Quasi limit theorems for open systems

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Survey and motivations

- I will present a few preliminary results of an ongoing paper with J Atnip, C Gonzalez-Tokman, G Froyland and Y Nakano.
- The following two contributions share our kind of investigation.
- M. M. Castro, J. S. W. Lamb, G. Olicon Mendez, and M. Rasmussen. *Existence and uniqueness of quasi-stationary and quasi-ergodic measures for absorbing Markov processes: a Banach lattice approach*, 2021. arXiv:2111.13791v6. To appear in Stochastic Processes and their Applications. FOR MARKOV CHAINS
- P. Collet, S. Martínez and J. San Martin, *Quasi-Stationary Distributions, Markov Chains, Diffusions and Dynamical Systems*, Springer, 2013. FOR GIBBS STATE
- Our results apply to deterministic dynamical systems and dynamical systems perturbed in a quenched way.
- Basic reference for quenched (fibred) systems: L Arnold: Random Dynamical Systems.

Random Dynamical Systems: quenched

- Suppose we have a complete metric space X and a probability space $(\Omega, \mathcal{F}, m)$ and an ergodic invertible map σ. Define I_ω (the FIBER) a closed subset of X and
- For each $\omega \in \Omega$ we consider the map
$$T_\omega : I_\omega \to I_{\sigma\omega}$$
- and the random composition
$$T^n_\omega = T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega.$$
- Put $I_c := \cup_{\omega\in\Omega}\{\omega\} \times I_\omega$; writing $B \in I_c$ means $\forall\omega, B_\omega \in I_\omega, \{\omega\} \times B_\omega = B \cap \{\omega\} \times I_\omega$.
- Ex. Take σ an irrational rotation of the circle $\sigma\omega = \overline{\omega+\alpha}$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\Omega = \cup_{j=1}^M \Omega_j$, a finite partition of the circle.
- On each Ω_j the map $\omega \in \Omega_j \to T_\omega$ is constant.
$$T^3_\omega(x) = T_{\overline{\omega+2\alpha}} T_{\overline{\omega+\alpha}} T_{\overline{\omega}}(x).$$
- In this context we will consider a probability measure satisfying the equivariant condition:
$$\int_{I_\omega} f \circ T_\omega \, d\mu_\omega = \int_{I_{\sigma\omega}} f \, d\mu_{\sigma\omega}, \qquad f \in L^1(I, \mu_{\sigma\omega}). \tag{1}$$

Another example: the i.i.d. case

- Take a probability space $(\Omega, \mathcal{F}, m)$ and associate to each $\omega \in \Omega$ a map T_ω.
- Then construct $(\Omega^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}}, m^{\mathbb{Z}})$ and take an element $\underline{\omega} \in \Omega^{\mathbb{Z}}$ as $\underline{\omega} = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$, where $\omega_j \in \Omega$.
- Define the shift
$$(\sigma\underline{\omega})_i = \underline{\omega}_{i+1}$$
and put
$$\underline{\omega} \to T_{\underline{\omega}} = T_{\omega_0},$$
where ω_0 is the 0-th coordinate of $\underline{\omega}$.
- Then
$$T^n_{\underline{\omega}} = T_{\sigma^{n-1}\underline{\omega}} \circ \cdots \circ T_{\sigma\underline{\omega}} \circ T_{\underline{\omega}} = T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0}.$$
- Suppose we define the shift σ over $\Omega^{\mathbb{N}}$. Then we could define the stationary measure μ_s satisfying
$$\int f(x) d\mu_s(x) = \int\int f(T_\omega(x) d\mu_s(x) dm(\omega).$$

It is the stationary measure of the associated Markov chain Z_n with transition probability
$$\mathbb{P}(Z_{n+1} \in B, Z_n = z) = m(\omega, T_\omega(z) \in B).$$

This is the ANNEALED scenario.

What makes randomness so different?

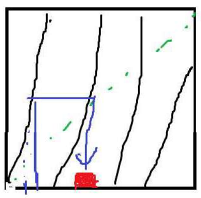
- In the following we could forget the presence of ω : the results will be still NEW.
- Invariance $\mu(T^{-1}A) = \mu(A)$ is replaced by equivariance $\mu_\omega(T_\omega^{-1}(A)) = \mu_{\sigma\omega}(A)$.
- The rate of decay could be affected by norms of functions along fibers $f_{\sigma^k\omega}(x), x \in I_{\sigma^k\omega}$. Use ergodicity of σ an especially temperedess. The function $a(\omega)$ is tempered if for ωa.e. we have $\lim_{|n|\to\infty} \frac{1}{n}\log|a(\sigma^n\omega)| = 0$, which is equivalent to say that $\forall\varepsilon, \exists A_\omega > 0, \forall n : a(\sigma^n\omega) \leq A_\omega e^{\varepsilon|n|}$.
- More important powers are replaced by compositions or cocycles; quasicompactness of operators are replaced by quasicompactness of cocycles.

Open systems

- Given a dynamical system (X, μ, T) with μ which is T-invariant, Birkhoff ergodic theorem states that for any $f \in L^1(\mu)$:

$$\lim_{n\to\infty} \frac{1}{n} S_n(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \hat{f}(x), \ \mu - a.s.$$

with $\int f d\mu = \int \hat{f} d\mu$; $\hat{f}(Tx) = \hat{f}(x)$. If μ is ergodic then $\hat{f}(x) = \int f d\mu$, a.s.
- How could we formulate it for **open systems**?
- Namely for systems which admit a set (the hole), which absorbs the trajectories?



QUESTIONS and Holes

- When the orbit enters the hole it disappears. Do we have points whose orbit never disappears? The surviving set.
- How does mass escape into the hole? Escape rate.
- How could we describe the metastable statistics of convergence to the equilibrium state on the surviving set?
- The hole. Let H be a measurable set in I_c. For each $\omega \in \Omega$ the fiber sets $H_\omega \subseteq I_\omega$ are uniquely determined by the condition that $\{\omega\} \times H_\omega = H \cap (\{\omega\} \times I_\omega)$. For each $\omega \in \Omega$ and $n \geq 0$ we define

$$X_{\omega,n} := \left\{x \in I_\omega : T_\omega^j(x) \notin H_{\sigma^j\omega} \text{ for all } 0 \leq j \leq n\right\} = \bigcap_{j=0}^{n} T_\omega^{-j}(X_{\sigma^j\omega,0}) \quad (2)$$

to be the set of points in $X_{\omega,0}$ whose trajectories avoid the hole for n iterates. Note that $X_{\omega,0} = I_\omega \backslash H_\omega$ and $X_{\omega,n} = \{\tau(\omega,\cdot) > n\}$, where τ is the first hitting time to H (i.e. $\tau(\omega, x) = \inf\{j \geq 1 : T_\omega^j(x) \in H_{\sigma^j\omega}\}$). We call $X_{\omega,n}$ the *nth level surviving set*. We then naturally define

$$X_{\omega,\infty} := \bigcap_{n=0}^{\infty} X_{\omega,n} = \bigcap_{n=0}^{\infty} T_\omega^{-n}(X_{\sigma^n\omega,0}) \quad (3)$$

the *ω-surviving set*.

Measures

- A probability measure ζ on $(\Omega \times X, F \otimes B)$ is said to be a RANDOM PROBABILITY MEASURE if it satisfies the following properties:
  1. For every $B \in B$, the map $\Omega \ni \omega \longmapsto \zeta_\omega(B) \in [0,1]$ is measurable,
  2. For m-a.e. $\omega \in \Omega$, the map $B \ni B \longmapsto \zeta_\omega(B) \in [0,1]$ is a Borel probability measure.
- CONDITIONALLY INVARIANT

  Det $\forall n : \eta(B) = \frac{\eta(T^{-n}B\cap X_n)}{\eta(X_n)}$

  Ran $\forall n : \rho_{\sigma^n\omega}(B) = \frac{\rho_\omega(T_\omega^{-n}B\cap X_{\omega,n})}{\rho_\omega(X_{\omega,n})}$
- YAGLOM LIMIT

  Det $\frac{\zeta(T^{-n}B\cap X_n)}{\zeta(X_n)} \to \eta(B)$

  Ran $\frac{\zeta_\omega(T_\omega^{-n}B\cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})} - \rho_{\sigma^n\omega}(B) \to 0$
- CONDITIONING LIMIT

  Det $\frac{\zeta(B\cap X_n)}{\zeta(X_n)} \to$ some measure

  Ran $\frac{\zeta_\omega(B\cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})} \to$ some random measure

Operators

- From the abstract point of view, we will assume the existence of a measure (CONFORMAL) and of operators (transfer or Perron-Fröbenius) verifying:
- Det: $\int L_c(\phi)\psi d\nu_c = \lambda \int \phi \ \psi \circ T d\nu_c, \ \lambda = \int L_c(1) \ d\nu_c$
- Ran:
  $\int L_{\omega,c}(\phi_\omega)\psi_{\sigma\omega} d\nu_{\sigma\omega,c} = \lambda_\omega \int \phi_\omega \ \psi_{\sigma\omega} \circ T_\omega d\nu_{\omega,c}, \ \lambda_\omega = \int L_{\omega,c}(1_\omega) \ d\nu_{\sigma\omega,c}$
- **Warning: the choice of the functional spaces is essential.**
- (Ex: for billiard systems and hyperbolic diffeos with singularities,
  $$L_c : \mathcal{B} \to \mathcal{B}, L_c(h)(\phi) = h(\phi \circ T),$$
  where $h \in \mathcal{B}$, and ϕ is a test function.)
- We will consider iterates of these operators
  $$L^n_{\omega,c}(f) = L_{\sigma^{n-1}\omega,c} \cdots L_{\omega,c}(f)$$
  which become powers in the deterministic setting
  $$L^n_c(f).$$

The functional space

- We will consider a weight function $g : X \to (0, \infty)$.
- We assume there exists a Banach space $(\mathcal{B}, \|\cdot\|)$ of complex-valued functions on X such that
- there is $C_0 > 0$ for which
  $$\|f \cdot v\| \le C_0 \|f\| \|v\|, \qquad \|f\|_\infty \le C_0 \|f\| \tag{4}$$
  for any $f, v \in \mathcal{B}$,
- $\mathcal{B}$ is dense in $L^1(\nu_{\omega,c})$ for any $\omega \in \Omega$,
- $\hat{X}_n, g \in \mathcal{B}$ a.s. for any $n \in \mathbb{N}$ (recall that $\hat{A}$ is the indicator function of A),
- the fiberwise transfer operator $\mathcal{L}_{\omega,c} : \mathcal{B} \to \mathcal{B}$ (associated with g) given by
  $$\mathcal{L}_{\omega,c}(f)(x) := \sum_{T_\omega(y)=x} f(y) g_\omega(y), \quad f \in \mathcal{B}, \ x \in X_{\sigma\omega} \tag{5}$$
  is well-defined
  for each $\omega \in \Omega$,
- the map
  $$(\omega, x) \mapsto (\mathcal{L}_c f)_\omega(x) \tag{6}$$
  is measurable for any $\mathcal{B}$-valued random variable f, where
  $$(\mathcal{L}_c f)_\omega := \mathcal{L}_{\sigma^{-1}\omega,c} f_{\sigma^{-1}\omega} \qquad \text{for } \omega \in \Omega.$$

Operators on the open system I

- We now introduce the operators for the open systems
  $$L_\omega f = L_{\omega,c}(f 1_{X_\omega,0})$$
  and its iterates
  $$L^n_\omega f = L^n_{\omega,c}(f 1_{X_\omega,n-1})$$
- Suppose one could prove the existence of a random measure $\nu_{\omega,\infty}$ supported on $X_{\omega,\infty}$ and such that (Assumption **A**):
  1. $\lambda_\omega \int f_\omega d\nu_{\omega,\infty} = \int L_\omega f_\omega d\nu_{\sigma\omega,\infty}, \ \lambda_\omega = \int L_\omega 1 d\nu_{\sigma\omega,\infty}$
  2. $L_\omega \phi_\omega = \lambda_\omega \phi_{\sigma\omega}, \ \int \phi_\omega d\nu_{\omega,\infty} = 1.$
  3. $\|L^n_\omega f_\omega - \nu_{\omega,\infty}(f_\omega)\phi_{\sigma^n\omega}\|_\infty \le D\|f\|\kappa^n$
- Then a few interesting results follow, notably
- The measure $\mu_{\omega,\infty} = \phi_\omega \nu_{\omega,\infty}$ is equivariant
  $$\int f_{\sigma\omega} \circ T_\omega d\mu_{\omega,\infty} = \int f_{\sigma\omega} d\mu_{\sigma\omega,\infty}$$
- **Assumption A** is basically the quasi-compactness in the deterministic case and the quasi-compactness for random cocycles using MET.

Operators on the open system II

- The measure
  $$\eta_\omega(B) = \frac{\nu_{\omega,c}(1_B \phi_\omega 1_{X_{\omega,0}})}{\nu_{\omega,c}(\phi_\omega 1_{X_{\omega,0}})}$$
  is conditionally invariant.
- η_ω is the Yaglom limit of any other measure absolutely continuous wrt $\nu_{\omega,c}$ and with positive Radon-Nykodim derivative, class $P_{(\nu_{\omega,c,+})}$
- Each element of $P_{(\nu_{\omega,c,+})}$ has a (conditioning) limit and in particular $\nu_{\omega,\infty}$ is the conditioning limit of $\nu_{\omega,c}$ and $\mu_{\omega,\infty}$ of η_ω
- The previous results were known in a few cases for deterministic systems (see Demers, Young, Liverani, Maume, Todd, etc, they are new for random systems. We now begin to state results which are new in both framework).
- In the case of billiards quoted above, Demers proved that the conditionally invariant measure ν is simply the eigenmeasure of $L = 1_{X_0} L_c(1_{X_0})$
  $$L\nu = \lambda_e \nu,$$
  where $\nu(X_n) = \lambda_e^n$, λ_e being the escape rate.
- In that case ν is singular wrt to the Lebesgue measure, but it is its Yaglom limit, so the escape rate could be practically computed.

New results I

- We remind that $\zeta_{\omega,n}(B) = \frac{\zeta_\omega(B\cap X_{\omega,n})}{\zeta_\omega(X_{\omega,n})}$ and η_ω is the conditionally invariant. Choose $\zeta \in P_{(\nu_{\omega,c},+)}$.
- We already said that $\zeta_{\omega,n}$ has a (conditioning) limit $\zeta_{\omega,\infty}$.

Theorem: Quasi Exponential Mixing

Under **Assumption A**, there exist are $\kappa \in (0,1)$ and a measurable function $C : \Omega \to (0,\infty)$ such that

$$\left|\zeta_{\omega,n}\left(\left(f_{\sigma^k\omega} \circ T^k_\omega\right) v_\omega\right) - \eta_{\sigma^k\omega,n-k}(f_{\sigma^k\omega})\zeta_{\omega,\infty}(v_\omega)\right| \le \frac{C(\omega)\|\psi_\omega\|\|\phi_\omega\|}{\inf \phi_{\sigma^k\omega} \inf \psi_\omega}\|f_{\sigma^k\omega}\|_\infty\|v\|\kappa^k$$

for every $\zeta \in P_{(\nu_{\omega,c},+)}$ with density ψ, measurable $f\colon \mathcal{I}_c \to \mathbb{R}$, $v \in \mathcal{B}$, integers $n \ge 1$, $1 \le k \le n$ and m-a.e. $\omega \in \Omega$. Moreover, if $\omega \mapsto \log\|f_\omega\|_\infty$ is in $L^1(m)$ then the right-hand side goes to zero exponentially fast as $k, n \to \infty$.

- We should add $\inf \phi_\omega$ is tempered; moreover $\log\|f_\omega\|_\infty$ is also tempered.

New results II

- We could now state the ergodic theorem for our open systems

Theorem: Quasi Strong Law of Large Numbers

Under **Assumption A** Then, $\mu_{\omega,\infty}$ is the unique quasi-ergodic probability measure with respect to $P_{(\nu_{\omega,c},+)}$.
Furthermore, for each $\zeta_\omega \in P_{(\nu_{\omega,c},+)}$ and $\mathcal{B}$-valued random variable $f_\omega : I_\omega \to \mathbb{R}$ with $\|f\|_\infty \in L^1(m)$, there exist sets $\Gamma_{\omega,n} \subseteq X_{\omega,n}$ with $\zeta_{\omega,n}(\Gamma_{\omega,n}) = 1$ a.s. for each $n \ge 0$ such that for m-a.e. $\omega \in \Omega$ and any $\{x_n\}_{n\ge 0} \in \prod_{n=0}^\infty \Gamma_{\omega,n}$, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T^j_\omega(x_n) = \int f_\omega \, d\mu_{\omega\infty}.$$

New results III

- We now establish a few limit theorems
- We first define the dynamical variance.
- Let $(S_n f)_\omega(x) := \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T^j_\omega(x)$.
  Assume that
  $$\mu_{\omega,\infty}(f_\omega) = 0 \quad m\text{-almost surely.}$$
  Then,
  $$(s^2(f))_\omega := \mu_{\omega,\infty}\left(f^2_\omega\right) + 2\sum_{n=1}^\infty \mu_{\sigma^{-n}\omega,\infty}\left(f_\omega \circ T^n_{\sigma^{-n}\omega} \cdot f_{\sigma^{-n}\omega}\right)$$
  exists m-almost surely. Set
  $$\Sigma^2(f) := \int s^2(f)\, dm.$$
  Furthermore,
  $\Sigma^2(f) > 0$ if and only if f is not cohomologous to a constant;

New results IV

Theorem: Quasi Central Limit Theorem

Under **Assumption A**. Let $f\colon X \to \mathbb{R}$ be a $\mathcal{B}$-valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m-a.s. and $\Sigma^2(f) > 0$. Then, for any $\zeta \in P_{(\nu_{\omega,c},+)}$ and $a \in \mathbb{R}$,

$$\lim_{n\to\infty} \zeta_{\omega,n}\left(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T^j_\omega \le a\right) = \frac{1}{\sqrt{2\pi\Sigma^2(f)}}\int_{-\infty}^a \exp\left(-\frac{z^2}{2\Sigma^2(f)}\right) dz \tag{7}$$

We emphasise that the right-hand-side of (7) is a (non-random) constant.

New results V

Theorem: Quasi Large Deviation Principle

Under **Assumption A**. Let $f : X \to \mathbb{R}$ be a $\mathcal{B}$-valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m-a.s. and $\Sigma^2(f) > 0$. Then, for any $\zeta \in P_{(\nu_{\omega,c},+)}$, there is a nonnegative, non-random, continuous, strictly convex function $c : (-\delta_0, \delta_0) \to \mathbb{R}$ with some $\delta_0 > 0$ such that c vanishes at 0 and that for any $\zeta \in P_{\nu_{\omega,c},+}(\nu_c)$ and $\delta \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \zeta_{\omega,n} \left(\frac{1}{n} \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T^j_\omega > \delta \right) = -c(\delta) \quad , \ m - a.s.$$

New results VI

Theorem: Quasi Berry-Essen Theorem

Under **Assumption A**. Let $f : X \to \mathbb{R}$ be a $\mathcal{B}$-valued random variable. Assume that $\mu_{\omega,\infty}(f) = 0$ m-a.s. and $\Sigma^2(f) > 0$. Then for any $\zeta \in P_{(\nu_{\omega,c},+)}$,, there is $C = C_{f,\zeta}$ such that

$$\sup_{a\in\mathbb{R}} \left| \zeta_{\omega,n} \left(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_{\sigma^j\omega} \circ T^j_\omega(\omega, x) \leq a \right) - \frac{1}{\sqrt{2\pi\Sigma^2(f)}} \int_{-\infty}^{a} \exp\left(-\frac{z^2}{2\Sigma^2(f)} \right) dz \right| \leq \frac{C}{\sqrt{n}}$$

Twsted operator: the Nagaev-Guivarc'h approach I

- The limit theorem are proved by adapting the Nagaev-Guivarc'h method in terms of the twisted operator to the cocycle of transfer operators.
- This theory, which used an implicit function result of Hennion, was proposed in our two papers:
  1. D. DRAGICEVIC, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, A spectral approach for quenched limit theorems for random expanding dynamical systems, Comm.Math. Phys., (2018).
  2. D . DRAGICEVIC, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, A spectral approach for quenched limit theoems for random hyperbolic dynamical systems, Trans. Amer Math. Soc, (2020).
- We used the quasi-compactness of the cocycle (and therefore the Multiplicative Ergodic Theorem) plus the implicit function theorem .

Twsted operator: the Nagaev-Guivarc'h approach II

- Focusing on the CLT, we will try to prove it by the convergence of the characteristic function of our process. By denoting with S_n the (random) Birkoff sum, we must control a term like

$$\zeta_{\omega,n}(e^{\theta(S_nf)_\omega}) = \zeta_{\omega,n}(e^{\theta \sum_{j=0}^{n-1} f_{\sigma^j\omega}\circ T^j_\omega}) =$$

$$\frac{\nu_{\omega,c}(e^{\theta(S_nf)_\omega} \hat{X}_{\omega,n} \cdot \psi_\omega)}{\nu_{\omega,c}(\hat{X}_{\omega,n} \cdot \psi_\omega)},$$

where ψ is the density of ζ_ω w.r.t. $\nu_{\omega,c}$ and $\hat{F} = 1_F$.
- By using duality one reduces the previous quantity to

$$\frac{\nu_{\sigma^n\omega,c}(\tilde{\mathcal{L}}^{\theta,n}_\omega \psi_\omega)}{\nu_{\sigma^n\omega,c}(\tilde{\mathcal{L}}^n_\omega \psi_\omega)},$$

where

$$\tilde{\mathcal{L}}^\theta_\omega(\varphi) := \tilde{\mathcal{L}}_\omega(e^{\theta f_\omega} \cdot \varphi) = \lambda_\omega^{-1} \mathcal{L}_{\omega,c}(e^{\theta f_\omega} \hat{X}_\omega \cdot \varphi), \qquad \varphi \in \mathcal{B}.$$

is the twisted operator. For the CLT $\theta = -\frac{it}{\sqrt{n}}$

Twisted operator III

- There are an open set $U \subset \mathbb{C}$ containing 0, C^1 maps $\phi : U \to L^\infty(\Omega, \mathcal{B})$, $\nu : U \to L^\infty(\Omega, \mathcal{B}^*)$, and a C^3 map $\lambda : U \to L^\infty(m)$ such that the following hold:
  (1) For every $\theta \in U$ and m-almost every $\omega \in \Omega$,
  $$\tilde{\mathcal{L}}^\theta_\omega \phi^\theta_\omega = \lambda^\theta_\omega \phi^\theta_{\sigma\omega}, \quad \text{and}$$
  $\lambda^\theta = \nu_\omega(e^{\theta f}\phi^\theta_\omega)$
  (2) For every $\theta \in U$, there is an $\mathcal{L}^\theta_\omega$-equivariant splitting $\mathcal{B} = \mathbb{C}\phi^\theta_\omega \oplus \mathrm{Ker}(\nu^\theta_\omega)$, where $\mathrm{Ker}(\nu^\theta_\omega) = \{u \in \mathcal{B} : \nu^\theta_\omega(u) = 0\}$. Moreover, there is $\kappa_1 \in (0,1)$ such that for every $\theta \in U$ and m-almost every $\omega \in \Omega$,
  $$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^{\theta,n}_\omega|_{\mathbb{C}\phi^\theta_\omega}\|_{\mathcal{B}} = \int \widehat{\Lambda}(\theta)\, dm$$
  and
  $$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^{\theta,n}_\omega|_{\mathrm{Ker}(\nu^\theta_\omega)}\|_{\mathcal{B}} \le \kappa_1 \int \widehat{\Lambda}(\theta)\, dm,$$
  where $\widehat{\Lambda}$ is the function $\widehat{\Lambda} : U \to L^\infty(m)$ given by $\widehat{\Lambda}(\theta)_\omega = \log \lambda^\theta_\omega$ for $\theta \in U$ and $\omega \in \Omega$.
  (3) It holds that
  $$\widehat{\Lambda}(0) = \widehat{\Lambda}'(0) = 0 \quad \int \widehat{\Lambda}''(0)\, dm = \Sigma^2(f).$$
  Furthermore, by shrinking U if necessary, there are $C > 0$, $r \in (0,1)$ such that for every $\theta \in U$ and m-almost every $\omega \in \Omega$,
  $$\|\mathcal{L}^{\theta,n}_\omega|_{\mathrm{Ker}(\nu^\theta_\omega)}\|_{\mathcal{B}} \le Cr^n.$$

Proof of the strong law of large numbers

- We show quasi strong law of large numbers in the following abstract form. We introduce the correlation function
$$C_{\zeta,f,v}(n,k) := \zeta_n\left(f \circ T^k \cdot v\right) - (\mu_\infty(f))_k\, \zeta_n(v)$$
for each $\zeta \in P_{\nu_c}$, each $\mathcal{B}$-valued random variable f and each $v \in \mathcal{B}$.

Proposition

Let $\zeta \in P_{\nu_c}$, and f be a $\mathcal{B}$-valued random variable such that $\|f\|_\infty \in L^1(m)$. Suppose that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |C_{\zeta,f,v}(n,k)| = 0$$
for every $v \in \mathcal{B}$. Then, there exist sets $\Gamma_n \subseteq X_n$ with $\zeta_n(\Gamma_n) = 1$ a.s. for each $n \ge 0$ such that for m-a.e. $\omega \in \Omega$ and any $\{x_n\}_{n\ge0} \in \prod_{n=0}^\infty \Gamma_{\omega,n}$, we have
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^j_\omega(x_n) = \int f\, d\mu_\infty.$$

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Proof.

Suppose that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |C_{\zeta,f,v}(n,k)| = 0$$
for every $v \in \mathcal{B}$. Then, since $\mathcal{B}$ is dense in $L^1(\nu_{\omega,c})$ for each $\omega \in \Omega$, we can assume that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |C_{\zeta,f,v}(n,k)| = 0$$
for every $v \in L^1(\zeta_{\omega,n})$. For each $\omega \in \Omega$, let $Z_\omega := \prod_{l=0}^\infty \zeta_{\omega,l}$, which is a probability measure on $\prod_{l=0}^\infty X_{\omega,l}$. To get the conclusion, it is enough to show that
$$\int \left(\frac{1}{n}\sum_{j=0}^{n-1}(f_{\sigma^k\omega} \circ T^k_\omega(x_n) - \mu_{\sigma^k\omega,\infty}(f_{\sigma^k\omega})\right) V(x_0, x_1, \ldots)\, dZ_\omega(x_0, x_1, \ldots) \to 0$$
as $n \to \infty$, for and $V \in L^1(Z_\omega)$ and m-a.e. $\omega \in \Omega$ (recall that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mu_{\sigma^k\omega,\infty}(f_{\sigma^k\omega}) = \int f\, d\mu_\infty$ by the ergodicity of (σ, m)). □

On the other hand, for every $n \in \mathbb{N}$,
$$\left|\int \left(\frac{1}{n}\sum_{j=0}^{n-1}(f_{\sigma^k\omega} \circ T^k_\omega(x_n) - \mu_{\sigma^k\omega,\infty}(f_{\sigma^k\omega})\right) V dZ_\omega\right| \le \frac{1}{n}\sum_{j=0}^{n-1} |C_{\zeta,f,\tilde{v},\omega}(n,k)|$$
where
$$\tilde{v}(x) = \int V(x_0, x_1, \ldots, x_{n-1}, x_{n+1}, \ldots)\, d\left(\prod_{l\ne n} \zeta_{\omega,l}\right).$$
On the other hand, by construction $\tilde{v}$ belongs to $L^1(\zeta_{\omega,n})$. Hence, from the claim at the beginning of this proof, we get the desired convergence.
The quasi-ergodicity follows from the above estimate (just Lebesgue dominated convergence theorem).

Examples

- J. ATNIP, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, Thermodynamic formalism for random weighted covering systems, Communication Mathematical Physics, 386, pp. 819–902 (2021)
- J. ATNIP, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, Equilibrium states for non-transitive random open and closed dynamical systems, Ergodic Theory and Dynamical systems, 43, (10), pp. 3193-3215, (2023)
- J. ATNIP, G. FROYLAND, C. GONZALEZ-TOKMAN, S. VAIENTI, Thermodynamic Formalism and Perturbation Formulae for Quenched Random Open Dynamical Systems, submitted, 160p, to appear on Disser. Math.

Checking the general assumptions for a large class of examples I

- Suppose that the spaces $\mathcal{J}_\omega = [0,1]$ for each $\omega \in \Omega$ and the maps $T_\omega : [0,1] \to [0,1]$ are surjective, finitely-branched, piecewise monotone, nonsingular (with respect to Lebesgue measure), and that there exists $C \geq 1$ such that

$$\text{ess sup}_\omega |T'_\omega| \leq C \qquad \text{and} \qquad \text{ess sup}_\omega\, d(T_\omega) \leq C, \tag{8}$$

where $d(T_\omega) := \sup_{y\in[0,1]} \# T_\omega^{-1}(y)$. We let $\mathcal{Z}_{\omega,0}$ denote the (finite) monotonicity partition of T_ω and for each $k \geq 2$ we let $\mathcal{Z}_{\omega,0}^{(k)}$ denote the partition of monotonicity of T_ω^k.

- The map $\sigma : \Omega \to \Omega$ is a homeomorphism, the skew-product map $T : \Omega \times [0,1] \to \Omega \times [0,1]$ is measurable, and $\omega \mapsto T_\omega$ has countable range.
- Recall that the *variation* of $f : [0,1] \to \mathbb{R}_+$ on $Z \subset [0,1]$ be

$$\mathrm{var}_Z(f) = \sup_{x_0 < \cdots < x_k,\, x_j \in Z} \sum_{j=0}^{k-1} |f(x_{j+1}) - f(x_j)|,$$

and $\mathrm{var}(f) := \mathrm{var}_{[0,1]}(f)$. We let $\mathrm{BV} = \mathrm{BV}([0,1])$ denote the set of complex-valued functions on $[0,1]$ that have bounded variation.
We define the Banach space $\mathrm{BV}_1 \subseteq L^\infty()$ to be the set of (equivalence classes of) functions of bounded variation on $[0,1]$, with norm given by

$$\|f\|_{\mathrm{BV}_1} := \inf_{\tilde{f}=f \text{ a.e.}} \mathrm{var}(\tilde{f}) + (|f|).$$

Checking the general assumptions for a large class of examples II

- We set $J_\omega := |T'_\omega|$ and define the random Perron–Frobenius operator, acting on functions in BV

$$P_\omega(f)(x) := \sum_{y \in T_\omega^{-1}(x)} \frac{f(y)}{J_\omega(y)}.$$

The operator P satisfies the well-known property that

$$\int_{[0,1]} P_\omega(f)\, d = \int_{[0,1]} f d \tag{9}$$

for m-a.e. $\omega \in \Omega$ and all $f \in \mathrm{BV}$. Recall that $g_0 = \{g_{\omega,0}\}_{\omega\in\Omega}$ and that

$$\mathcal{L}_{\omega,0}(f)(x) := \sum_{y \in T_\omega^{-1}(x)} g_{\omega,0}(y) f(y), \quad f \in \mathrm{BV}.$$

- We assume that the weight function $g_{\omega,0}$ lies in BV for each $\omega \in \Omega$ and satisfies

$$\text{ess sup}_\omega \|g_{\omega,0}\|_{\infty,1} < \infty, \tag{F2}$$

and

$$\text{ess inf}_\omega \inf g_{\omega,0} > 0. \tag{F3}$$

- We also assume a uniform covering condition: For every subinterval $J \subset [0,1]$ there is a $k = k(J)$ such that for a.e. ω one has $T_\omega^k(J) = [0,1]$. We will use it with the further requirement that there are only finitely many maps T_ω.

Checking the general assumptions for a large class of examples III

- There is a uniform-in-n and uniform-in-ω upper bound $\mathfrak{h} \geq 1$ on the number of connected components of $H_{\omega,n}$
- We also assume that

$$\lim_{n\to\infty} \text{ess sup}_\omega (H_{\omega,n}) = 0. \tag{F6}$$

- Further we suppose that there exists $N' \geq 1$ such that (contracting potential)

$$(9 + 12\mathfrak{h}N') \cdot \text{ess sup}_\omega \|g_{\omega,0}^{(N')}\|_{\infty,1} < \text{ess inf}_\omega \inf \mathcal{L}_{\omega,0}^{N'} \mathbb{1}. \tag{F8}$$

- With these assumptions we can prove the quasi-compactness of the unperturbed operator by using two important facts:
  1. The construction of the closed measure $\nu_{\omega,0}$ given in our paper *Thermodynamic Formalism for Random Weighted Covering Systems, CMP (2021).*
  2. The generalisation to general potentials of the quasi-compactness decomposition obtained for geometric ponential in *D . Dragicevic, G. Froyland, C. Gonzalez-Tokman, S.V, Almost Sure Invariance Principle for random piecewise expanding maps, Nonlinearity*

Checking the general assumptions for a large class of examples IV

- We now use quasi-compactness (or hyperbolicity) of the unperturbed transfer operator cocycle to guarantee that we have hyperbolic cocycles for large n for the perturbed operators. This requires
  1. $\hat{\mathcal{L}}_{\omega,0}$ is a hyperbolic transfer operator cocycle on BV_1 with norm $\|\cdot\|_{\mathrm{BV}_1}$ and a one-dimensional leading Oseledets space and slow and fast growth rates .
  2. The family of cocycles $\{\hat{\mathcal{L}}_{\omega,n,s}\}_{n\geq n_0}$ satisfy a uniform Lasota–Yorke inequality
  $$\|\hat{\mathcal{L}}^k_{\omega,n,s}f\|_{\mathrm{BV}_1} \leq A\alpha^k\|f\|_{\mathrm{BV}_1} + B^k\|f\|_1$$
  for a.e. ω and $n \geq n_0$, where $\alpha \leq \gamma < \Gamma \leq B$.
  3. $\lim_{n\to\infty} \operatorname{ess\,sup}_\omega \left|\left|\left|\hat{\mathcal{L}}_{\omega,0} - \hat{\mathcal{L}}_{\omega,n,s}\right|\right|\right| = 0$, where $\cdot$ is the $\mathrm{BV} - L^1()$ triple norm.
- The main technical problem is to translate to BV_1 what we got for $\nu_{,0}$.
- We may now apply Theorem 4.8 in *H. Crimmins, Stability of hyperbolic Oseledets splittings for quasi-compact operator cocycles*, to conclude that given $\delta > 0$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ the cocycle generated by $\hat{\mathcal{L}}_{\omega,n,s}$ is hyperbolic, with
  - (i) the existence of an equivariant family $\hat{\phi}_{\omega,n,s} \in \mathrm{BV}$ with $\operatorname{ess\,sup}_\omega \|\hat{\phi}_{\omega,n,s} - \phi_{\omega,0}\|_{\mathrm{BV}_1} < \delta$,
  - (ii) existence of corresponding Lyapunov multipliers $\hat{\lambda}_{\omega,n,s}$ satisfying $|\hat{\lambda}_{\omega,n,s} - 1| < \delta$,
  - (iii) operators $\hat{Q}_{\omega,n,s}$ satisfying $\|(\hat{Q}_{\omega,n,s})^n\|_{\mathrm{BV}_1} \leq K'(\gamma+\delta)^n$, where γ is the decay rate for $Q_{\omega,0}$.