Large deviation estimates related to arcsine laws for intermittent maps

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Intermittent maps are typical examples of infinite ergodic transformations. They are also toy models of intermittent phenomena in statistical physics, such as intermittent turbulence. A variety of asymptotic behaviors of intermittent maps have been studied, for example, in [10, 11, 12, 3, 4, 5, 8, 9, 7, 1]. In the following, we consider large deviation estimates related to arcsine laws for intermittent maps. This manuscript is based on [6].

1 Setting

For simplicity, we focus only on the following map. Let $0 < \alpha < 1$. The map $T = T_\alpha : [0,1] \to [0,1]$ is defined by

$$Tx = \begin{cases} x + 2^{1/\alpha} x^{1+1/\alpha}, & x \le 1/2 \\ 2x - 1, & x > 1/2. \end{cases}$$

Then $T0 = 0$, $T'0 = 1$ and $T''x > 0$ $(0 < x < 1/2)$. Hence the point $x = 0$ is a weakly repelling fixed point. For almost every starting point x, the orbit

$$(T^k x)_{k=0}^{\infty} = (x, Tx, T^2x = T(Tx), \dots)$$

spends long time on $[0, 1/2]$, and it bursts into $(1/2, 1]$ intermittently.

2 Earlier studies

Let us recall the Dynkin–Lamperti theorem for intermittent maps. Set

$$G_n(x) = \max\{k \le n \;:\; T^k x > 1/2\}, \quad n \in \mathbb{N},\ x \in [0,1].$$

In other words, $G_n(x)$ denotes the last arrival time of the orbit to $(1/2, 1]$ before time n. Thaler [10] obtained the following result, imitating the Dynkin–Lamperti theorem for renewal processes.

Theorem A (Thaler [10]). For any absolutely continuous probability measure $\nu(dx)$ on $[0,1]$ and for any $0 \le t \le 1$,

$$\nu\left[x \in [0,1] \;:\; \frac{G_n(x)}{n} \le t\right] \to \frac{\sin(\pi\alpha)}{\pi}\int_0^t \frac{ds}{s^{1-\alpha}(1-s)^{\alpha}}, \quad \text{as } n \to \infty. \quad (1)$$

In other words, the random variable $G_n(x)/n$ under $\nu(dx)$ converges in distribution to the beta distribution $B(\alpha, 1-\alpha)$.

If $\alpha = 1/2$, then $B(1/2, 1/2)$ is the usual arcsine distribution. This kind of limit theorems are called the Dynkin–Lamperti theorem, and also hold for renewal processes, 1-dimensional diffusion processes and 1-dimensional Lévy processes under suitable settings. Theorem A was further extended and deepened in [11, 12, 3, 4, 5, 8, 9].

3 Main result

Note that

$$(\text{the right-hand sinde of (1)}) \sim \frac{\sin(\pi\alpha)}{\pi\alpha} t^{\alpha}, \quad \text{as } t \to 0+.$$

We now explain the main result.

Theorem B. Assume that an absolutely continuous measure $\nu(dx)$ on $[0,1]$ admits a Riemann-integrable density $\nu(dx)/dx$ and its support $\operatorname{supp}(\nu)$ does not include 0. Let $\{c(n)\}_{n=0}^{\infty}$ be a sequence on $[0,1]$ satisfying $c(n) \to 0$ and $nc(n) \to \infty$, as $n \to \infty$. Then

$$\nu\left[x \in [0,1] \;:\; \frac{G_n(x)}{n} \le c(n)\right] \asymp c(n)^{\alpha}, \quad \text{as } n \to \infty.$$

For the proof, we adopt a method of double Laplace transforms, imitating a proof of similar large deviation estimates for 1-dimensional diffusion processes [2].

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