Ergodic properties of random rotations with mixing Lasota-Yorke noise

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1. Introduction

We start with the definition of random rotation on the one-dimensional torus $\mathbb{T} = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Given a measure-preserving transformation τ and a $\mathbb{T}$-valued random variable α on a probability space $(\Omega, \mathcal{F}, P)$, consider a $\mathbb{T}$-valued stationary sequence $\{\alpha_n\}_{n\geq 0}$ defined by $\alpha_n = \alpha \circ \tau^n$ $(n \geq 0)$. Our random rotation is a family of maps $\mathcal{X} = \{X_n\}_{n\geq 0}$ determined by the following rule:

$$\begin{cases} X_0(\omega)x = x \in \mathbb{T} \\ \\ X_{n+1}(\omega)x = X_n(\omega)x + \alpha_n(\omega) \quad (n \geq 0), \end{cases}$$

where '+' denotes the usual addition on $\mathbb{T}$. $\mathcal{X}$ will be called the random rotation with dice variable α and noise system τ. As mentioned later, random walks on $\mathbb{T}$ are thought of as random rotations with $\{\alpha_n\}$ independent and identically distributed.

It is a classical fact that ergodic properties of a random walk $\mathcal{X}$ as a Markov chain with respect to the normalized Haar measure m are classified in terms of transition probability ν. To be concrete, (1) $\mathcal{X}$ is not ergodic if and only if there exists a positive integer N such that $\nu\{(0), (1/N), \ldots, ((N-1)/N)\} = 1$, (2) $\mathcal{X}$ is ergodic but not mixing if and only if there exist a positive integer N and an irrational θ such that $\nu\{(\theta), (\theta + 1/N), \ldots, (\theta + (N-1)/N)\} = 1$, and (3) $\mathcal{X}$ is mixing in the other cases, where for $a \in \mathbb{R}$, (a) denotes the coset in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ containing a.

Further we note that if the transition probability ν is discrete, the random walk $\mathcal{X}$ can be realized as a random rotation such that $(\Omega, \mathcal{F}, P)$ is the usual Lebesgue space with $\Omega = [0, 1)$ and the noise system τ is an expanding piecewise linear map mixing with respect to the Lebesgue measure P. In addition, we have to note that an expanding piecewise linear map is a typical example of the so called Lasota-Yorke map (abbrev. L-Y map hereafter).

Based on the above observation, first we consider an analogous classification problem for ergodic properties of the random rotations having a mixing L-Y map as its noise system in Section 6. The result is a sort of annealed one since it is stated in terms of the skew product transformation on the product space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m) \times (\Omega, \mathcal{F}, P)$ defined by

$$T_{\mathcal{X}}(x, \omega) = (X_1(\omega)x, \tau\omega) \qquad (x, \omega) \in \mathbb{T} \times \Omega.$$

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By identifying the unit interval with the one-dimensional torus, $T_{\mathcal{X}}$ can be regarded as the skew product transformation on $\mathbb{T}^2$. We note that Siboni [5] studied the criterion of ergodicity of $T_{\mathcal{X}}$ in the cases when τ is a β-transformation with integral $\beta \geq 2$. In Section 7 we also discuss about some sample-wise ergodic properties and show that any random rotation cannot be quenched 'weak-mixing' by using the product random rotation.

2. Random walks on the one-dimensional torus

In this section we recall the classification result on ergodic properties of random walks on $\mathbb{T}$. Let ν be a Borel probability measure on $\mathbb{T}$. Consider the product probability space $(\Omega, \mathcal{F}, P) = \prod_{n\geq 0}(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)$, the shift transformation $\tau : \Omega \to \Omega$ defined by $(\tau\omega)_n = \omega_n$ $(n \geq 0)$ for $\omega = (\omega_n)_{n\geq 0} \in \Omega$, and the coordinate function $\alpha : \Omega \to \mathbb{T}\,;\, \alpha(\omega) = \omega_0$. Then the random rotation $\mathcal{X} = \{X_n\}_{n\geq 0}$ given by dice variable α and noise system τ is nothing but the random walk on $\mathbb{T}$ with transition probability ν. Since any rotation preserves the normalized Haar measure m, m is a stationary measure for the random walk $\mathcal{X}$.

Recall the ergodic properties of a Markov chain $\mathcal{X}$ with respect to its stable distribution m. Define the Markov operator M of $\mathcal{X}$ by $Mf(x) = \displaystyle\int_\Omega f(X_n(\omega)x)\,dP$ for a Borel measurable function f on $\mathbb{T}$. The random walk $\mathcal{X}$ is said to be ergodic with respect to m if for any $f \in L^1(m)$, $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} M^k f(x) = \int_{\mathbb{T}} f\,dm$ m-a.e. $\mathcal{X}$ is said to be mixing with respect to m if for any $f,\, g \in L^2(m)$, $\displaystyle\lim_{n\to\infty}\int_{\mathbb{T}} M^n f(x)g(x)\,dm = \int_{\mathbb{T}} f\,dm \int_{\mathbb{T}} g\,dm.$

Since the skew product transformation $T_{\mathcal{X}} : \mathbb{T}\times\Omega \to \mathbb{T}\times\Omega\,;\,(x,\omega) \mapsto (X_1(\omega)x, \tau\omega)$ preserves the product measure $m \times P$, the corresponding ergodic properties of the system $(T_{\mathcal{X}}, m\times P)$ are defined as usual (see [6]). The following result holds for any Markov chain with stable distribution as well as random walks on $\mathbb{T}$ (see Theorem 4.1 in [3] (cf.[2])).

Proposition 2.1. *For a random walk $\mathcal{X}$ we have the following.*

(1) *$\mathcal{X}$ is ergodic with respect to m if and only if $(T_{\mathcal{X}}, m \times P)$ is ergodic.*

(2) *$\mathcal{X}$ is mixing with respect to m if and only if $(T_{\mathcal{X}}, m \times P)$ is mixing.*

The following classification is a consequence of Theorem 4.2 in [3].

Theorem 2.2. (1) *$(T_{\mathcal{X}}, m\times P)$ is not ergodic if and only if there exists a positive integer N such that $\nu(\{(0),\,(1/N),\,\ldots,\,((N-1)/N)\}) = 1$.*

(2) *$(T_{\mathcal{X}}, m \times P)$ is ergodic but not weak-mixing if and only if there exist a positive integer N and an irrational θ such that $\nu(\{(\theta),\,(\theta+1/N),\,\ldots,\,(\theta+(N-1)/N)\}) = 1$.*

(3) *$(T_{\mathcal{X}}, m \times P)$ is weak-mixing if and only if it is exact.*

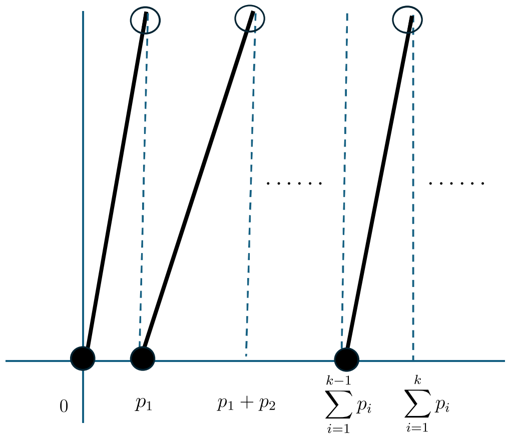
3. Random walks as random rotations with mixing L-Y noise

In this section we show that a random walk on $\mathbb{T}$ with discrete transition probability ν can be realized as a random rotation with noise system given by mixing piecewise linear expanding map of the unit interval.

For $I = \mathbb{N}$ or $I = \{1, 2, \cdots r\}$, let $\{a_i\}_{i\in I}$ be a set of distinct elements in $\mathbb{T}$ and let $\{p_i\}_{i\in I}$ be a family of positive numbers such that $\sum_{i\in I} p_i = 1$ with $p_i = \nu(\{a_i\})$ $(i \in I)$. Choose the unit interval $[0, 1)$ as the sample space Ω for our noise system. Define the map $\tau : \Omega \to \Omega$ by

$$\tau\omega = \frac{1}{p_k}\left(\omega - \sum_{i=1}^{k-1} p_i\right) \qquad \left(\omega \in \left[\sum_{i=1}^{k-1} p_i, \sum_{i=1}^{k} p_i\right),\ k \in I\right).$$

The graph of such a map is given as follows.



Clearly τ preserves the Lebesgue measure P on $\Omega = [0, 1)$. Next we define $\alpha : \Omega \to \mathbb{T}$ by

$$\alpha(\omega) = a_k \qquad \left(\omega \in \left[\sum_{i=1}^{k-1} p_i, \sum_{i=1}^{k} p_i\right),\ k \in I\right).$$

Then it is easy to see that $\mathbb{T}$-valued stationary sequence $\{\alpha_n = \alpha\circ\tau^n\}_{n\geq 0}$ is independent and identically ν-distributed. Therefore the random rotation given by the dice variable α and the noise system τ becomes the random walk on $\mathbb{T}$ with the transition probability ν. The piecewise linear expanding map τ is obviously an L-Y map satisfying the mixing condition (M) in [4].Thus our next aim is to examine whether an analogous result is valid for random rotations with mixing L-Y noise system.

4. Mixing L-Y maps and their Perron-Frobenius operators

In this section we summarize basic results on L-Y maps and their Perron-Frobenius operators following [4]. In what follows $(\Omega, \mathcal{F}, P)$ be the usual Lebesgue space i.e. $\Omega = [0, 1)$ and P is the completed Lebesgue measure. P-almost everywhere defined map $\tau : \Omega \to \Omega$ is called an L-Y map if it satisfies the following conditions.

There exist a family $\mathcal{P}$ of closed intervals with nonempty interior and a family $\mathcal{T} = \{\tau_J\}_{J\in\mathcal{P}}$ of maps such that

$(\mathcal{P}, 1)$ If $J, K \in \mathcal{P}$ $(J \neq K)$, then $\mathrm{int}J \cap \mathrm{int}K = \emptyset$.

$(\mathcal{P}, 2)$ $P\left(\Omega \setminus \bigcup_{J\in\mathcal{P}} J\right) = 0$.

$(\mathcal{T}, 1)$ For $J \in \mathcal{P}$, $\tau_J : J \to \tau_J J$ is C^2-diffeomorphism.

$(\mathcal{T}, 2)$ For $J \in \mathcal{P}$, $\tau|_{\mathrm{int}J} = \tau_J|_{\mathrm{int}J}$.

$(\mathcal{T}, 3)$ Except for a finite number of $J \in \mathcal{P}$, $\tau_J J = [0, 1]$.

(E)(expanding) There exists $n \in \mathbb{N}$, $\mathrm{ess.inf}|(\tau^n)'| > 1$.

(R)(Rényi condition (bounded distortion)) $\mathrm{ess.sup}\left|\dfrac{\tau''}{(\tau')^2}\right| < \infty$.

In the above 'ess.inf' and 'ess.sup' mean the essential infimum and the essential supremum with respect to the Lebesgue measure P.

An L-Y map τ is called a mixing L-Y map if it satisfies the following condition .

(M) τ has a fully-supported P-absolutely continuous invariant probability measure Q such that the system (τ, Q) is mixing.

The next remark tells us that the mixing condition (M) is not so restrictive for L-Y maps.

Remark 4.1. As above let P denote the Lebesgue measure on $\Omega = [0, 1)$. Any L-Y map we can show that there exists a finite number (say r) of P-absolutely continuous invariant probability measures $Q_1, \ldots, Q_r$ with density of bounded variation such that the following hold

(1) Any P-absolutely continuous complex-valued τ-invariant measure can be expressed as a linear combination of Q_i's.

(2) For each i $(1 \leq i \leq r)$, the system (τ, Q_i) is ergodic.

(3) For each i $(1 \leq i \leq r)$, there exist a positive integer N_i and a disjoint family of measurable sets L_{ij} $(0 \leq j \leq N_i - 1)$ satisfying $\tau L_{ij} = L_{ij+1} \pmod{N_i}$ and the system $(\tau^{N_i}, N_i Q_i|_{L_{ij}})$ is exact.

In the below we briefly review about Perron-Frobenius operators. Given a nonsingular transformation T on a probability space $(M, \mathcal{M}, \mu)$, the Perron-Frobenius operator (abbrev. P-F operator) $\mathcal{L}_{T,\mu} : L^1(\mu) \to L^1(\mu)$ for T with respect to μ is determined by the following identity.

$$\int_M f(\mathcal{L}_{T,\mu}g)\,d\mu = \int_M (f\circ T)g\,d\mu \qquad (f\in L^\infty(\mu) \text{ and } g\in L^1(\mu)).$$

The following is an easy consequence of this characterization.

PROPOSITION 4.2. (1) *For $h\in L^1(\mu)$, $h\mu$ is a T-invariant complex-valued measure if and only if $\mathcal{L}_{T,\mu}h = h$ in $L^1(\mu)$.*

(2) *Let $\nu = hm$ be a μ-absolutely continuous T-invariant probability measure, φ an S^1-valued measurable function on M, and $g\in L^1(\nu)$. Then the following are equivalent.*

(i) *$\mathcal{L}_{T,\mu}(\varphi gh) = gh$ in $L^1(\mu)$.*

(ii) *$\mathcal{L}_{T,\nu}(\varphi g) = g$ in $L^1(\nu)$.*

(iii) *$g\circ T = \varphi g$ in $L^1(\nu)$.*

For a real-valued measurable function u and $t\in\mathbb{R}$, we consider a perturbed P-F operator $\mathcal{L}_{T,\mu,\sqrt{-1}u} : L^1(\mu)\to L^1(\mu)$ defined by

$$\mathcal{L}_{T,\mu,\sqrt{-1}tu}g = \mathcal{L}_{T,\mu}(e^{\sqrt{-1}tu}g) \qquad (g\in L^1(\mu)).$$

For $n\in\mathbb{N}$, it is easy to see that

$$\int_M \mathcal{L}^n_{T,\mu,\sqrt{-1}tu}g\,d\mu = \int_M \exp\left(\sqrt{-1}tS_nu\right)g\,d\mu.$$

Therefore the family $\{\mathcal{L}^n_{T,\mu,\sqrt{-1}tu}g,\ t\in\mathbb{R}\}$ has all information of distribution of the sum $S_nu = \sum_{j=0}^{n-1} u\circ T^j$ with respect to the μ-absolutely continuous measure with density g.

Now we go back to the review of results on mixing L-Y maps. Let BV denote the subspace of $L^1(P)$ consisting of all elements with version of bounded variation. BV turns out to be a Banach algebra endowed with the norm defined by

$$\|\varphi\|_{BV} = \bigvee\varphi + \|\varphi\|_{\infty,P} \qquad (\varphi\in BV),$$

where $\bigvee\varphi$ is the infimum of the total variations $\bigvee\tilde\varphi$ taken over all the versions of $\tilde\varphi$ of φ. We summarize some of results in Lemma 2.1 $\sim$ Lemma 2.3 in [4] as the following proposition.

PROPOSITION 4.3. *Let Q be the unique P-absolutely continuous invariant probability measure for a mixing L-Y map τ. For a real-valued element α in BV and $t\in\mathbb{R}$, define the operator $U_{\sqrt{-1}t} : L^1(Q)\to L^1(Q)$ by. $U_{\sqrt{-1}t}\varphi = e^{\sqrt{-1}t\alpha}\varphi$ ($\varphi\in L^1(Q)$). Then we have:*

(1) *For $\lambda\in S^1$, λ is an eigenvalue of $L_{\sqrt{-1}t} = \mathcal{L}_{\tau,Q,\sqrt{-1}t\alpha} : L^1(Q)\to L^1(Q)$ if and only if $\bar\lambda = \lambda^{-1}$ is an eigenvalue of $U_{\sqrt{-1}t}$.*

(2) *If h is an eigenvector of $U_{\sqrt{-1}t}$ on $L^1(Q)$ corresponding to an eigenvalue λ with modulus 1, $|h|$ is a constant Q-a.e.*

(3) *For $\lambda \in S^1$, $h \in L^1(Q)$ is an eigenvector of $L_{\sqrt{-1}t}$ corresponding to λ if and only if hh_0 is an eigenvector of $\mathcal{L}_{\sqrt{-1}t} = \mathcal{L}_{\tau,P,\sqrt{-1}t\alpha}$ on $L^1(P)$ corresponding to λ, where h_0 denotes the density of Q with respect to P.*

(4) *If $\lambda \in S^1$ is an eigenvalue of $U_{\sqrt{-1}t}$, then it is simple.*

(5) *$U_{\sqrt{-1}t}$ has at most one eigenvalue of modulus 1.*

(6) *$\mathcal{L}_{\sqrt{-1}t}$ is a bounded linear operator on BV as well as a bounded linear operator on $L^1(P)$.*

(7) *If $\mathcal{L}_{\sqrt{-1}t}$ on $L^1(P)$ does not have an eigenvalue of modulus 1, then $\mathcal{L}_{\sqrt{-1}t}$ on BV has the spectral radius less than 1.*

(8) *If $\mathcal{L}_{\sqrt{-1}t}$ on $L^1(P)$ has an eigenvalue of modulus 1, say $\lambda(\sqrt{-1}t)$, then $\mathcal{L}_{\sqrt{-1}t}$ has the following spectral decomposition as an operator on BV*

$$\mathcal{L}^n_{\sqrt{-1}t} = \lambda(\sqrt{-1}t)^n E_{\sqrt{-1}t} + R^n_{\sqrt{-1}t} \qquad (n \geq 1) \tag{$*$}$$

satisfying:

(8-i) *$E_{\sqrt{-1}t}$ is the projection onto the one-dimensional eigenspace corresponding to $\lambda(\sqrt{-1}t)$ given by*

$$E_{\sqrt{-1}t}g = \int_\Omega gh\, dP \int_\Omega \bar{h} h_0\, dP \qquad (g \in L^1(P)),$$

where h is any eigenvector of $L_{\sqrt{-1}t}$ corresponding $\lambda(\sqrt{-1}t)$ with modulus 1.

(8-ii) *$R_{\sqrt{-1}t}$ is the bounded operator on BV with spectral radius less than 1 such that*

$$E_{\sqrt{-1}t}R_{\sqrt{-1}t} = R_{\sqrt{-1}t}E_{\sqrt{-1}t} = O_{BV}.$$

(8-iii) *The spectral decomposition $(*)$ is valid on $L^1(P)$ and $\|R^n_{\sqrt{-1}t}g\|_{1,P} \to 0$ $(n \to \infty)$ for any $g \in L^1(P)$.*

5. Classification of functions of bounded variation with respect to a mixing L-Y map

In this section we recall the classification theorem in [4] on functions of bounded variation based on the spectral property of the Perron-Frobenius operator for an L-Y map with mixing condition (M). For a real-valued $\alpha \in BV$, define the following.

$$\Lambda(\alpha) = \{t \in \mathbb{R} : \mathcal{L}_{\sqrt{-1}t} \text{ has an eigenvalue with modulus } 1\},$$

$$G(\alpha) = \{\lambda \in S^1 : \lambda = \lambda(\sqrt{-1}t) \text{ for some } t \in \Lambda(\alpha)\},$$

$$H(\alpha) = \{(h) : h \in H_0(\alpha)\},$$

where

$$H_0(\alpha) = \{h \in L^1(Q) \,:\, h \text{ is } S^1\text{-valued eigenvector of } U_{\sqrt{-1}t} \text{ for some } t \in \Lambda(\alpha)\}.$$

By virtue of Proposition 4.3, we can easily verify the following.

PROPOSITION 5.1. (1) *$\Lambda(\alpha)$ is a subgroup of $\mathbb{R}$ with the usual addition.*

(2) *$G(\alpha)$ is a subgroup of S^1 with the usual multiplication.*

(3) *$H(\alpha)$ is a group with multiplication $(h_1)(h_2) = (h_1 h_2)$ for h_1, $h_2 \in H_0(\alpha)$.*

We are in a position to state the classification theorem in [4].

THEOREM 5.2. *Let τ be a mixing L-Y map on $\Omega = [0, 1)$ and Q τ-invariant probability measure absolutely continuous with respect to the Lebesgue measure P. Consider the subspace BV_0 of BV defined by*

$$BV_0 = \{f \in BV \,:\, f \text{ is real-valued and } \int_\Omega f\, dQ = 0\}.$$

Put

$$a = a(\alpha) = \begin{cases} \inf\{t > 0 \,:\, t \in \Lambda(\alpha)\} & \text{if } \Lambda(\alpha) \setminus \{0\} \neq \emptyset \\ +\infty & \text{if } \Lambda(\alpha) \setminus \{0\} = \emptyset. \end{cases}$$

Then BV_0 can be expressed as a disjoint union of the subsets B_j $(0 \le j \le 5)$ having the following characterization.

(1) *$B_0 = \{\alpha \in BV_0 \,:\, \Lambda(\alpha) = \mathbb{R}\}$ and $\alpha \in B_0$ if and only if there exists a $\beta \in L^2(Q)$ such that $\alpha = \beta \circ \tau - \beta$. In particular, $\alpha \in B_0$ yields $a = 0$ and $G(\alpha) = \{1\}$.*

(2) *$B_1 = \{\alpha \in BV_0 \,:\, \Lambda(\alpha) \cong \mathbb{Z},\, G(\alpha) \cong \mathbb{Z}/p\mathbb{Z},\, H(\alpha) \cong \mathbb{Z}/q\mathbb{Z}$ for some p, $q \in \mathbb{N}\}$ and $\alpha \in B_1$ if and only if there exist $b > 0$ and an integer-valued function $K \in BV_0$ such that $b\alpha = 2\pi K$. In particular, $\alpha \in B_1$ yields $a > 0$, $\Lambda(\alpha) = a\mathbb{Z}$, and $b = apq$.*

(3) *$B_2 = \{\alpha \in BV_0 \,:\, \Lambda(\alpha) \cong \mathbb{Z},\, G(\alpha) \cong \mathbb{Z}/p\mathbb{Z},\, H(\alpha) \cong \mathbb{Z}$ for some $p \in \mathbb{N}\}$ and $\alpha \in B_2$ if and only if there exist $b > 0$, an integer-valued function $K \in BV_0 \setminus B_0$, and a real-valued bounded function g such that ng can not be an integer-valued for any $n \in \mathbb{Z}\setminus\{0\}$ and $b\alpha = 2\pi(g \circ \tau - g + K)$. In particular, $\alpha \in B_2$ yields $a > 0$, $\Lambda(\alpha) = a\mathbb{Z}$, and $b = ap$.*

(4) *$B_3 = \{\alpha \in BV_0 \,:\, \Lambda(\alpha) \cong \mathbb{Z},\, G(\alpha) \cong \mathbb{Z},\, H(\alpha) \cong \mathbb{Z}/q\mathbb{Z}$ for some $q \in \mathbb{N}\}$ and $\alpha \in B_3$ if and only if there exist $b > 0$, $\theta \in (0, 1) \cap \mathbb{Q}^c$, and an integer-valued function K with $K + \theta \in BV_0 \setminus B_0$ such that $b\alpha = 2\pi(K + \theta)$. In particular, $\alpha \in B_3$ yields $a > 0$, $\Lambda(\alpha) = a\mathbb{Z}$, and $b = aq$.*

(5) *$B_4 = \{\alpha \in BV_0 \,:\, \Lambda(\alpha) \cong \mathbb{Z},\, G(\alpha) \cong \mathbb{Z},\, H(\alpha) \cong \mathbb{Z}\}$ and $\alpha \in B_4$ if and only if there exist $b > 0$, $\theta \in (0, 1) \cap \mathbb{Q}^c$, an integer-valued function K with $K + \theta \in BV_0$, and a real-valued bounded function g such that ng can not be an integer-valued for any $n \in \mathbb{Z}\setminus\{0\}$ and $b\alpha = 2\pi(g \circ \tau - g + K + \theta)$. In particular, $\alpha \in B_4$ yields $a > 0$, $\Lambda(\alpha) = a\mathbb{Z}$, and $b = a$.*

(6) *$B_5 = \{\alpha \in BV_0 \,:\, \Lambda(\alpha) = \{0\},\, G(\alpha) = \{1\},\, H(\alpha) \cong \{1\}\}$.*

Sketch of Proof. First we consider the decomposition $BV_0 = B_0 \cup (BV_0 \setminus B_0)$, where B_0 is the totality of element α with vanishing limit variance. $\alpha \in B_0$ if and only if $a = 0$. Indeed, by the well known result on the central limit theorem for dynamical system, there exists $\beta \in L^2(Q)$ such that $\alpha = \beta \circ \tau - \beta$. $e^{\sqrt{-1}t\beta\circ\tau} = e^{\sqrt{-1}t\beta}e^{\sqrt{-1}t\alpha}$ for all $t \in \mathbb{R}$. Consequently, we have $\Lambda(\alpha) = \mathbb{R}$.

For α with $a = \infty$, we can easily see that $\Lambda(\alpha) = \{0\}$, $G(\alpha) = \{1\}$, and $H(\alpha) \cong \{1\}$.

It remains to classify the case when $0 < a < \infty$. Clearly, we have $\Lambda(\alpha) = a\mathbb{Z} \cong \mathbb{Z}$.

B_1 corresponds to the case when $h \circ \tau = \bar{\lambda}e^{\sqrt{-1}a\alpha}h$ for some $h \in L^1(Q)$ with $h^q = 1$ and $\lambda \in S^1$ with $\lambda^p = 1$. Therefore $e^{\sqrt{-1}apq\alpha} = 1$. Thus $K = apq\alpha/2\pi$ is $\mathbb{Z}$-valued.

B_2 corresponds to the case when $h \circ \tau = \bar{\lambda}e^{\sqrt{-1}a\alpha}h$ for some $h \in L^1(Q)$ with $h^n \neq 1$ for any $n \in \mathbb{N}$ and $\lambda \in S^1$ with λ a pth root of 1. Put $g = \arg(h^p)/(2\pi)$ and $K = ap\alpha/(2\pi) - (g \circ \tau - g)$. K is a $\mathbb{Z}$-valued bounded function. Moreover, since $hh_0 \in BV$, g has a version such that it has at most countably many points of discontinuity and of the first kind. Thus $K \in BV$.

B_3 corresponds to the case when $h \circ \tau = \bar{\lambda}e^{\sqrt{-1}a\alpha}h$ for some $h \in L^1(Q)$ with $h^q = 1$ for some $q \in \mathbb{N}$ and $\lambda \in S^1$ with λ not root of 1. We have $e^{\sqrt{-1}aq} = \lambda^q$. Put $\theta = \arg(\lambda^q)/(2\pi)$ and $K = aq\alpha/(2\pi) - \theta$. Then we have K is $\mathbb{Z}$-valued and BV. Obviously $\theta \in (0,1)$ is irrational.

B_4 corresponds to the case when $h \circ \tau = \bar{\lambda}e^{\sqrt{-1}a\alpha}h$ for some $h \in L^1(Q)$ with $h^n \neq 1$ for any $n \in \mathbb{N}$ and $\lambda \in S^1$ with λ not root of 1. Put $g = (\arg(h))/(2\pi)$ and $\theta = \arg(\lambda)/(2\pi)$. Then $a\alpha = 2\pi(g \circ \tau - g + K + \theta)$. Clearly g is bounded function such that ng is not $\mathbb{Z}$-valued for any $n \in \mathbb{Z} \setminus \{0\}$ and $\theta \in (0,1)$ is irrational. In the same manner as above, we see that K is an element in BV. □

6. Classification of random rotations with mixing L-Y noise

In this section we show the following.

THEOREM 6.1. *Consider a mixing L-Y map τ and $\alpha \in BV_0$. Let Q be a unique P-absolutely continuous invariant probability measure for τ. Then for the skew product transformation $T_{\mathcal{X}}$ associated to the random rotation $\mathcal{X}$ with dice variable α and noise system τ, the following hold:*

(1) *$(T_{\mathcal{X}}, m \times Q)$ is not ergodic if and only if there exist $N \in \mathbb{N}$, a real-valued measurable function β, and $\mathbb{Z}$-valued function K such that $\alpha = \beta \circ \tau - \beta + (1/N)K$.*

(2) *$(T_{\mathcal{X}}, m \times Q)$ is ergodic but not weak-mixing if and only if there exist $N \in \mathbb{N}$, θ irrational, a real-valued measurable function β, and $\mathbb{Z}$-valued function K such that $\alpha = \beta \circ \tau - \beta + \theta + (1/N)K$.*

(3) *$(T_{\mathcal{X}}, m \times Q)$ is weak-mixing if and only if it is exact.*

Proof. Both assertions (1) and (2) are essentially proved in Anzai [1]. So we just prove (2) assuming the assertion (1) is valid. We may identify $\mathbb{T}$ with the unit interval for notational convenience.

First suppose that there exist $N \in \mathbb{N}$, θ irrational, a real-valued measurable function β, and $\mathbb{Z}$-valued function K such that $\alpha = \beta \circ \tau - \beta + \theta + (1/N)K$. Since θ is irrational, (1) yields $(T_{\mathcal{X}}, m \times Q)$ is ergodic. Next, define a function Φ by $\Phi(x, \omega) = e^{\sqrt{-1}2\pi N(x-\beta(\omega))}$ for $(x, \omega) \in \mathbb{T} \times \Omega$. Then $\Phi(T_{\mathcal{X}}(x, \omega)) = e^{\sqrt{-1}2\pi N(x+\alpha(\omega)-\beta(\tau\omega))} = e^{\sqrt{-1}2\pi N(x+\theta-\beta(\omega))} = e^{\sqrt{-1}2\pi N\theta}\Phi(x, \omega)$. Therefore the induced operator $U_{T_{\mathcal{X}}}$ has an eigenvalue $e^{\sqrt{-1}2\pi N\theta} \neq 1$. Thus $(T_{\mathcal{X}}, m \times Q)$ is not weak-mixing.

Conversely, if $(T_{\mathcal{X}}, m \times Q)$ is ergodic but not weak-mixing. There exist $\Phi \in L^2(m \times Q)$ and $\lambda \in S^1 \setminus \{1\}$ such that $\Phi(x + \alpha(\omega), \tau\omega) = \lambda\Phi(x, \omega)$ $(m \times P)$-a.e.(x, ω). By the ergodicity we may assume that $|\Phi(x, \omega)| = 1$ $(m \times P)$-a.e.(x, ω). i.e. We can find $\Gamma \in \mathcal{F}$ with $Q(\Gamma) = 1$ such that $\omega \in \Gamma$ yields $|\Phi(x, \omega)| = 1$ m-a.e.x.

For $n \in \mathbb{Z}$ and $\omega \in \Gamma$, we define $\hat{\Phi}_n(\omega)$ by

$$\hat{\Phi}_n(\omega) = \int_{\mathbb{T}} \Phi(x, \omega) e^{-\sqrt{-1}2\pi nx}\, dx$$

For P-a.e.ω, we have

$$\begin{aligned}
\hat{\Phi}_n(\tau\omega)e^{\sqrt{-1}2\pi n\alpha(\omega)} &= \int_{\mathbb{T}} \Phi(x, \tau\omega)e^{-\sqrt{-1}2\pi n(x-\alpha(\omega))}\, m(dx) \\
&= \int_{\mathbb{T}} \Phi(x + \alpha(\omega), \tau\omega)\lambda e^{-\sqrt{-1}2\pi nx}\, m(dx) \quad (\because \text{ rotation invariance of } m) \\
&= \lambda \int_{\mathbb{T}} \Phi(x, \omega)e^{-\sqrt{-1}2\pi nx}\, dx = \lambda\hat{\Phi}_n(\omega)
\end{aligned}$$

holds. Since (τ, Q) is ergodic and P and Q are equivalent, $|\hat{\Phi}_n(\omega)| = c_n$ Q-a.e.ω for some constant c_n. Since Φ is not a constant function, there exists $n \in \mathbb{Z} \setminus \{0\}$ with $c_n \neq 0$. By using $\overline{\Phi}$ if necessary, we may assume $c_{-N} \neq 0$ for some $N \in \mathbb{N}$. Therefore, we have $\hat{\Phi}_{-N}(\tau\omega) = \lambda e^{\sqrt{-1}2\pi N\alpha}\hat{\Phi}_{-N}(\omega)$ Q-a.e.ω. Thus if we put $\beta = \arg(\hat{\Phi}_{-N})/(2\pi N)$ and $\theta = (-\arg(\lambda))/(2\pi N)$, and $K = N\alpha + (\arg\lambda)/(2\pi) - \arg(\hat{\Phi}_{-N})/(2\pi) \circ \tau + \arg(\hat{\Phi}_{-N})/(2\pi)$ then K is Z-valued and we have

$$\alpha = \beta \circ \tau - \beta + \theta + K/N.$$

If θ could be rational, $(T_{\mathcal{X}}, m \times Q)$ is not ergodic by (1). Thus θ is irrational. Hence we have arrived at the result.

It remains to prove the assertion (3). Comparing with the classification in Theorem 5.2, it is not hard to see that $(T_{\mathcal{X}}, m \times Q)$ is weak-mixing if and only if $\Lambda(\alpha) \cap 2\pi\mathbb{Z} = \{0\}$. By (7) in Proposition 4.3 the condition implies that spectral radius of $\mathcal{L}_{\sqrt{-1}2\pi N}$ is less than 1 for any $N \in \mathbb{Z} \setminus \{0\}$. By virtue of Proposition 2.3 in [3], it is enough to complete the table below.

<table>
<tr><th colspan="2">$\Lambda(\alpha)$</th><th>$G(\alpha)$</th><th>$(T_{\mathcal{X}}, m \times Q)$</th></tr>
<tr><td colspan="2">$\mathbb{R}$</td><td>$\{I\}$</td><td rowspan="2">not ergodic</td></tr>
<tr><td rowspan="3">$a\mathbb{Z}$</td><td rowspan="2">$a \in 2\pi\mathbb{Q} \setminus \{0\}$</td><td>$\sharp G(\alpha) < \infty$</td></tr>
<tr><td>$\sharp G(\alpha) = \infty$</td><td>ergodic but not weak-mixing</td></tr>
<tr><td>$a \notin 2\pi\mathbb{Q} \setminus \{0\}$</td><td>any</td><td rowspan="2">exact</td></tr>
<tr><td colspan="2">$\{0\}$</td><td>$\{I\}$</td></tr>
</table>

Now we have to show that for any $\Phi \in L^1(m \times P)$

$$\left\| \mathcal{L}^n_{T, m\times P}\Phi - \int_{\mathbb{T}\times\Omega} \Phi \, d(m \times P) h_0 \right\|_{1, m\times P} \to 0 \quad (n \to \infty), \tag{$**$}$$

where h_0 is the density of the unique P-absolutely continuous probability measure Q. Then it is enough to prove $(**)$ for Φ having the form $\Phi(x, \omega) = e_k(x)\varphi(\omega) \quad ((x, \omega) \in \mathbb{T}\times\Omega)$ with $e_k(x) = e^{\sqrt{-1}2\pi k x} \quad (k \in \mathbb{Z})$ and φ being of bounded variation. By an easy calculation we obtain

$$\mathcal{L}^n_{m\times P}(e_k\varphi)(x, \omega) = e_k(x)\mathcal{L}^n_{-\sqrt{-1}2\pi k}\varphi(\omega),$$

where $\mathcal{L}_{-\sqrt{-1}2\pi k} = \mathcal{L}_{\tau, P, -\sqrt{-1}2\pi k\alpha}$ as before. Therefore by (7) in Theorem 4.3 we have

$$\begin{aligned} \mathcal{L}^n_{m\times P}(e_k\varphi) \to & \begin{cases} \left(\int_\Omega \varphi \, dP\right) h_0 & (k = 0) \\ 0 & (k \neq 0) \end{cases} \\ & = \left(\int_{\mathbb{T}\times\Omega} e_k\varphi d(m \times P)\right) h_0 \end{aligned}$$

in $L^1(m \times P)$. Now the proof is completed. □

7. Sample-wise ergodic properties

Finally, we consider briefly some sample-wise properties. The following theorem tells us that for the random rotation $\mathcal{X}$ with mixing L-Y noise, the ergodicity of the corresponding skew product transformation $T_{\mathcal{X}}$ yields quenched ergodicity but we can not expect much stronger quenched ergodic properties even if $T_{\mathcal{X}}$ is exact.

Theorem 7.1. *Let $\mathcal{X} = \{X_n\}_{n\geq 0}$ be a random rotation with dice variable α of bounded variation and mixing L-Y noise τ. Then we have the following.*

(1) *If $(T_{\mathcal{X}}, m\times Q)$ is ergodic, there exists a measurable set $\Gamma\in\mathcal{F}$ with $P(\Gamma)=1$ such that if $\omega\in\Gamma$, then for any initial data $x\in\mathbb{T}$, the sequence $\{X_n(\omega)x\}_{n\geq 0}$ is uniformly distributed in $\mathbb{T}$, i.e.*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(X_n(\omega)x) \to \int_{\mathbb{T}} f\,dm \qquad \textit{uniformly in } x\in\mathbb{T} \quad (n\to\infty)$$

for any $f\in C(\mathbb{T})$.

(2) *Further we suppose that $(T_{\mathcal{X}}, m\times Q)$ is exact. Consider the product system $\mathcal{X}^2=\mathcal{X}\times\mathcal{X}=\{(X_n,X_n)\}_{n\geq 0}$ with dice variable α and noise system τ. Then there exists a measurable set $\Delta\in\mathcal{F}$ with $P(\Delta)=1$ such that if $\omega\in\Delta$*

$$\frac{1}{n}\sum_{j=0}^{n-1} F(X_n(\omega)x, X_n(\omega)y) \to \int_{\mathbb{T}} F(x-y+u,u)\,m(du) \ \textit{uniformly in } (x,y)\in\mathbb{T}\times\mathbb{T}\,(n\to\infty)$$

for any $F\in C(\mathbb{T}\times\mathbb{T})$. In particular, the system $(T_{\mathcal{X}^2}, m^2\times Q)$ can not be ergodic, where $T_{\mathcal{X}^2}$ is the skew product transformation corresponding to the product system $\mathcal{X}^2$.

Sketch of Proof. It is crucial that by virtue of the Ascoli-Arzelà theorem, for $f\in C(\mathbb{T})$ and $F\in C(\mathbb{T}\times\mathbb{T})$ and for any ω, $\{f(X_n(\omega)\cdot)\}_{n\geq 0}$ and $\{F(X_n(\omega)\cdot, X_n(\omega)\cdot)\}_{n\geq 0}$ are relatively compact in $C(\mathbb{T})$ and $C(\mathbb{T}\times\mathbb{T})$, respectively. Since both (1) and (2) are proved in the similar way by the help of this fact and the ergodic theorem, we only explain about the idea of the proof of (2).

If $F\in C(\mathbb{T}\times\mathbb{T})$, by the ergodic theorem $A_nF(x,y,\omega)=(1/n)\sum_{j=0}^{n-1}F(X_n(\omega)x,X_n(\omega)y)$ converges $m^2\times P$-a.e. to some $F^*(x,y,\omega)\in L^1(m^2\times Q)$. Since $\{A_nF(\cdot,\cdot,\omega)\}_{n\geq 0}$ is relatively compact in $C(\mathbb{T}\times\mathbb{T})$, the identification $F^*(x,y,\omega)$ with the deterministic function $\int_{\mathcal{T}} F(x-y+u,u), m(du)$ is essential for the proof. This is verified as follows.

As in the above, we regard $\mathbb{T}$ as $[0,1)$ and use the $e_k(x)=e^{\sqrt{-1}2\pi kx}$ for our convenience. Let us consider the case when the Fourier series $F(x,y)=\sum_{k,l}\hat{F}_{kl}e_k(x)e_l(y)$ converges fast enough. Take any real-valued $G\in C(\mathbb{T}\times\mathbb{T})$ whose Fourier series also converges sufficiently fast and $\varphi\in BV$ on Ω. Then we have

$$\begin{aligned}
&\int_{\mathbb{T}\times\mathbb{T}\times\Omega} F(X_n(\omega)x, X_n(\omega)y)G(x,y)\varphi(\omega)\,d(m^2\times P)\\
&=\sum_{k,l}\hat{F}_{kl}\int_{\mathbb{T}\times\mathbb{T}} G(x,y)e_k(x)e_l(y)\,dm^2\int_\Omega \mathcal{L}^n_{\sqrt{-1}2\pi(k+l)}\varphi(\omega)\,dP\\
&=\sum_{k,l}\hat{F}_{kl}\overline{\hat{G}}_{kl}\int_\Omega \mathcal{L}^n_{\sqrt{-1}2\pi(k+l)}\varphi(\omega)\,dP. \to \sum_k \hat{F}_{k,-k}\overline{\hat{G}}_{k,-k}\int_\Omega \varphi(\omega)\,dP
\end{aligned}$$

since $(T_{\mathcal{X}}, m \times Q)$ is exact. By the Parseval theorem, we obtain

$$\begin{aligned}
&\sum_k \hat{F}_{k,-k}\overline{\hat{G}}_{k,-k} \int_\Omega \varphi(\omega)\, dP \\
&= \int_{\mathbb{Z}} \left(\int_{\mathbb{T}} F(x+u,u)\, m(du) \right) \left(\int_{\mathbb{T}} G(v, y+v)\, m(dv) \right) \, m(dx) \int_\Omega \varphi(\omega)\, dP \\
&= \int_{\mathbb{T}\times\mathbb{T}\times\Omega} \left(\int_{\mathbb{T}} F(x-y+u,u) m(du) \right) G(x,y)\varphi(\omega)\, d(m^2 \times P).
\end{aligned}$$

Therefore we have shown that the limit function F^* has a deterministic and continuous version if the Fourier series of F converges first enough. From this fact it is easy to see that there exists a measurable set $\Delta \in \mathcal{F}$ with $P(\Delta) = 1$ such that if $\omega \in \Delta$

$$\frac{1}{n}\sum_{j=0}^{n-1} F(X_n(\omega)x, X_n(\omega)y) \to \int_{\mathbb{T}} F(x-y+u,u)\, m(du) \text{ uniformly in } (x,y) \in \mathbb{T}\times\mathbb{T}\, (n \to \infty)$$

for any F in a countable dense subset of $C(\mathbb{T}\times\mathbb{T})$. Thus the usual approximation argument lead us to the desired result. □

Remark 7.2. For a single measure-preserving system (T, μ) it is well known that (T, μ) is weak-mixing if and only if the product system $(T \times T, \mu \times \mu)$ is ergodic (e.g. Walters [6]). So Theorem 7.1 may be interpreted that the random rotation cannot be quenched 'weak-mixing' even if the noise system is exact. This is an analogue of the fact that any rotation cannot be weak-mixing.

References

[1] H. Anzai, Ergodic skew product transformations on the torus, Osaka Math. J. **3** (1951), 83–99.

[2] S. Kakutani, Random ergodic theorem and Markoff processes with a stable distribution, Proc. 2nd. Berkeley (1957), 241–261.

[3] T. Morita, Deterministic version lemmas in ergodic theory of random dynamical systems, Hiroshima Math. J. **1**8 (1988), 15–29.

[4] T. Morita, Generalized local limit theorem for Lasota-Yorke transformations, Osaka J. Math. **26** (1989), 579–595, Correction Osaka J. Math. **30** (1993), 611–612.

[5] S. Siboni, Statistical properties of skew-endomorphisms with Bernoulli bases, Physica D **99** (1997),407–427.

[6] P. Walters, An introduction to ergodic theory, Springer, Berlin-Hedelberg-New York 1982.

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