Invariant measures of random maps with the small entrances to the neighborhood of a fixed point

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Abstract: We study a random dynamical system such that one transformation is randomly selected from a family of transformations and then applied on each iteration. The selection of transformations is according to a probability density function which may depend on the position in the state space. We mainly consider piecewise monotonic random maps from $[0, 1]$ to $[0, 1]$ with fixed point 0. In the case when the entrances to the neighborhood of the fixed point 0 are small, we study the estimates of absolutely continuous invariant measures of random maps.

1 Introduction: Examples of deterministic maps

First we make clear the meaning of the entrance to the neighborhood of a fixed point for deterministic maps.

Roughly speaking, the entrance to the neighborhood of a fixed point is a part of the inverse image of the neighborhood of a fixed point. To make clear the meaning of this, we first consider some examples of deterministic maps.

Let $\varepsilon > 0$ be sufficiently small. Let $T : [0,1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} \dfrac{4}{3}x & \text{for } x \in \left[0, \frac{3}{4}\right), \\ 4x - 3 & \text{for } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

For this one-dimensional map T, the entrance to $[0,\varepsilon]$ is $T^{-1}([0,\varepsilon]) \cap [\frac{3}{4}, 1] = T|_{[\frac{3}{4},1]}^{-1}([0,\varepsilon])$.

In this example, the order of the size of the entrance to $[0,\varepsilon]$ becomes small, if the derivative of T becomes unbounded in the right neighborhood of $3/4$. To make clear this fact, we are going to consider the following map T_s. Let $T_s : [0,1] \to [0,1]$ be defined by

$$T_s(x) = \begin{cases} \dfrac{4}{3}x & \text{for } x \in \left[0, \frac{3}{4}\right), \\ (4x - 3)^{1/s} & \text{for } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

For this one-dimensional map T_s, the entrance to $[0,\varepsilon]$ is

$$T_s^{-1}([0,\varepsilon]) \cap \left[\frac{3}{4}, 1\right] = T_s|_{[\frac{3}{4},1]}^{-1}([0,\varepsilon]) = \left[\frac{3}{4}, \frac{3+\varepsilon^s}{4}\right].$$

If $s > 1$ becomes larger, the entrance to $[0,\varepsilon]$ becomes smaller.

T_s in this example has an absolutely continuous invariant probability measure μ_s with $\mu_s([0,\varepsilon]) \approx \varepsilon^s$ if $1 < s < 4$, where $f(\varepsilon) \approx g(\varepsilon)$ means that there exist positive constants c_1 and c_2 such that $c_1 g(\varepsilon) \leq f(\varepsilon) \leq c_2 g(\varepsilon)$ for any small $\varepsilon > 0$.

On the estimate of the size of an absolutely continuous invariant measure on $[0,\varepsilon]$, the map in the neighborhood of 0 is also important. We know an absolutely continuous invariant measure of the following deterministic map T_t with an indifferent fixed point.

Let $t \geq 1$ be a constant and let $T_t : [0,1] \to [0,1]$ be defined by $T_t(x) = x + x^t \mod 1$. Then, T_t has an absolutely continuous σ-finite invariant measure μ with $\mu_t([0,\varepsilon]) = \infty$ for $t \geq 2$, and $\mu_t([0,\varepsilon]) \approx \varepsilon^{2-t}$ for $1 \leq t < 2$.

If you would learn more about invariant measures of one-dimensional maps with indifferent fixed points, you should see [T].

2 Examples of random maps

We are going to consider the following three examples of random maps with indifferent fixed points and with unbounded derivatives.

Example 1A. Let a_0 and b_0 be constants with $1 < a_0 < b_0$ and let c_0 be a constant with $0 < c_0 < 1$ and $a_0 c_0 \geq 1$. Define $T_{t,s} : [0,1] \to [0,1]$ by

$$T_{t,s}(x) = \begin{cases} x + \dfrac{1}{t}\left(\dfrac{t}{t-1}\right)^t x^t & \text{for } x \in \left[0, \frac{t-1}{t}\right), \\ (tx - t + 1)^{1/s} & \text{for } x \in \left[\frac{t-1}{t}, 1\right], \end{cases}$$

where (t, s) is randomly selected from the parameter set

$$W = \{(t,s) \,|\, a_0 \leq t \leq b_0,\, a_0 c_0 \leq s \leq c_0 t\,\}$$

accordingly to the probability density function $p(t, s, x) = 2/\{c_0(b_0 - a_0)^2\}$.

Each map $T_{t,s}$ in Example 1A has an unbounded derivative. In the following example, the parameter set W is slightly changed.

Example 1Aδ. Let a_0, b_0 and c_0 be as in Example 1A and let $T_{t,s}$ be as in Example 1A. Change the parameter set W of the previous example to

$$W = \{(t,s) \,|\, a_0 \leq t \leq b_0,\, a_0 c_0 + \delta(t - a_0) \leq s \leq c_0 t\,\}.$$

The probability density function is also changed to $p(t, s, x) = 2/\{(c_0 - \delta)(b_0 - a_0)^2\}$.

In the following example, the parameter s can be chosen from only two values.

Example 1B. Let a_0 and b_0 be constants with $1 < a_0 < b_0$. Let a_1 and b_1 be constants with $1 < a_1 < b_1 < a_0$. Let $T_{t,s}$ be as in Example 1A. The pair of parameters (t, s) is uniformly randomly selected from the parameter set $[a_0, b_0] \times \{a_1, b_1\}$.

Later, we will consider the sizes of absolutely continuous invariant measures on $[0, \varepsilon]$ for these random maps with the indifferent fixed point 0 and with unbounded derivatives.

We are also going to consider the following three examples of random maps without indifferent fixed points and with unbounded derivatives.

Example 2A. Let a_0, b_0 and c_0 be as in Example 1A. Define $T_{t,s} : [0,1] \to [0,1]$ by

$$T_{t,s}(x) = \begin{cases} \dfrac{t}{t-1}x & \text{for } x \in \left[0, \frac{t-1}{t}\right), \\ (tx - t + 1)^{1/s} & \text{for } x \in \left[\frac{t-1}{t}, 1\right], \end{cases}$$

where the pair of parameters (t, s) is randomly selected as in Example 1A.

Example 2Aδ. Let a_0, b_0 and c_0 be as in Example 1A and let $T_{t,s}$ be as in Example 2A. The pair of parameters (t, s) is randomly selected as in Example 1Aδ.

Example 2B. Let a_0 and b_0 be constants with $1 < a_0 < b_0$. Let a_1 and b_1 be constants with $1 < a_1 < b_1 < a_0$. Define $T_{t,s} : [0,1] \to [0,1]$ by Let $T_{t,s}$ be as in Example 2A. The pair of parameters (t, s) is randomly selected as in Example 1B.

We will study the estimates of the sizes of the invariant measures of these random maps.

3 Definition of random maps

Before studying the estimates, following [I1] (cf. [Ba-G]), we make clear the definition of random maps and invariant measures.

Let $(W, \mathcal{B}, \nu)$ be a σ-finite measure space. We use W as a parameter space. Let $(X, \mathcal{A}, m)$ be a σ-finite measure space. We use X as a state space. Let $T_{\boldsymbol{t}} : X \to X$ ($\boldsymbol{t} \in W$) be a nonsingular transformation, which means that $m(T_{\boldsymbol{t}}^{-1}D) = 0$ if $m(D) = 0$

for any $D \in \mathcal{A}$. Assume that $T_{\boldsymbol{t}}(x)$ is a measurable function of $\boldsymbol{t} \in W$ and $x \in X$. Let $p : W \times X \to [0, \infty)$ be a measurable function which is a probability density function of $\boldsymbol{t} \in W$ for each $x \in X$, that is, $\int_W p(\boldsymbol{t}, x)\nu(d\boldsymbol{t}) = 1$ for each $x \in X$.

We define the position dependent random map $T = \{T_{\boldsymbol{t}}; p(t, x) : \boldsymbol{t} \in W\}$ as a Markov process with the following transition function:

$$\mathbf{P}(x, D) := \int_W p(\boldsymbol{t}, x) 1_D(T_{\boldsymbol{t}}(x))\, \nu(d\boldsymbol{t})$$

for any $x \in X$ and for any $D \in \mathcal{A}$, where 1_D is the indicator function of D.

The transition function $\mathbf{P}$ induces an operator $\mathbf{P}_*$ on measures on X defined by

$$\begin{aligned} \mathbf{P}_*\mu(D) &= \int_X \mathbf{P}(x, D)\, \mu(dx) \\ &= \int_X \int_W p(\boldsymbol{t}, x) 1_D(T_{\boldsymbol{t}}(x))\, \nu(d\boldsymbol{t})\, \mu(dx) \end{aligned}$$

for any measure μ on X and for any $D \in \mathcal{A}$.

If $\mathbf{P}_*\mu = \mu$, μ is called an *invariant measure* for the random map $T = \{T_{\boldsymbol{t}}; p(\boldsymbol{t}, x) : \boldsymbol{t} \in W\}$.

4 Estimates of invariant measures

We state the results for the estimates of the invariant measures of random maps with indifferent fixed points and with unbounded derivatives.

Theorems 1A and 1B follow from Theorems 7.2 and 7.3 in [I3], respectively.

Theorem 1A. *Let T be a random map as in Example 1A. Then, the random map T has an absolutely continuous σ-finite invariant measure μ which satisfies the following (1) and (2) :*

(1) If $a_0 \geq a_1 + 1$, then $\mu([0, \varepsilon]) = \infty$ for any small $\varepsilon > 0$.

(2) If $a_0 < a_1 + 1$*, then*

$$\frac{C_1 \varepsilon^{a_1+1-a_0}}{-\log \varepsilon} \le \mu([0,\varepsilon]) \le \frac{C_2 \varepsilon^{a_1+1-a_0-\kappa}}{-\log \varepsilon}$$

for any small $\varepsilon > 0$ *and any small* $\kappa > 0$*, where* C_1 *and* C_2 *are constants.*

Theorem 1B. *Let* T *be a random map as in Example 1B. Then, the conclusion (1) of Theorem 1A remains valid. However, under this situation, the conclusion (2) of Theorem 1 is changed to the following:*

*(2)** *If* $a_0 < a_1 + 1$*, then*

$$C_1^* \varepsilon^{a_1+1-a_0} \le \mu([0,\varepsilon]) \le C_2^* \varepsilon^{a_1+1-a_0-\kappa}$$

for any small $\varepsilon > 0$ *and any small* $\kappa > 0$*, where* C_1^* *and* C_2^* *are constants.*

We feel that there is not big difference between Examples 1A and 1Aδ. The size of an invariant measure of the random map of Example 1Aδ is slightly different from it of Example 1A. We can show the following:

Theorem 1Aδ. *Let* T *be a random map as in Example 1Aδ. Then, the random map* T *has an absolutely continuous* σ*-finite invariant measure* μ *which satisfies the following (1) and (2) :*

(1) If $a_0 \ge a_1 + 1$*, then* $\mu([0,\varepsilon]) = \infty$ *for any small* $\varepsilon > 0$*.*

(2) If $a_0 < a_1 + 1$*, then*

$$\frac{C_1 \varepsilon^{a_1+1-a_0}}{(\log \varepsilon)^2} \le \mu([0,\varepsilon]) \le \frac{C_2 \varepsilon^{a_1+1-a_0-\kappa}}{(\log \varepsilon)^2}$$

for any small $\varepsilon > 0$ *and any small* $\kappa > 0$*, where* C_1 *and* C_2 *are constants.*

Now, we state the results for random maps without indifferent fixed points and with unbounded derivatives.

Theorems 2A and 2B follow from Theorems 7.6 and 7.7 in [I3], respectively.

Theorem 2A. *Let T be a random map as in Example 2A. Then, the random map T has an absolutely continuous invariant probability measure μ which satisfies $\mu([0,\varepsilon]) \approx \varepsilon^{a_1}/(-\log\varepsilon)$.*

Theorem 2B. *Let T be a random map as in Example 2B. Then, the random map T has an absolutely continuous invariant probability measure μ which satisfies $\mu([0,\varepsilon]) \approx \varepsilon^{a_1}$.*

We feel that there is not big difference between Examples 2A and 2Aδ, which is the same situation as Examples 1A and 1Aδ. The size of an invariant measure of the random map of Example 2Aδ is slightly different from it of Example 2A. We can show the following:

Theorem 2Aδ. *Let T be a random map as in Example 2Aδ. Then, the random map T has an absolutely continuous invariant probability measure μ which satisfies $\mu([0,\varepsilon]) \approx \varepsilon^{a_1}/(\log\varepsilon)^2$.*

Remark. In Examples 1A, 1Aδ, 2A and 2Aδ, the probability density functions $p(t,s,x)$ are constants. However, we can obtain the same results, if $0 < \inf p(t,s,x) \le \sup p(t,s,x) < \infty$.

5 Idea of the proofs of the theorems

Here, we give the idea of the proofs of Theorems 1Aδ and 2Aδ, which are similar to Theorems 7.2, 7.3, 7.6 and 7.7 in [I3].

Consider the first return map R of the random map T on $[c,1]$, where c is a constant $0 < \varepsilon < c < 1$. The definition of the first return map of a random map is given in [I2] and [井 2].

To state how to make an invariant measure of the original random map T, we define two operators $\mathcal{U}_T$ and $\widetilde{\mathcal{U}}_T : L^\infty(m) \to L^\infty(m)$ by

$$\begin{aligned} \mathcal{U}_T f(x) &= \int_W p(t,x) f(T_t(x))\ \nu(dt) \ \text{ and} \\ \widetilde{\mathcal{U}}_T f &= \mathcal{U}(1_{X\setminus A} \cdot f) \ \text{ for } \ f \in L^\infty(m). \end{aligned}$$

Using these operators, we have the following theorem which was shown in [I2].

Theorem A. *Let T be a random map. Assume that the first return random map R of T on A is well defined. Further, assume that R has an invariant probability measure μ_A. Define the measure μ as*

$$\mu(D) = \sum_{n=0}^{\infty} \int_A \widetilde{\mathcal{U}}_T^n \left(\mathcal{U}_T 1_D\right)(x)\ \mu_A(dx) \quad \textit{for any } D \in \mathcal{A}.$$

Then, μ is a σ-finite invariant measure for T.

In Theorem A, the existence of an invariant probability measure μ_A of R is assumed, which can be obtained by the result in [I1] (We can see the summary in [井 1]).

Making suitable two new random maps which are deterministic in a small neighborhood of 0 and using the two comparison theorems (one is for the upper estimate and the other is for the lower estimate) in [I3], we can estimate the absolutely continuous invariant measures on $[0, \varepsilon]$.

References

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