consistent:

$$\Gamma^* \cup \bigcup_{j<\alpha} \text{`}X_j \text{ is indiscernible over } \bigcup_{i\neq j} X_i \cup Y\text{'}.$$

Therefore, there is a mutually indiscernible set that realizes Γ. □

3 Main results

As we saw in the previous section, we often consider trees of tuples in classification theory. Then, similarly to indiscernible sequences, indiscernible trees can be useful in classification theory.

We first see a proof of the existence of indiscernible trees without the Erdös-Rado theorem. In the last section, we show that forking and dividing are equivalent over a model in NTP$_2$ theories by using indiscernible trees.

3.1 A construction of indiscernible trees

Let T be a complete L-theory. Throughout this section, we work in a monster model $\mathbb{M}$ of T.

We consider the ordering $\lessdot$ on $\omega^{<\omega}$, where $\lessdot$ is defined by

$$\nu \lessdot \mu :\Leftrightarrow \ \nu \text{ is a proper initial segment of } \mu \ .$$

For $\nu \in \omega^{<\omega}$, we define $N(\nu) = \{\mu \mid \nu \not\lessdot \mu\}$.

Lemma 3.1. *Let $P(\{x_i\}_{i<\omega})$ and $Q(\{x_i\}_{i<\omega})$ be sets of formulas, both possessing the subsequence property. Let $\Gamma(X)$ be a set of formulas with free variables from $X = \{x_\nu \mid \nu \in \omega^{<\omega}\}$ expressing the following properties:*

1. *for every infinite path $\eta \in \omega^\omega$, the sequence $\{x_{\eta \restriction n} \mid n < \omega\}$ satisfies the property P,*
2. *for every finite sequence $\nu \in \omega^{<\omega}$, the sequence of its immediate successors $\{x_{\nu^\frown\langle i\rangle} \mid i < \omega\}$ satisfies the property Q.*

We assume that $\Gamma(X)$ is consistent. Then, the set $\Gamma(X)$ augmented by the following conditions $(*)_\nu$ for all $\nu \in \omega^{<\omega}$ is consistent.

$(*)_\nu$ *The sequences $I_\nu := \{x_{\nu^\frown\langle i\rangle} \mid i < \omega\}$ is indiscernible over $x_{N(\nu)} = \{x_\mu \mid \mu \in N(\nu)\}$.*

Proof. Let $\{\nu_j\}_{j<\omega}$ be an enumeration of $\omega^{<\omega}$ such that there is no $i < j < \omega$ with $\nu_j \lessdot \nu_i$. Let $\Gamma_n(X)$ be the set $\Gamma(X)$ augmented by the conditions $(*)_{\nu_j}$ $(j < n)$. In order to prove that $\Gamma^*(X) := \bigcup_{n<\omega} \Gamma_n(X)$ is consistent, it suffices to show that $\Gamma_n(X)$ is consistent for any $n < \omega$ by compactness. We prove by induction on n. We suppose the consistency of $\Gamma_n(X)$ has been established.

Claim. *$\Gamma_n(X)$ has the subsequence property with respect to $\{x_{\nu_n{}^\frown\langle i\rangle}\}_{i<\omega}$ over $x_{N(\nu_n)}$.*

Choose a realization $A = \{a_\nu\}_{\nu\in\omega^{<\omega}}$ of $\Gamma_n(X)$. Let $f\colon \omega \to \omega$ be a strictly increasing function. We define the mapping $\sigma\colon \omega^{<\omega} \to \omega^{<\omega}$ as follows:

- For $\nu \gtrdot \nu_n$, if $\nu = \nu_n{}^\frown\langle k\rangle^\frown\mu$, $\sigma(\nu) = \nu_n{}^\frown\langle f(k)\rangle^\frown\mu$
- For $\nu \in N(\nu_n)$, $\sigma(\nu) = \nu$

The mapping σ on $\omega^{<\omega}$ is naturally considered as a mapping on X or A, both of which are also denoted by σ. Observe that for $j < n$,

- $\sigma(\nu) = \nu$ for every immediate successor ν of ν_j
- $\sigma(\nu) \in N(\nu_j)$ for $\nu \in N(\nu_j)$
- $\sigma(\nu_n{}^\frown\langle i\rangle) = \nu_n{}^\frown\langle f(i)\rangle$

Also, the mapping σ preserves $\lessdot$ relation and sibling relation. From these, we see that $\{a_{\sigma(\nu)}\}_\nu$ satisfies Γ_n and that Γ_n has the subsequence property. (End of the proof of Claim)

Therefore, there is a realization $B = \{b_\nu\}_{\nu\in\omega^{<\omega}}$ of $\Gamma_n(X)$ such that $\{b_{\nu_n{}^\frown\langle i\rangle} \mid i<\omega\}$ is indiscernible over $b_{N(\nu_n)}$. This implies that $\Gamma_{n+1}(X)$ is consistent. $\square$

Lemma 3.2. *Let $\Delta(X)$ be the set $\Gamma(X)$ in Lemma 3.1 augmented by the following condition:*

($*$) *For all $\nu \in \omega^{<\omega}$, $I_\nu = \{x_{\nu^\frown\langle i\rangle}\}_{i<\omega}$ is indiscernible over $x_{N(\nu)}$.*

We assume that $\Delta(X)$ is consistent. Then, the set $\Delta(X)$ augmented by the following condition (†) is consistent.

(†) *the sequences $J_n = \{x_{\langle n\rangle^\frown\langle 0^j\rangle} \mid j \in \omega\backslash\{0\}\}$ $(n<\omega)$ are mutually indiscernible.*

Proof. It is enough to show that $\Delta(X)$ has the multiple subsequence property with respect to $\{J_n\}_{n<\omega}$. Choose a set $A = \{a_\nu\}_{\nu\in\omega^{<\omega}}$ realizing $\Delta(X)$. For every $n<\omega$, let $f_n\colon \omega\backslash\{0\} \to \omega\backslash\{0\}$ be an increasing function. We define the mapping $\tau\colon \omega^{<\omega} \to \omega^{<\omega}$ by

$$\tau(\nu) = \langle \nu(0), 0^{f_m(1)-1}, \nu(1), 0^{f_m(2)-f_m(1)-1}, \nu(2), \ldots, 0^{f_m(k)-f_m(k-1)-1}, \nu(k)\rangle,$$

where $m = \nu(0)$.

The mapping τ on $\omega^{<\omega}$ is naturally considered as a mapping on X or A, both of which are also denoted by τ. Observe that

- $\tau(N(\nu)) \subseteq N(\tau(\nu))$ for any $\nu \in \omega^{<\omega}$, and
- $\tau(\langle n\rangle^\frown\langle 0^j\rangle) = \langle n\rangle^\frown\langle 0^{f_n(j)}\rangle$ for any $n<\omega$ and $j>0$.

In addition, the mapping τ preserves $\lessdot$ relation and sibling relation. From these, we see that $\{a_{\tau(\nu)}\}_\nu$ satisfies $\Delta(X)$ and that $\Delta(X)$ has the multiple subsequence property. Therefore, there is a realization of $\Delta(X)$ which satisfies (†). $\square$

Lemma 3.3. *Let $\Theta(X)$ be the set $\Delta(X)$ in Lemma 3.2 augmented by the following condition:*

(†) *the sequences $J_n = \{x_{\langle n\rangle^\frown\langle 0^j\rangle} \mid j \in \omega\backslash\{0\}\}$ $(n<\omega)$ are mutually indiscernible.*

We assume that $\Theta(X)$ is consistent. Then, the set $\Theta(X)$ augmented by the following condition (#) is consistent:

(#) *the sequence $\{K_n \mid n<\omega\}$ are indiscernible, where K_n denotes the set $\{x_\nu \mid \nu \gtrdot \langle n\rangle$ or $\nu = \langle n\rangle\}$.*

Proof. It is enough to show that $\Theta(X)$ has the subsequence property with respect to $\{K_n\}_{n<\omega}$. Choose a set $A = \{a_\nu\}_{\nu\in\omega^{<\omega}}$ realizing $\Theta(X)$. Let $f\colon\omega\to\omega$ be an increasing function. We define the mapping $\rho\colon \omega^{<\omega}\to\omega^{<\omega}$ by

$$\rho(\nu) = \big\langle f\big(\nu(0)\big)\big\rangle^\frown\big\langle \nu(1), \ldots, \nu(l(\nu)-1)\big\rangle$$

The mapping ρ on $\omega^{<\omega}$ is naturally considered as a mapping on X or A, both of which are also denoted by τ. Observe that

- $\rho(N(\nu)) \subseteq N(\rho(\nu))$ for any $\nu \in \omega^{<\omega}$, and
- $\rho(J_n) = J_{f(n)}$ and $\rho(K_n) = K_{f(n)}$ for any $n < \omega$.

In addition, the mapping ρ preserves $\lessdot$ relation and sibling relation. From these, we see that $\{a_{\rho(\nu)}\}_\nu$ satisfies $\Theta(X)$ and that $\Theta(X)$ has the subsequence property. Therefore, there is a realization of $\Theta(X)$ which satisfies (#). □

The following theorem follows immediately from Lemma 3.1, 3.2 and 3.3.

Theorem 3.4. *Let $P(\{x_i\}_{i<\omega})$ and $Q(\{x_i\}_{i<\omega})$ be sets of formulas, both possessing the subsequence property. Let $\Gamma(X)$ be a set of formulas with free variables from $X = \{x_\nu \mid \nu \in \omega^{<\omega}\}$ expressing the following properties:*

- *For every infinite path $\eta \in \omega^\omega$, the sequence $\{x_{\eta\restriction n} \mid n < \omega\}$ satisfies the property P.*
- *For every finite sequence $\nu \in \omega^{<\omega}$, the sequence of its immediate successors $\{x_{\nu^\frown\langle i\rangle} \mid i < \omega\}$ satisfies the property Q.*

Suppose that $\Gamma(X)$ is consistent. Then, there exists a realization $A = \{a_\eta\}_\eta$ of Γ with the following properties:

1. *For all $\nu \in \omega^{<\omega}$, $I_\nu = \{a_{\nu^\frown\langle i\rangle}\}_{i<\omega}$ is indiscernible over $a_{N(\nu)}$.*
2. *The sequences $J_n = \{a_{\langle n\rangle^\frown\langle 0^j\rangle}\}_{j>0}$ $(n < \omega)$ are mutually indiscernible.*
3. *The sequence $\{K_n \mid n < \omega\}$ are indiscernible, where K_n denotes the set $\{x_\nu \mid \nu \gtrdot \langle n\rangle \text{ or } \nu = \langle n\rangle\}$.*

3.2 Forking and dividing in NTP$_2$ theories

Let T be a complete theory and $\mathbb{M}$ a monster model of T. In this section, models are supposed to be elementary submodels of $\mathbb{M}$.

Definition 3.5. Let M be a model and $p(x)$ a complete type over M. Suppose that $I = \{a_i\}_{i<\alpha}$ is a sequence of tuples in $\mathbb{M}$, where α is an ordinal, and let $p_i(x) = \mathrm{tp}(a_i/Ma_{<i})$. The sequence I is called a *coheir sequence* in $p(x)$ if the following properties are satisfied:

1. $p(x) = p_0(x)$ and $p_i(x) \subset p_j(x)$ $(i < j < \alpha)$.
2. Each p_i is finitely satisfiable in M.

Note that a coheir sequence in some $p \in S(M)$ is an indiscernible sequence over M.

Lemma 3.6. *Let $I = \{a_j\}_{j<\omega}$ be an indiscernible sequence over M. Let $P(X) = \mathrm{tp}(I/M)$, where $X = \{x_j\}_{j<\omega}$ and x_j is a variable tuple for a_j. Then, there exists a coheir sequence $\{I_i\}_{i<\omega}$ over M in $P(X)$ such that each $I_i = \{a_{i,j}\}_{j<\omega}$ is an indiscernible over $MI_{<i} = M \cup \bigcup_{k<i} I_k$.*

Proof. Starting with $I_0 = I$, we inductively choose I_n as follows. Suppose we have chosen I_n. Let $P_n(X) = \mathrm{tp}(I_n/MI_{<n})$. Notice that $P_n(X)$ is finitely satisfiable in M, and has the subsequence property with respect to $X = \{x_i\}_{i<\omega}$. Let $\Phi(X)$ be the set of all $L(MI_{\leq n})$-formulas $\varphi(\bar{x})$ with $\bar{x} \in X$ such that $\varphi(\bar{x})$ is not satisfied by any tuple in M. The set of $L(MI_{\leq n})$-formulas

$$Q(X) = P_n(X) \cup \{\neg\varphi(\bar{x}) \mid \varphi(\bar{x}) \in \Phi\}.$$

is consistent since $P_n(X)$ is finitely satisfiable in M. Let $J = \{b_i\}_{i<\omega}$ be a realization of $Q(X)$. Then, any subsequence of J also satisfies $Q(X)$. Therefore, it has the subsequence property with respect to X, and it is realized by an indiscernible sequence $I_{n+1} = \{a_{n+1,j}\}_{j<\omega}$ over $MI_{\leq n}$. The sequence I_{n+1} satisfies the required conditions. □

Remark 3.7. Let I_n $(n < \omega)$ be as obtained in the above lemma. Then, for all $\eta \in \omega^\omega$, $J_\eta = \{a_{i,\eta(i)}\}_{i<\omega}$ is a coheir indiscernible sequence over M, and the type $\mathrm{tp}(J_\eta/M)$ is fixed, independently of η.

The following lemma is proved in the same way as Fact 1.10 in [5].

Lemma 3.8. *Let $I = \{a_i\}_{i<\omega}$ and $J = \{b_j\}_{j<\omega}$ be indiscernible sequences over M in $p(x) \in S(M)$. We assume that I is a coheir sequence over M. Then, there exists a sequence $J' = \{b'_j\}_{j<\omega}$ such that*

i) $b'_0 = a_0$,

ii) $J' \equiv_M J$,

iii) $\{b'_j\} \cup \{a_i\}_{i>0} \equiv_M I$, *for all* $j < \omega$, *and*

iv) J' *is indiscernible over* $M(I \setminus \{a_0\})$.

Proof. Let $p(X, x) = \mathrm{tp}(\{a_i\}_{i>0}, a_0/M)$, where $X = \{x_i\}_{i>0}$. Since $p(X, a_0)$ is finitely satisfiable in M, it also does not divide over M. Hence, the following set is consistent:

$$\bigcup_{j<\omega} p(X, b_j)$$

Let $\{a'_i\}_{i>0}$ be a realization of this set, and then the following set is consistent.

$$\bigcup_{j<\omega} p(\{a'_i\}_{i>0}, y_j) \cup \text{`}\{y_j\}_{j<\omega} \equiv_M J\text{'}.$$

Since $\{a_i\}_{i>0} \equiv_M \{a'_i\}_{i>0}$, the following set is also consistent:

$$q(\{y_j\}_{j<\omega}) := \bigcup_{j<\omega} p(\{a_i\}_{i>0}, y_j) \cup \text{`}\{y_j\}_{j<\omega} \equiv_M J\text{'}.$$

Because J is indiscernible over M, q is realized by an indiscernible sequence $J'' = (b''_j)_{j<\omega}$ over $M(I \setminus \{a_0\})$. In addition, as we have $a_0 \equiv_{M(I\setminus\{a_0\})} b''_0$, there is a realization $J' = (b'_j)_{j<\omega}$ of q such that J' is $M(I \setminus \{a_0\})$-indiscernible and $b'_0 = a_0$. Therefore, the conditions i)-iv) are satisfied. □

Proposition 3.9. *Let $I = \{a_i\}_{i<\omega}$ and J be M-indiscernible sequences as given in Lemma 3.8. Let $\{a_i\}_{i\in\mathbb{Z}}$ be an M-indiscernible sequence (of order type $\mathbb{Z}$) extending I. By repeatedly applying the lemma and using compactness, we obtain a set $\{a_\nu \mid \nu \in \omega^{<\omega}\}$ (indexed by a tree) with the following properties:*

1. *For all paths* η, $\{a_{\eta|n}\}_{n<\omega} \equiv_M I^*$, *where* $I^* = \{a_{-i}\}_{i<\omega}$;

2. *For all* $\nu \in \omega^{<\omega}$, $\{a_{\nu^\frown\langle n\rangle} \mid n < \omega\} \equiv_M J$.

Now, we consider NTP_2 theories. The argument in this section is a variant of the arguement in [5] on simple theories.

Proposition 3.10. *Let* T *be an* NTP_2 *theory. Suppose that* $\varphi(x, a)$ *divides over a model* M. *Then, the dividing of* $\varphi(x, a)$ *is witnessed by a coheir sequence over* M.

Proof. Choose an M-indiscernible sequence $I = \{a_i\}_{i<\omega}$ with $a_0 = a$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. By Lemma 3.6, there is a coheir sequence $\{I_i \mid i < \omega\}$ over M starting with $I_0 = I$ such that each I_{i+1} is indiscernible over $MI_{\leq i}$. Let $I_i = (a_{i,j})_{j<\omega}$. Since $(a_{i,0})_{i<\omega}$ is a coheir sequence over M, it is sufficient to prove the following claim.

Claim. $\{\varphi(x, a_{i,0}) \mid i < \omega\}$ *is inconsistent.*

Suppose not. Since the type $\mathrm{tp}(\{a_{i,\eta(i)}\}_{i<\omega}/M)$ is fixed independently of $\eta \in \omega^\omega$,, $\{\varphi(x, a_{i,\eta(i)}) \mid i < \omega\}$ is consistent for all η. Thus, the set $\{\varphi(x, a_{i,j}) \mid i, j < \omega\}$ witnesses the TP_2, which is a contradiction. □

Proposition 3.11. *Let* T *be an* NTP_2 *theory. Suppose that both* $\varphi(x, a)$ *and* $\psi(x, a)$ *divide over* M. *Then,* $\varphi(x, a) \vee \psi(x, a)$ *divides over* M.

Proof. Choose M-indiscernible sequences I and J that witness the dividing of $\varphi(x, a)$ and $\psi(x, a)$, respectively. We can assume I is a coheir sequence in $\mathrm{tp}(a/M)$. By Proposition 3.9, we obtain a tree $\{a_\eta \mid \eta \in \omega^{<\omega}\}$ with the following properties:

- For all $\eta \in \omega^\omega$, $\{a_{\eta|n}\}_{n<\omega} \equiv_M I^*$.
- For all $\nu \in \omega^{<\omega}$, $\{a_{\nu^\frown\langle n\rangle} \mid n < \omega\} \equiv_M J$.

We prepare a set $X = \{x_\nu\}_{\nu\in\omega^{<\omega}}$ of variables and let $\Gamma(X)$ be the set of $L(M)$-formulas describing the above two properties (where a_ν is replaced with x_ν). By Theorem 3.4, we can find $B = \{b_\nu\}_\nu$ that realizes $\Gamma(X)$ and satisfies the properties 1, 2 and 3 described therein. We consider the sequence $K := \{b_{\langle n,0\rangle} \mid n < \omega\}$. Note that the sequence K is indiscernible over M (condition 3).

Assume, for a contradiction, that $\varphi(x, a) \vee \psi(x, a)$ does not divide over M. If K witness the dividing of both $\varphi(x, a)$ and $\psi(x, a)$, then it will also witness the dividing of $\varphi(x, a) \vee \psi(x, a)$. Thus, (a) $\{\varphi(x, d) \mid d \in K\}$ is consistent, or (b) $\{\psi(x, d) \mid d \in K\}$ is consistent.

Suppose (a) is the case. We consider the set $\{\varphi(x, b_{\langle n\rangle^\frown\langle 0^i\rangle}) \mid i > 0, n < \omega\}$. By (a) and the mutual indiscernibility (condition 2), for all $f : \omega\backslash\{0\} \to \omega\backslash\{0\}$, $\{\varphi(x, b_{\langle n\rangle^\frown\langle 0^{f(n)}\rangle}) \mid n < \omega\}$ is consistent. On the other hand, since all subsets of B corresponding to an infinite path are isomorphic to I^*, for all n, $\{\varphi(x, b_{\langle n\rangle^\frown\langle 0^i\rangle}) \mid i > 0\}$ is k-inconsistent for some fixed k. This means that T has TP_2, leading to a contradiction.

For case (b), we consider the set $\{\psi(x, b_{\langle n,i\rangle}) \mid n < \omega, i < \omega\}$. By the mutual indiscernibility of the sequences $J_n := \{b_{\langle n,i\rangle} \mid i < \omega\}$ $(n < \omega)$, it follows that for all functions $f\colon \omega \to \omega$, the set $\{\psi(x, b_{\langle n,f(n)\rangle}) \mid n < \omega\}$ is consistent, since $\{b_{\langle n,f(n)\rangle}\}_{n<\omega} \equiv_M K$. On the other hand, because $J_n \equiv_M J$ for all $n < \omega$, the set $\{\psi(x, b_{\langle n,i\rangle}) \mid i < \omega\}$ is k-inconsistent for some fixed k. Therefore, T has TP_2, which leads to a contradiction. □

By Proposition 3.11, we can prove the equivalence of dividing and forking over models in NTP_2 theories.

Corollary 3.12 (Chernikov and Kaplan, [3])**.** *Let T be an NTP_2 theory. Then, forking and dividing are equivalent over models, namely, a formula $\varphi(x, a)$ forks over a model M if and only if it divides over it.*

Remark 3.13. Forking and dividing are not necessarily equivalent over a set in NTP_2 theories, unlike simple theories. For example, a typical example $\mathrm{Th}(\mathbb{Q}, \mathrm{cyc})$ (the theory of dense cyclic order) is NIP, especially NTP_2. See [6]. Further examples are in [3].

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